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# THE IMPACT OF TWO INDEPENDENT GAUSSIAN WHITE NOISES ON THE BEHAVIOR OF A STOCHASTIC EPIDEMIC MODEL

***Mehmet Yavuz*[1,2]*, Lahcen Boulaasair*[3]*, Hassane Bouzahir*[3]**
***Mamadou Abdoul Diop*[4]*, Brahim Benaid*[3]**

[1] *Necmettin Erbakan University, Department of Mathematics and Computer Sciences, 42090 Meram Konya, Turkey*
[2] *Centre for Environmental Mathematics, Faculty of Environment, Science and Economy University of Exeter, Cornwall TR10 9FE, UK*
[3] *ISTI Lab, Ibn Zohr University, ENSA, PO Box 1136, Agadir, Morocco*
[4] *UFR Applied Sciences and Technologies, Department of Mathematics Gaston Berger University, PO Box 234, Saint-Louis, Senegal*
*M.Yavuz@exeter.ac.uk, boulaasair@gmail.com, hbouzahir@yahoo.fr, ordydiop@gmail.com brahim.benaid@gmail.com*

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**Abstract.** The aim of this paper is to investigate a stochastic SIS (Susceptible, Infected, Susceptible) epidemic model in which the disease transmission coefficient and the death rate are subject to random disturbances. Using the convergence theorem for local martingales and solving the Fokker-Planck equation associated with the one-dimensional stochastic differential equation, we demonstrate that the disease will almost surely persist in the mean. In the case of global asymptotic stability of the endemic equilibrium for a SIS deterministic epidemic model, we formulate suitable conditions guaranteeing that the stochastic SIS model has a unique ergodic stationary distribution. Furthermore, we deal with the exponential extinction of the disease. Finally, some numerical simulations are provided to illustrate the obtained analytical results.

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## 1. Introduction

Compartmental epidemic models are used to describe the transmission of infectious diseases within a population. These models involve categorizing the population into distinct compartments corresponding to their health status. This mathematical modeling approach was pioneered by Kermack and McKendrick, who utilized a system of three deterministic differential equations to scrutinize the population dynamics as individuals move among the susceptible, infected, and recovered compartments [1].

Epidemiological systems exhibit complexity which arises from the inherent randomness in nature (see [2–11] and the references therein). The way infectious diseases are transmitted can vary significantly due to various factors such as the nature of the diseases, environmental conditions, individual behaviors, and the efficacy of disease prevention and control measures [12]. In the literature on stochastic compartmental models, the basic one was formulated by Gray et al. [4] using a system of two stochastic differential equations

$$\begin{cases} dS(t) = [\mu(S(0)+I(0)) - \mu S(t) - \beta S(t)I(t) + \lambda I(t)\,]dt - \sigma S(t)I(t)\,dB(t), \\ dI(t) = [\beta S(t)I(t) - (\mu+\lambda)I(t)\,]dt + \sigma S(t)I(t)\,dB(t). \end{cases} \tag{1}$$

Herein, $S(t)$ and $I(t)$ are the numbers of susceptible and infected individuals with initial values $S(0)$ and $I(0)$, respectively. The parameter $\mu$ is the per capita birth and death rate, $\beta$ is the disease transmission coefficient between susceptible and infected individuals, $\lambda$ the recovery rate, and $\sigma$ is the white noise intensity associated with the Brownian motion $B(t)$ $(t \geq 0)$. From model (1), the total population $S(t)+I(t) = S(0)+I(0)$ is constant for all $t > 0$. Therefore, instead of analyzing system (1), it is sufficient to study the following one-dimensional stochastic differential equation

$$dI(t) = \Big(\big[\beta(S(0)+I(0)-I(t)) - (\mu+\lambda)\big]dt + \sigma\big(S(0)+I(0)-I(t)\big)dB(t)\Big)I(t). \tag{2}$$

In [4], the authors demonstrated the existence, uniqueness, and positivity of a solution to equation (2). Conditions governing the extinction and persistence of the disease have been established based on the intensity of white noise. In the case of persistence, the paper [4] shows the existence of a unique stationary distribution for (2) and provides expressions for its mean and variance. Another investigation of equation (2) was conducted by Xu [6], where the stochastic extinction threshold of (2) is determined by Feller's test for explosions [13]. Solving the Fokker-Planck equation related to (2), the study establishes the existence and uniqueness of the invariant density for (2). Furthermore, it presents conditions ensuring the prevalence of the disease in terms of this invariant density.

