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EXACT SOLUTIONS OF FRACTIONAL OSCILLATOR EIGENFUNCTION PROBLEM WITH FIXED MEMORY LENGTH

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Abstract. In this paper, the eigenproblem for a fractional oscillator under homogeneous Dirichlet and Neumann boundary conditions is considered. Key properties of fractional operators with fixed memory length are established, such as the connection between left and right operators, the product rule for fractional integrals, and the fractional integration by the parts rule for periodic/antiperiodic functions. Explicit solutions in the form of discrete sets of sine/cosine eigenfunctions are derived. The impact of fractional order and memory length on eigenvalues is presented on graphs. Finally, a comparison of eigenvalues of oscillator with a fixed memory length and infinite memory length is shown.

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1. Introduction

Fractional calculus, a mathematical framework extending traditional calculus to non-integer orders, has gained prominence across scientific disciplines for modeling systems with memory and hereditary properties, e.g.: elasticity [1–3], viscoelasticity [4–6], plasticity [7], viscoplasticity [8], nonlocal continuum mechanics and physics [9, 10], diffusion [11], heat conduction [12, 13] or biological systems [14].

Fractional derivatives are known for their long-term memory characteristics as they are linked to all the historical data. In the case where $0 < \alpha < 1$, as the data becomes more distant, the weighting coefficient decreases. This concept, known as ”the principle of the dissipation of hereditary action”, was introduced by Volterra [15] and renamed by Podlubny in [16] as the ”short memory principle” (SMP). The main idea of SMP is to neglect the older data that had appeared. This idea was developed by Wei et al. in [17], where they presented some interesting properties of the short/fixed memory principle for Riemann-Liouville and Caputo cases. SMP has also been introduced for nabla discrete fractional calculus [18]. More detailed analysis of some properties of fractional operators with fixed memory length has been conducted by Ledesma et al. [19, 20]. However, all the results mentioned are only devoted to left fractional operators. The right fractional derivative with fixed memory length has been studied in [21], where apart from the derivative approximation, the series representation of left and right fractional derivatives is also presented.

SMP is significant and interesting not just for fractional calculus itself but also for the diverse fields that, over time, have incorporated tools from fractional calculus. The model based on the left Caputo derivative with fixed memory length (time length scale) has been used to the analysis of an abdominal aortic aneurysm [22]. Another example of the application of SMP can be found in the paper [23], where the Caputo--Almeida fractional derivative has been used to describe the evolution of damage (in time) to the Cylindrical Lithium-Ion Battery. In the paper [24], the non-local fractional Euler-Bernoulli beam theory has been formulated as a generalisation of classical Euler-Bernoulli beams, utilising fractional calculus (i.e. fractional Riesz-Caputo derivative with fixed memory length). The same fractional operator has been used to formulate the self-consistent space-Fractional Kirchhoff-Love Plate (see [25]).

Among the challenges posed by fractional calculus and its applications, the eigenvalue and eigenfunction problem for fractional operators with fixed memory length stands out as a key yet under-explored area.

2. Preliminaries

The aim of the paper is to formulate a fractional oscillator eigenproblem within the framework of fractional calculus for operators with fixed memory length. We shall show that it can be considered and explicitly solved in any interval $[-ML, ML]$, where $L > 0$ is the memory length and M an arbitrary natural number. First, we recall the notion of left and right fractional operators with a fixed memory length [18, 21].

Definition 1. Let $\alpha > 0, L > 0$. The left and right-sided fractional integrals with fixed memory length look as follows:

$$
{}_{t-L}I_t^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t-L}^{t} (t-s)^{\alpha-1} f(s) ds = \frac{1}{\Gamma(\alpha)} \int_{t-L}^{t} |t-s|^{\alpha-1} f(s) ds, \tag{1}
$$

$$
{}_{t}I_{t+L}^{\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{t}^{t+L} (s-t)^{\alpha-1} f(s) ds = \frac{1}{\Gamma(\alpha)} \int_{t}^{t+L} |s-t|^{\alpha-1} f(s) ds. \tag{2}
$$

The left and right fractional derivatives with fixed memory length are defined below

$$
{}_{t-L}D_t^{\alpha} f(t) := \frac{d}{dt} {}_{t-L}I_t^{1-\alpha} f(t), \tag{3}
$$

$$
{}_{t}D_{t+L}^{\alpha} f(t) := -\frac{d}{dt} {}_{t}I_{t+L}^{1-\alpha} f(t). \tag{4}
$$

As we see, these operators are a modification of fractional derivatives to operators dependent only on the close neighbourhood of point t. In case of the left operators, it is the left neighbourhood: $(t-L, t)$ and for the right ones we have the right neighbourhood: $(t, t+L)$ respectively.

Let us introduce the following reflection operator

$$Qf(t) := f(L-t) \tag{5}$$

and prove the relations between the left and right fractional operators with fixed memory length given in the proposition below.

Proposition 1 *Let $\alpha > 0$, $L > 0$ and reflection operator Q be defined by formula (5). Then, the following relation holds for integral*

$$Q \ {}_{t-L}I_t^{\alpha} Qf(t) = {}_tI_{t+L}^{\alpha} f(t), \tag{6}$$

while for $\alpha \in (0,1)$ we have for derivative of order α

$$Q \ {}_{t-L}D_t^{\alpha} Qf(t) = {}_tD_{t+L}^{\alpha} f(t). \tag{7}$$

Proof: we begin with the formula for integrals. Starting from the left-hand side and applying the change of variables: $\tau = L-s$, we get

$$Q \ {}_{t-L}I_t^{\alpha} Qf(t) = \frac{1}{\Gamma(\alpha)} \int_{-t}^{L-t} |L-t-s|^{\alpha-1} f(L-s) ds = -\frac{1}{\Gamma(\alpha)} \int_{t+L}^{t} |t-\tau|^{\alpha-1} f(\tau) d\tau$$
$$= \frac{1}{\Gamma(\alpha)} \int_{t}^{t+L} |t-\tau|^{\alpha-1} f(\tau) d\tau = {}_tI_{t+L}^{\alpha} f(t).$$

In turn, we apply the obtained relation in the proof of (6) for derivatives:

$$Q \ {}_{t-L}D_t^{\alpha} Qf(t) = Q\frac{d}{dt} QQ \, {}_{t-L}I_t^{\alpha} Qf(t) = -\frac{d}{dt} {}_tI_{t+L}^{1-\alpha} f(t) = {}_tD_{t+L}^{\alpha} f(t).$$

For the left derivatives and integrals, some explicit results were derived for analytic functions [17, 19, 20], such as polynomials, exponential and trigonometric functions. In particular, we recall here the respective derivation and integration formulas for trigonometric functions which we will apply in further considerations:

$$ {}_{t-L}D_t^{\alpha} \sin(\lambda t) = B_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,\alpha}(\lambda) \sin(\lambda(t-L)), \tag{8}$$
$$ {}_{t-L}D_t^{\alpha} \cos(\lambda t) = A_{L,\alpha}(\lambda) \cos(\lambda(t-L)) - B_{L,\alpha}(\lambda) \sin(\lambda(t-L)), \tag{9}$$
$$ {}_{t-L}I_t^{\alpha} \sin(\lambda t) = (-A_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + B_{L,\alpha}(\lambda) \sin(\lambda(t-L)))/\lambda, \tag{10}$$
$$ {}_{t-L}I_t^{\alpha} \cos(\lambda t) = (B_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,\alpha}(\lambda) \sin(\lambda(t-L)))/\lambda. \tag{11}$$

Constants in the above formulas depend on parameters $L > 0$, $\alpha \in (0,1)$ and $\lambda \in R$ values and look as follows:

$$A_{L,\alpha}(\lambda) = L^{-\alpha} \left(E_{2,1-\alpha}(-\lambda^2 L^2) - \frac{1}{\Gamma(1-\alpha)} \right), \tag{12}$$
$$B_{L,\alpha}(\lambda) = L^{1-\alpha} \lambda \, E_{2,2-\alpha}(-\lambda^2 L^2), \tag{13}$$

where $E_{\beta,\gamma}$ is the two-parameter Mittag-Leffler function [19] defined for arbitrary $z \in C$ by the formula given below:

$$E_{\beta,\gamma}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+\gamma)}.$$

We apply the above relations between the left and right operators (6), (7) as well as the properties of trigonometric functions to calculate analogous formulas for the right derivatives and integrals.