To enhance the analysis of the SIS epidemic model class, we add Gaussian white noise around the parameter $\mu$ in model (1). This leads to a modified version of the SIS stochastic model (1), defined by the following:

$$\begin{cases} dS(t) = \Big[\mu(S(0)+I(0)) - \mu S(t) - \beta S(t)I(t) + \lambda I(t)\Big]dt - \xi_1 S(t)dB_1(t) \\ \qquad\qquad - \xi_2 S(t)I(t)dB_2(t), \\ dI(t) = \Big[\beta S(t)I(t) - (\mu+\lambda)I(t)\Big]dt - \xi_1 I(t)dB_1(t) + \xi_2 S(t)I(t)dB_2(t), \end{cases} \tag{3}$$

where $B_1(t)$ and $B_2(t)$ are two independent Brownian motions, $\xi_i$ the intensity of their corresponding white noises ($i \in \{1,2\}$).
In contrast to the model (1), a significant characteristic of the stochastic model (3) is that the total population $\mathscr{N}(t) := S(t) + I(t)$ is no longer constant, due to the following equation:

$$d\mathscr{N}(t) = \Big[\mu(S(0)+I(0)) - \mu\mathscr{N}(t)\Big]dt - \xi_1\mathscr{N}(t)dB_1(t). \tag{4}$$

The innovation in this article lies in employing the convergence theorem for local martingales [14] and the stationary ergodic characteristics of the stochastic process (4), under specific conditions, to compute the persistence threshold of model (3). Additionally, the extinction threshold is determined by using the strong law of large numbers for martingales [15]. Moreover, conditions ensuring stationarity and ergodicity of the model (3) are given. To validate the analytical findings for each case study, we conduct some numerical simulations using the Matlab2015b software.

## 2. Main results

Throughout this paper, we define $(\Omega, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$ as a complete probability space with a filtration $(\mathscr{F}_t)_{t\geq 0}$ satisfying the usual conditions. The set of real numbers is denoted by $\mathbb{R}$. We define $\mathbb{R}^d_+$ as the set of $(x_1,...,x_d) \in \mathbb{R}^d$ such that $x_i > 0$ for $i = 1,...,d$, and we denote the time average of $\varphi(t)$ as $\langle\varphi(t)\rangle = t^{-1}\int_0^t \varphi(r)dr$, where $\varphi$ is a continuous function. Additionally, the abbreviation "a.s." stands for "almost surely".

Let us now consider the $d$-dimensional stochastic differential equation

$$d\mathscr{X}(t) = F(\mathscr{X}(t),t)dt + G(\mathscr{X}(t),t)d\mathscr{W}(t) \quad \text{for all } t \geq t_0, \tag{5}$$

with initial value $\mathscr{X}(t_0) \in \mathbb{R}^d$. The symbol $\mathscr{W}(t)$ is for an n-dimensional standard Brownian motion. We denote by $\mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0,\infty];\mathbb{R})$ the family of all functions $\mathscr{V}(x,t)$ defined on $\mathbb{R}^d \times [t_0,\infty]$ such that they are continuously twice differentiable in $\mathscr{X}$ and once in $t$. The differential operator $\mathscr{L}$ of Equation (5) is defined by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(\mathscr{X},t)\frac{\partial}{\partial \mathscr{X}_i} + \frac{1}{2}\sum_{i,j=1}^{d}[G^T(\mathscr{X},t)G(\mathscr{X},t)]_{ij}\frac{\partial^2}{\partial \mathscr{X}_i \partial \mathscr{X}_j}.$$

If $\mathscr{L}$ acts on $\mathscr{V} \in \mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0,\infty];\mathbb{R})$, then

$$\mathscr{L}\mathscr{V}(\mathscr{X},t) = \mathscr{V}_t(\mathscr{X},t) + \mathscr{V}_{\mathscr{X}}(\mathscr{X},t)F(\mathscr{X},t) + \frac{1}{2}trace[G^T(\mathscr{X},t)\mathscr{V}_{\mathscr{X}\mathscr{X}}(\mathscr{X},t)G(\mathscr{X},t)],$$

where

$$V_t = \frac{\partial \mathscr{V}}{\partial t},\ \mathscr{V}_{\mathscr{X}} = \Big(\frac{\partial \mathscr{V}}{\partial \mathscr{X}_1},...,\frac{\partial \mathscr{V}}{\partial \mathscr{X}_d}\Big), \mathscr{V}_{\mathscr{X}\mathscr{X}} = \Big(\frac{\partial^2 \mathscr{V}}{\partial \mathscr{X}_i \partial \mathscr{X}_j}\Big)_{d\times d}.$$

Here, $G^T(\mathscr{X},t)$ and $trace[G^T(\mathscr{X},t)\mathscr{V}_{\mathscr{X}\mathscr{X}}(\mathscr{X},t)G(\mathscr{X},t)]$ stand for the transpose of the vector $G(\mathscr{X},t)$ and the trace of the matrix $G^T(\mathscr{X},t)\mathscr{V}_{\mathscr{X}\mathscr{X}}(\mathscr{X},t)G(\mathscr{X},t)$, respectively.
The Itô formula is presented by the following equality:

$$d\mathscr{V}(\mathscr{X}(t),t) = \mathscr{L}\mathscr{V}(\mathscr{X}(t),t)dt + \mathscr{V}_{\mathscr{X}}(\mathscr{X}(t),t)G(\mathscr{X}(t),t)\,d\mathscr{W}(t).$$

The following theorem concerns the existence of a global positive solution for model (3). Its proof is similar to that in [16], and therefore we omit it here.