Proposition 2 *The following formulas are valid when fractional order $\alpha \in (0,1)$, parameters $\lambda \in R$, $L > 0$ and $t \in R$*

$$\begin{aligned}
{}_{t}D^{\alpha}_{t+L}\sin(\lambda t) &= -B_{L,\alpha}(\lambda)\cos(\lambda(t+L)) + A_{L,\alpha}(\lambda)\sin(\lambda(t+L)), &(14)\\
{}_{t}D^{\alpha}_{t+L}\cos(\lambda t) &= A_{L,\alpha}(\lambda)\cos(\lambda(t+L)) + B_{L,\alpha}(\lambda)\sin(\lambda(t+L)), &(15)\\
{}_{t}I^{\alpha}_{t+L}\sin(\lambda t) &= (A_{L,\alpha}(\lambda)\cos(\lambda(t+L)) + B_{L,\alpha}(\lambda)\sin(\lambda(t+L)))/\lambda, &(16)\\
{}_{t}I^{\alpha}_{t+L}\cos(\lambda t) &= (B_{L,\alpha}(\lambda)\cos(\lambda(t+L)) - A_{L,\alpha}(\lambda)\sin(\lambda(t+L)))/\lambda. &(17)
\end{aligned}$$

Proof: we begin calculation with the left-hand side of (14), apply (7)-(9) and obtain:

$$\begin{aligned}
{}_{t}D^{\alpha}_{t+L}\sin(\lambda t) &= Q\,{}_{t-L}D^{\alpha}_{t}\,Q\sin(\lambda t) = Q\,{}_{t-L}D^{\alpha}_{t}\sin(\lambda(L-t))\\
&= Q\,{}_{t-L}D^{\alpha}_{t}(\sin(\lambda L)\cos(\lambda t) - \cos(\lambda L)\sin(\lambda t))\\
&= Q[\sin(\lambda L)(A_{L,\alpha}(\lambda)\cos(\lambda(t-L)) - B_{L,\alpha}(\lambda)\sin(\lambda(t-L)))\\
&\quad - \cos(\lambda L)(B_{L,\alpha}(\lambda)\cos(\lambda(t-L)) + A_{L,\alpha}(\lambda)\sin(\lambda(t-L)))]\\
&= \sin(\lambda L)(A_{L,\alpha}(\lambda)\cos(\lambda(-t)) - B_{L,\alpha}(\lambda)\sin(\lambda(-t)))\\
&\quad - \cos(\lambda L)(B_{L,\alpha}(\lambda)\cos(\lambda(-t)) + A_{L,\alpha}(\lambda)\sin(\lambda(-t)))\\
&= \sin(\lambda L)(A_{L,\alpha}(\lambda)\cos(\lambda t) + B_{L,\alpha}(\lambda)\sin(\lambda t))\\
&\quad - \cos(\lambda L)(B_{L,\alpha}(\lambda)\cos(\lambda t) - A_{L,\alpha}(\lambda)\sin(\lambda t))\\
&= -B_{L,\alpha}(\lambda)\cos(\lambda(t+L)) + A_{L,\alpha}(\lambda)\sin(\lambda(t+L)),
\end{aligned}$$

which yields equality (14). The proof of formulas (15)-(17) is analogous.
It is a well-known fact in fractional calculus that under the scalar product constructed by using a standard integral the left and right fractional Riemann-Liouville integrals are connected in interval $[-L,L]$ by the formula below:

$$\int_{-L}^{L} f(t)\ {}_{-L}I^{\alpha}_{t}g(t)dt = \int_{-L}^{L} g(t)\ {}_{t}I^{\alpha}_{L}f(t)dt. \tag{18}$$

A similar relation is also valid for integrals with fixed memory length, provided functions f and g are periodic or antiperiodic respectively. Here, in the paper, we prove the version for periodic / antiperiodic functions determined on a set of real numbers.

Proposition 3 *Let us assume that functions f, g are determined on R locally integrable functions and that they are both periodic*

$$f(t \pm 2ML) = f(t) \quad g(t \pm 2ML) = g(t) \tag{19}$$

or simultaneously antiperiodic functions

$$f(t \pm 2ML) = -f(t) \quad g(t \pm 2ML) = -g(t) \tag{20}$$

with constant M being an arbitrary natural number $M \in N$. Then, the following relation holds for integrals of fractional order $\alpha > 0$:

$$\int_{-ML}^{ML} f(t) \; {}_{t-L}I_t^{\alpha} g(t) dt = \int_{-ML}^{ML} g(t) \; {}_tI_{t+L}^{\alpha} f(t) dt. \tag{21}$$

Proof: we consider the case when both functions, functions f and g, are periodic, which means they obey condition (19). First, we change the order of integration and split the obtained integral as follows

$$\begin{aligned}
\int_{-ML}^{ML} f(t) \; {}_{t-L}I_t^{\alpha} g(t) dt &= \frac{1}{\Gamma(\alpha)} \int_{-ML}^{ML} f(t) \int_{t-L}^{t} (t-s)^{\alpha-1} g(s) ds dt \\
&= \frac{1}{\Gamma(\alpha)} \left(\int_{-(M+1)L}^{-ML} g(s) \int_{-ML}^{s+L} (t-s)^{\alpha-1} f(t) dt ds \right. \\
&+ \left. \int_{-ML}^{(M-1)L} g(s) \int_{s}^{s+L} (t-s)^{\alpha-1} f(t) dt \right) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(s) \left(\int_{s}^{ML} (t-s)^{\alpha-1} f(t) dt ds \right) = I_1 + I_2 + I_3.
\end{aligned}$$

After changing variables in integral I_1: $u = s + 2ML$, $v = t + 2ML$, we apply the periodicity properties for functions f and g

$$\begin{aligned}
I_1 &= \frac{1}{\Gamma(\alpha)} \int_{-(M+1)L}^{-ML} g(s) \left(\int_{-ML}^{s+L} (t-s)^{\alpha-1} f(t) dt \right) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(u-2ML) \left(\int_{ML}^{u+L} (v-u)^{\alpha-1} f(v-2ML) dv \right) du \\
&= \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(u) \left(\int_{ML}^{u+L} (v-u)^{\alpha-1} f(v) dv \right) du.
\end{aligned}$$

Next, we sum integrals I_1 and I_3 and obtain

$$\begin{aligned}
I_1 + I_3 &= \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(s) \left(\int_{ML}^{s+L} (t-s)^{\alpha-1} f(t) dt \right) ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(s) \left(\int_{s}^{ML} (t-s)^{\alpha-1} f(t) dt \right) ds \\
&= \frac{1}{\Gamma(\alpha)} \int_{(M-1)L}^{ML} g(s) \left(\int_{s}^{s+L} (t-s)^{\alpha-1} f(t) dt \right) ds.
\end{aligned}$$

Adding integral I_2 to the above sum, we obtain formula (21):

$$\int_{-ML}^{ML} f(t)\ {}_{t-L}I_t^{\alpha} g(t)dt = I_1 + I_2 + I_3$$
$$= \frac{1}{\Gamma(\alpha)} \int_{-ML}^{ML} g(s) \left(\int_s^{s+L} (t-s)^{\alpha-1} f(t)dt \right) ds = \int_{-ML}^{ML} g(s)\ {}_sI_{s+L}^{\alpha} f(s)ds.$$

Let us note that assumptions concerning the periodicity of functions f and g (19) can be replaced correspondingly by conditions (20), and formula (21) is still valid.

The formula for derivatives is an extension of the obtained relation (21). Applying (21) and the standard integration by the parts rule, we arrive at the property connecting the left and right fractional derivatives with fixed memory length by the product formula.