**Theorem 1.** *For any given initial condition $(S(0),I(0)) \in \mathbb{R}^2_+$, the stochastic model (3) admits, almost surely, a global positive solution.* □

The proof of the theorem above shows that there is no explosion of the solution when $t < \infty$, but at infinity we can not be sure if the explosion can happen or not. To respond to this dilemma, we present the following lemma:

**Lemma 1** *Let $(S(t),I(t))$ be the solution of model (3) with initial value $(S(0),I(0)) \in \mathbb{R}^2_+$. Then*

$$\lim_{t\to\infty} \mathscr{N}(t) < \infty \qquad a.s.$$

*and*

$$\lim_{t\to\infty} \langle \mathscr{N}(t)\rangle = \mathscr{N}(0) \qquad a.s.$$

*Furthermore, $\mathscr{N}(t)$ has a unique ergodic stationary distribution with density $\pi$ given by*

$$\pi(x) = \left(\frac{2\mu\mathscr{N}(0)}{\xi_1^2}\right)^{1+\frac{2\mu}{\xi_1^2}} \Gamma^{-1}\Big(1+\frac{2\mu}{\xi_1^2}\Big)\, x^{-2\left(1+\frac{\mu}{\xi_1^2}\right)}\, e^{-\frac{2\mu\mathscr{N}(0)}{\xi_1^2 x}},$$

*where $\Gamma$ is the Gamma function.*

PROOF From the differential equation (4), we have

$$\begin{aligned}\mathscr{N}(t) =& \mathscr{N}(0) - \xi_1\int_0^t e^{-\mu(t-s)}\mathscr{N}(s)dB_1(s)\\ =& \mathscr{N}(0) - \mathscr{A}(t),\end{aligned}$$

where $\mathscr{A}(t) = \xi_1\int_0^t e^{-\mu(t-s)}\mathscr{N}(s)dB_1(s)$ is a martingale vanishing at $t=0$.

According to the convergence theorem for local martingales, we conclude that

$$\lim_{t\to\infty} \mathscr{N}(t) < \infty \qquad \text{a.s.} \tag{6}$$

On the other hand, one easily derives

$$\langle \mathscr{A}(t) \rangle = \frac{\xi_1}{\mu t}\left[\int_0^t N(r)dB_1(r) - \int_0^t e^{-\mu(t-r)} S(r)dB_1(r)\right].$$

Bearing in mind (6) and the strong law of large numbers for martingales, we get

$$\lim_{t\to\infty} \langle \mathscr{A}(t) \rangle = 0 \qquad \text{a.s.}$$

Therefore

$$\lim_{t\to\infty} \langle \mathscr{N}(t) \rangle = \mathscr{N}(0) \qquad \text{a.s.}$$

Next, we consider the mathematical stationarity and ergodicity of the stochastic process (4).

A stochastic process is stationary if its probability distribution varies more or less constantly over a certain period of time. Solving the Fokker-Planck equation associated with the stochastic equation (4) shows that the stochastic process $\mathscr{N}(t)$ has a unique stationary distribution with a density defined by

$$\pi(x) = \left(\frac{2\mu\,\mathscr{N}(0)}{\xi_1^2}\right)^{1+\frac{2\mu}{\xi_1^2}} \Gamma^{-1}\left(1+\frac{2\mu}{\xi_1^2}\right) x^{-2\left(1+\frac{\mu}{\xi_1^2}\right)} e^{-\frac{2\mu\,\mathscr{N}(0)}{\xi_1^2 x}}.$$

Assign $a(r) = \mu\mathscr{N}(0) - \mu r$ and $b(r) = -\xi_2 r$.
By direct calculation, we obtain

$$\int_c^z \frac{a(r)}{b^2(r)}du = \frac{\mu\mathscr{N}(0)}{\xi_1^2}\left(\frac{1}{c}-\frac{1}{z}\right) + \frac{\mu}{\xi_1^2}(\log c - \log z),$$

where $c$ is a positive constant.
Then

$$e^{-2\int_c^z \frac{a(r)}{b^2(r)}dr} = c^{-\frac{2\mu}{\xi_1^2}} e^{-2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{c}} z^{\frac{2\mu}{\xi_1^2}} e^{2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{z}}.$$

For all $s > 0$, we have

$$\begin{aligned}\int_0^s e^{-2\int_c^z \frac{a(r)}{b^2(r)}dr} dz &= c^{-\frac{2\mu}{\xi_1^2}} e^{-2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{c}} \int_0^s z^{\frac{2\mu}{\xi_1^2}} e^{2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{z}} dz \\ &\geq c^{-\frac{2\mu}{\xi_1^2}} e^{-2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{c}} e^{2\frac{\mu\mathscr{N}(0)}{\xi_1^2}\frac{1}{s}} \int_0^s z^{\frac{2\mu}{\xi_1^2}} dz.\end{aligned}$$

Therefore,

$$\lim_{s\to\infty}\int_0^s e^{-2\int_c^z \frac{a(r)}{b^2(r)}dr}\,dz=\infty.$$

One more integration gives

$$\int_0^\infty b^{-2}(z)\,e^{2\int_c^z \frac{a(r)}{b^2(r)}dr}dz=\xi_1^{-2}\,c^{\frac{2\mu}{\xi_1^2}}\,e^{\frac{2\mu\mathcal{N}(0)}{\xi_1^2}\frac{1}{c}}\int_0^\infty z^{-2\left(1+\frac{\mu}{\xi_1^2}\right)}\,e^{-\frac{2\mu\mathcal{N}(0)}{\xi_1^2 z}}.$$

Performing the variable changing: $y=-\dfrac{2\mu\mathcal{N}(0)}{\xi_1^2 z}$, we get

$$\int_0^\infty b^{-2}(z)\,e^{2\int_c^z \frac{a(r)}{b^2(r)}dr}dz=\xi_1^{-2}\,c^{\frac{2\mu}{\xi_1^2}}\,e^{\frac{2\mu\mathcal{N}(0)}{\xi_1^2}\frac{1}{c}}\left(\frac{2\mu\mathcal{N}(0)}{\xi_1^2}\right)^{-\left(1+\frac{2\mu}{\xi_1^2}\right)}\Gamma\left(1+\frac{2\mu}{\xi_1^2}\right)<\infty.$$

By Theorem 1.16 in [17], we conclude that the stochastic process (4) is ergodic. This completes the proof. ■

**Remark 1** The solutions of stochastic differential equations can be considered as a time series. Stationarity and ergodicity are important properties of the time series. A process is said to be stationary if its statistical properties do not vary over time, namely its mean, its variance, or even its covariance. However, it is said to be ergodic if its temporal average converges in mean square towards its statistical expectation.□

### 2.1. Disease persistence

Let $\mathscr{R}^P=\dfrac{1}{\mu+\lambda+\frac{\xi_1^2}{2}}\left(\beta\mathcal{N}(0)-\xi_2^2\dfrac{\mu\mathcal{N}^2(0)}{2\mu-\xi_1^2}\right)$ be the persistence threshold of the model (3).

**Theorem 2.** *If* $\mu>\dfrac{\xi_1^2}{2}$ *and* $\mathscr{R}^P>1$*, then the disease will almost surely be persistent.*

PROOF Applying the Itô formula yields

$$d\ln I(r)=\left[\beta S(r)-\left(\mu+\lambda+\frac{\xi_1^2}{2}\right)-\frac{\xi_2^2}{2}S^2(r)\right]dr-\xi_1 dB_1(r)+\xi_2 S(r)dB_2(r). \tag{7}$$

It follows that

$$\frac{\ln I(t)-\ln I(0)}{t}=\beta\langle S(t)\rangle-\left(\mu+\lambda+\frac{\xi_1^2}{2}\right)-\frac{\xi_2^2}{2}\langle S^2(t)\rangle-\xi_1\frac{B_1(t)}{t}+\frac{\xi_2}{t}\int_0^t S(r)dB_2(r). \tag{8}$$

From the stochastic model (3), we have

$$\langle S(t)\rangle = -\frac{\mathcal{N}(t)-\mathcal{N}(0)}{\mu t} + \mathcal{N}(0) - \langle I(t)\rangle - \frac{\xi_1}{\mu t}\int_0^t \mathcal{N}(r)dB_1(r). \tag{9}$$

Combining (9) with (8) gives

$$\begin{aligned}\beta\langle I(t)\rangle \geq & \beta\mathcal{N}(0) - \Big(\mu+\lambda+\frac{\xi_1^2}{2}\Big) - \frac{\xi_2^2}{2}\langle \mathcal{N}^2(t)\rangle - \beta\frac{\mathcal{N}(t)-\mathcal{N}(0)}{\mu t} - \frac{\ln I(t)-\ln I(0)}{t} \\ & -\xi_1\frac{B_1(t)}{t} + \frac{\xi_2}{t}\int_0^t S(r)dB_2(r) - \beta\frac{\xi_1}{\mu t}\int_0^t \mathcal{N}(r)dB_1(r). \end{aligned}\tag{10}$$

The ergodic property of the stochastic process $\mathcal{N}(t)$ entails

$$\lim_{t\to\infty}\langle \mathcal{N}^2(t)\rangle = \int_0^\infty x^2\pi(x)dx.$$

Achieving an integration by parts, we get

$$\int_0^\infty x^2\pi(x)dx = \left(\frac{2\mu\mathcal{N}(0)}{\xi_1^2}\right)^2 \frac{\Gamma\Big(\frac{2\mu}{\xi_1^2}-1\Big)}{\Gamma\Big(\frac{2\mu}{\xi_1^2}+1\Big)} = \frac{2\mu\mathcal{N}^2(0)}{2\mu-\xi_1^2},$$

together with (10), we obtain

$$\begin{aligned}\beta\liminf_{t\to\infty}\langle I(t)\rangle \geq & \beta\mathcal{N}(0) - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right) - \xi_2^2\frac{\mu\mathcal{N}^2(0)}{2\mu-\xi_1^2} \quad \text{a.s.,} \\ = & \left(\mu+\lambda+\frac{\xi_1^2}{2}\right)\Big(\mathscr{R}^P-1\Big) \quad \text{a.s.}\end{aligned}$$

If $\mathscr{R}^P > 1$, we deduce that

$$\liminf_{t\to\infty}\langle I(t)\rangle > 0 \quad \text{a.s.,}$$

which means that the disease will almost surely prevail. The proof is complete. ■

**Remark 1** Adding the condition $\mu > \frac{\xi_1^2}{2}$ in the statement of Theorem 2 is necessary for $\Gamma\Big(\frac{2\mu}{\xi_1^2}-1\Big)$ to be finite. □

## 2.2. Stationary distribution

The deterministic counterpart of the two stochastic systems (1) and (3) has a unique endemic equilibrium $(S_e, I_e) = \Big(\frac{\mu+\lambda}{\beta}, \frac{(\mu+\lambda)}{\beta}\big(\mathscr{R}-1\big)\Big)$ which is globally

asymptotically stable, if $\mathscr{R} := \dfrac{\beta \mathscr{N}(0)}{\mu+\lambda} > 1$ (see [18]).
This subsection is devoted to establish necessary conditions for the stochastic process (3) to be stationary. The proof of the following result is similar to that of Theorem 6 in [18]. Hence, we omit it here.

**Theorem 3.** *If* $\mathscr{R} > 1$ *and* $0 < \mathscr{A}_3 < \min\left(\mathscr{A}_1 S_e^2,\ \mathscr{A}_2 I_e^2\right)$*, then the model (3) has a unique ergodic stationary distribution, where*

$$\mathscr{A}_1 = \mu - \frac{3}{2}\xi_1^2 - \frac{2\mu}{\beta} I_e \xi_2^2,\ \ \mathscr{A}_2 = \mu - \frac{3}{2}\xi_1^2 \ \ \text{and}\ \ \mathscr{A}_3 = 3\xi_1^2\left(S_e^2 + I_e^2\right) + \frac{2\mu I_e}{\beta}\left(\frac{\xi_1^2}{2} + S_e^2 \xi_2^2\right).$$

### 2.3. Stochastic extinction

**Theorem 4.** *If* $\xi_2^2 \leq \dfrac{\beta}{\mathscr{N}(0)}$ *and* $\mathscr{R}^X := \dfrac{\mathscr{N}(0)}{\mu+\lambda+\frac{\xi_1^2}{2}}\left[\beta - \dfrac{\xi_2^2}{2}\mathscr{N}(0)\right] < 1$ *holds, then*

$$\lim_{t\to\infty} I(t) = 0 \quad \text{and} \quad \lim_{t\to\infty} \langle S(t)\rangle = \mathscr{N}(0) \qquad \text{a.s.}$$

□

PROOF Returning to (8), we get

$$\frac{\ln I(t)}{t} \leq \frac{\ln I(0)}{t} + \frac{1}{t}\int_0^t \left[ -\frac{\xi_2^2}{2}\left(S(r) - \frac{\beta}{\xi_2^2}\right)^2 + \frac{\beta^2}{2\xi_2^2} - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right)\right] dr - \xi_1 \frac{B_1(t)}{t} + \frac{\xi_2}{t}\int_0^t S(r)\,dB_2(r). \tag{11}$$

By Lemma 1, we have

$$\limsup_{t\to\infty} \langle S(t)\rangle \leq \mathscr{N}(0) \quad \text{a.s.}$$

It follows that, for any positive number $\varepsilon$, there exists $\mathscr{T}_0 > 0$ such that

$$\langle S(t)\rangle \leq \mathscr{N}(0) + \frac{\varepsilon}{\xi_2^2} \qquad \text{for all } t \geq \mathscr{T}_0.$$

We assume that $\xi_2^2 \leq \dfrac{\beta}{\mathscr{N}(0)}$. Then

$$\langle S(t)\rangle \leq \mathscr{N}(0) + \frac{\varepsilon}{\xi_2^2} \leq \frac{\beta+\varepsilon}{\xi_2^2} \qquad \text{for all } t \geq \mathscr{T}_0,$$

combined with (11), we obtain for all $t \geq \mathscr{T}_0$

$$\begin{aligned}\frac{\ln I(t)}{t} \leq & \frac{\ln I(0)}{t} - \frac{\xi_2^2}{2}\left[\frac{\beta+\varepsilon}{\xi_2^2} - \left\langle S(t)\right\rangle\right]^2 + \frac{(\beta+\varepsilon)^2}{2\xi_2^2} - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right) - \xi_1\frac{B_1(t)}{t} \\ & + \frac{\xi_2}{t}\int_0^t S(r)dB_2(r) \\ \leq & \frac{\ln I(0)}{t} - \frac{\xi_2^2}{2}\left[\frac{\beta}{\xi_2^2} - \mathscr{N}(0)\right]^2 + \frac{(\beta+\varepsilon)^2}{2\xi_2^2} - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right) - \xi_1\frac{B_1(t)}{t} \\ & + \frac{\xi_2}{t}\int_0^t S(r)dB_2(r) \\ = & \frac{\ln I(0)}{t} + \beta\mathscr{N}(0) - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right) - \frac{\xi_2^2}{2}\mathscr{N}^2(0) + \varepsilon\frac{2\beta+\varepsilon}{2\xi_2^2} - \xi_1\frac{B_1(t)}{t} \\ & + \frac{\xi_2}{t}\int_0^t S(r)dB_2(r).\end{aligned}$$

Letting $\varepsilon \to 0$ leads to

$$\begin{aligned}\limsup_{t\to\infty}\frac{\ln I(t)}{t} \leq & \beta\mathscr{N}(0) - \left(\mu+\lambda+\frac{\xi_1^2}{2}\right) - \frac{\xi_2^2}{2}\mathscr{N}^2(0) \qquad \text{a.s.,} \\ = & \left(\mu+\lambda+\frac{\xi_1^2}{2}\right)\left(\mathscr{R}^X - 1\right) \qquad \text{a.s.}\end{aligned}$$

If $\mathscr{R}^X < 1$, then

$$\limsup_{t\to\infty}\frac{\ln I(t)}{t} < 0 \qquad \text{a.s.}$$

Thus

$$\lim_{t\to\infty} I(t) = 0 \qquad \text{a.s.}$$

Hence, for any positive number $\delta$, there exists a $\mathscr{T}_1 > 0$ such that $I(t) \leq \delta$ for all $t \geq \mathscr{T}_1$.
Integrating the first equation of system (3), for all $t \geq \mathscr{T}_1$, gives

$$\begin{aligned}\frac{S(t)-S(\mathscr{T}_1)}{t} \geq & \frac{t-\mathscr{T}_1}{t}\mu\,\mathscr{N}(0) - \frac{\delta\beta}{t}\int_{\mathscr{T}_1}^t S(r)dr - \frac{\mu}{t}\int_{\mathscr{T}_1}^t S(r)dr \\ & - \frac{\xi_1}{t}\int_{\mathscr{T}_1}^t S(r)dB_1(r) - \frac{\xi_2}{t}\int_{\mathscr{T}_1}^t S(r)I(r)dB_2(r).\end{aligned}$$

Letting $\delta \to 0$, leads to

$$\liminf_{t\to\infty}\langle S(t)\rangle \geq \mathscr{N}(0) \quad \text{a.s.}$$

By Lemma 1, we have

$$\limsup_{t\to\infty}\langle S(t)\rangle \leq \mathscr{N}(0) \quad \text{a.s.}$$

Therefore

$$\lim_{t\to\infty}\langle S(t)\rangle = \mathscr{N}(0) \quad \text{a.s.}$$

## 3. Numerical simulations

Numerical verification helps to validate the theoretical results and understand the underlying phenomena. By means of the Milstein method mentioned in Higham [19], we consider the following discretized version of the stochastic model (3):

$$\begin{aligned} S(i+1) = &S(i) + \Big(\mu\mathscr{N}(0) - \mu S(i) - \beta S(i)I(i) + \lambda I(i)\Big)dt - \sigma_1 S(i)\sqrt{dt}\, rn_1 \\ &- \sigma_2 S(i)I(i)\sqrt{dt}\, rn_2, \\ I(i+1) = &I(i) + \Big(\beta S(i)I(i) - (\mu+\lambda)I(i)\Big)dt - \sigma_1 I(i)\sqrt{dt}\, rn_1 + \sigma_2 S(i)I(i)\sqrt{dt}\, rn_2, \end{aligned}$$

where $dt > 0$ is the time increment and $rn_j$ $(j = 1,2)$ are normally distributed random variables. We present the following examples in order to check the analytical results of Theorems 2, 3 and 4.

**Example 1** (Disease persistence – Stationary distribution)
We consider the following initial data: $(S(0), I(0)) = (0.7, 0.3)$. We choose

$$\xi_1 = 0.1,\ \xi_2 = 0.2,\ \beta = 0.65,\ \mu = 0.35,\ \lambda = 0.1.$$

Then

$$\mathscr{R}^P = 1.384,\ \ \mu - \frac{\xi_1^2}{2} = 0.345$$

and

$$\liminf_{t\to\infty}\langle I(t)\rangle = 0.2688 > 0 \qquad \text{a.s.}$$

Consequently, the conditions of Theorem 2 are verified, and the disease will almost surely prevail (Fig. 1).
Moreover

$$\mathscr{R} = 1.4444,\ \mathscr{A}_1 = 0.3309,\ \mathscr{A}_2 = 0.335,\ \mathscr{A}_3 = 0.0252,$$

and

$$\mathscr{A}_3 - \min\left(\mathscr{A}_1 S_e^2,\ \mathscr{A}_2 I_e^2\right) = -0.0065.$$

According to Theorem 3, the model (3) has a unique ergodic stationary distribution and the deterministic endemic equilibrium $(S_e, I_e) = (0.6923, 0.3077)$ is globally asymptotically stable, as shown in Figures 1 and 2. □

**Example 2** (Disease extinction)
Let

$$S(0) = 0.3,\ I(0) = 0.7,\ \xi_1 = 0.7,\ \xi_2 = 0.6,\ \beta = 0.9,\ \mu = 0.1,\ \lambda = 0.8.$$

We compute that

$$\mathscr{R}^X = 0.6288 \ \text{and} \ \frac{\xi_2^2}{2} - \frac{\beta}{\mathscr{N}(0)} = -0.72.$$

Then

$$\limsup_{t\to\infty} \frac{\ln I(t)}{t} = -0.425 < 0 \qquad \text{a.s.}$$

From Figure 3, the result of Theorem 4 is confirmed. □

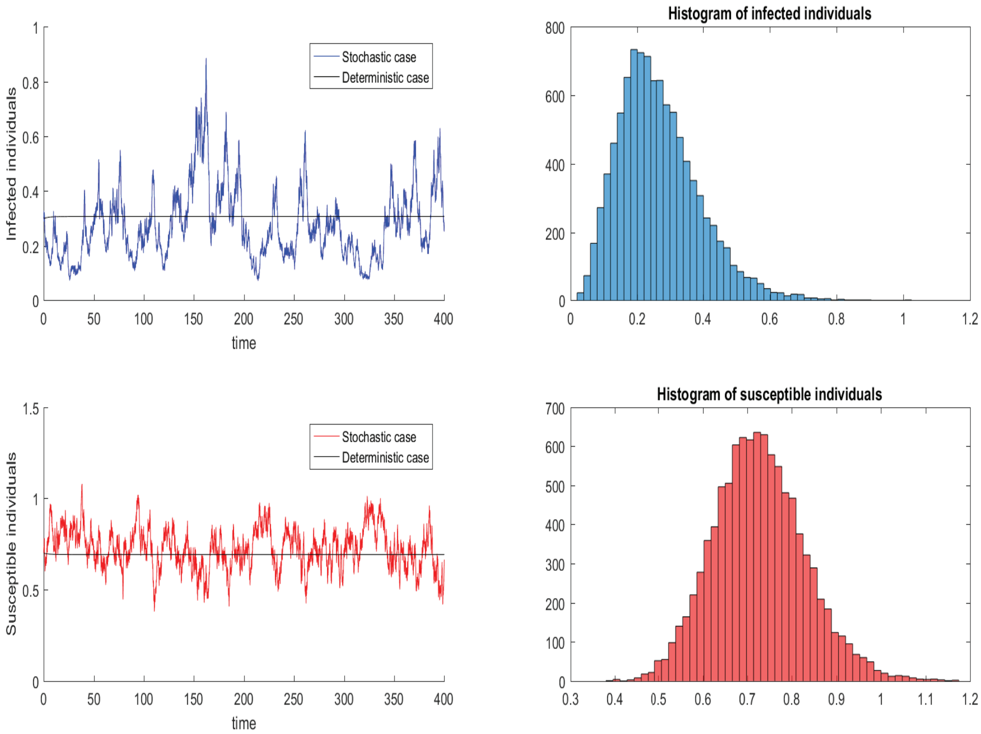


Fig. 1. The left-hand column presents the trajectories of individuals $S$ and $I$ of stochastic system (3), and its deterministic system, respectively. The right-hand column presents the frequency histogram fitting curves at time $t = 400$

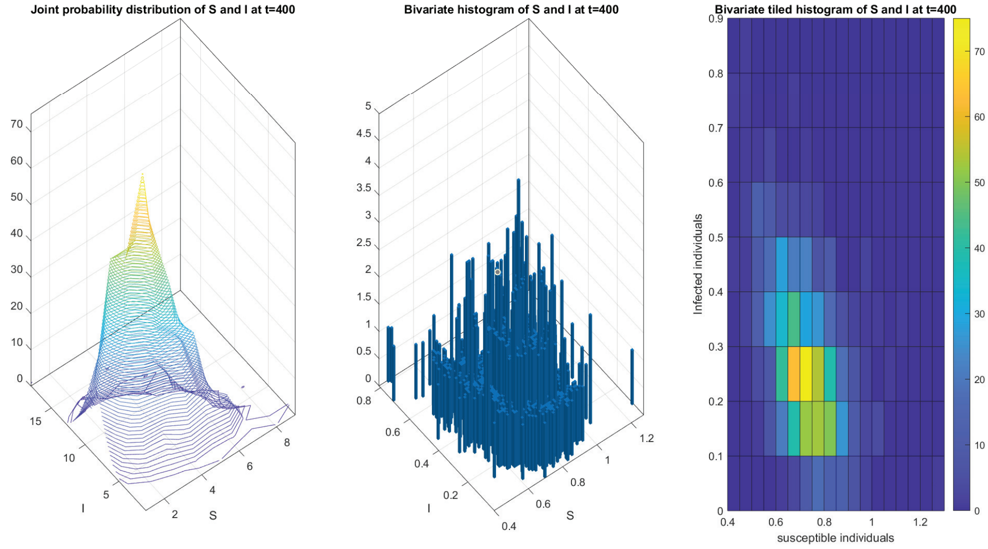


Fig. 2. From left to right: The 3D representation of the joint densities of $S$ and $I$, the bivariate distribution of $S$ and $I$, and the 2D upper view of the bivariate distribution of $S$ and $I$ at $t = 400$, respectively

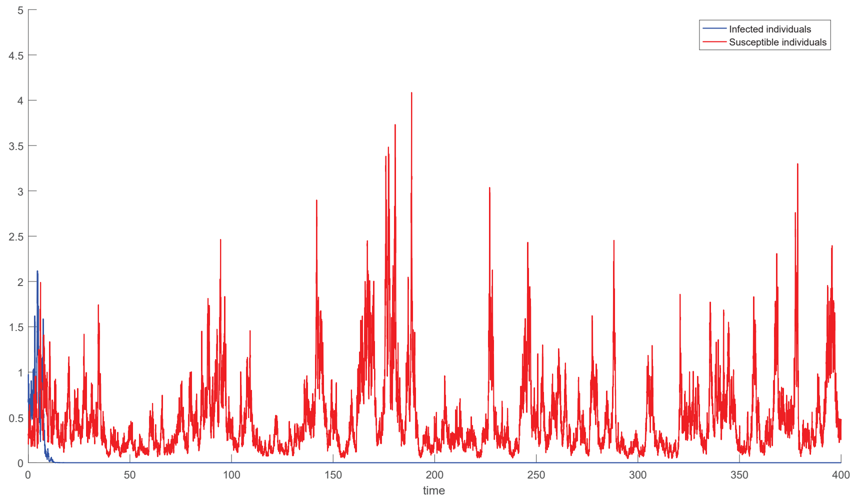


Fig. 3. Random paths of $S(t)$ and $I(t)$ up to the time horizon $t = 400$

## 4. Conclusion and future remarks

In this paper, we investigated a class of stochastic SIS epidemic models with two independent white noises. The only stochastic component is due to environmental variability as white noises around the disease transmission rate $\beta$ and the death rate $\mu$. By solving the Fokker-Plank equation associated with the one-dimensional stochastic differential equation (4), we skillfully showed that the disease will almost surely be persistent. After, conditions ensuring the existence of a unique ergodic invariant distribution for the model (3) were established. In this case, it should be noted that the endemic equilibrium $(S_e, I_e)$ of the deterministic counterpart of (3) is globally asymptotically stable. Therefore, the invariance of the distribution of the stochastic model (3) retains the prevalence of the disease when the intensities $\xi_1$ and $\xi_2$ are very small. Finally, when random fluctuations are very large, the disease goes away exponentially. Some topics deserve further investigation. We are looking to the use of Lévy processes and other related investigations (see [20–23]) in stochastic model (3). We leave them for our future works.

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