Proposition 4 *Let assumptions of Proposition 3 be fulfilled, fractional order* $\alpha \in (0,1)$ *and derivatives* ${}_{t-L}D_t^{\alpha} f$ *and* ${}_{t-L}D_t^{\alpha} g$ *be continuous functions in* $[-ML, ML]$. *Then, the following formula holds*

$$\int_{-ML}^{ML} f(t)\ {}_{t-L}D_t^{\alpha} g(t)dt = \int_{-ML}^{ML} g(t)\ {}_tD_{t+L}^{\alpha} f(t)dt. \tag{22}$$

Proof: we start with the left-hand side of the above formula, apply the definition of left fractional derivative with fixed memory length and the integration by the parts rule:

$$\int_{-ML}^{ML} f(t)\ {}_{t-L}D_t^{\alpha} g(t)dt = \int_{-ML}^{ML} f(t)\ \frac{d}{dt} {}_{t-L}I_t^{1-\alpha} g(t)dt =$$
$$= -\int_{-ML}^{ML} \left(\frac{d}{dt} f(t) \right)\ {}_{t-L}I_t^{1-\alpha} g(t)dt + f(t)\ {}_{t-L}I_t^{1-\alpha} g(t)|_{t=-ML}^{ML}$$
$$= -\int_{-ML}^{ML} \left({}_tI_{t+L}^{1-\alpha} \frac{d}{dt} f(t) \right)\ g(t)dt + f(t)\ {}_{t-L}I_t^{1-\alpha} g(t)|_{t=-ML}^{ML}$$
$$= \int_{-ML}^{ML} g(t)\ {}_tD_{t+L}^{\alpha} f(t)dt + f(t)\ {}_{t-L}I_t^{1-\alpha} g(t)|_{t=-ML}^{ML} = \int_{-ML}^{ML} g(t)\ {}_tD_{t+L}^{\alpha} f(t)dt,$$

where we applied the fact that for pairs of periodic / antiperiodic functions the boundary terms in the above formula vanish.

3. Main results

Let us now introduce the notion of fractional differential operator with fixed memory length for an oscillator. Much like the Sturm-Liouville fractional operators introduced in [26] and studied in many papers by Klimek and collaborators (compare [27] and the references therein), the differential operator includes the left and right fractional derivatives with fixed memory length:

$$\mathscr{L} := {}_tD_{t+L}^{\alpha}\ {}_{t-L}D_t^{\alpha}. \tag{23}$$

Adopting the notation (23), the studied fractional oscillator equation takes the form:

$$\mathscr{L} y_\lambda(t) = \rho_{L,\alpha}(\lambda) y_\lambda(t). \tag{24}$$

Now, we check that sine and cosine functions are eigenfunctions of this operator for any value of parameters fulfilling $L > 0$, $\alpha \in (0,1)$ and $\lambda \in R$.

$$\mathscr{L}\ \sin(\lambda t) = [(A_{L,\alpha}(\lambda))^2 + (B_{L,\alpha}(\lambda))^2] \sin(\lambda t), \tag{25}$$

$$\mathscr{L}\ \cos(\lambda t) = [(A_{L,\alpha}(\lambda))^2 + (B_{L,\alpha}(\lambda))^2] \cos(\lambda t). \tag{26}$$

The above formulas result from Proposition 1, formulas (8)-(11), and properties of trigonometric functions. We start with the left-hand side of formula (25) and obtain

$$\begin{aligned}
\mathscr{L}\ \sin(\lambda t) &= {}_tD^\alpha_{t+L}\ {}_{t-L}D^\alpha_t \sin(\lambda t) \\
&= {}_tD^\alpha_{t+L} \left(B_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,\alpha}(\lambda) \sin(\lambda(t-L))\right) \\
&= Q\ {}_{t-L}D^\alpha_t Q \left(B_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,\alpha}(\lambda) \sin(\lambda(t-L))\right) \\
&= Q\ {}_{t-L}D^\alpha_t \left(B_{L,\alpha}(\lambda) \cos(\lambda t) - A_{L,\alpha}(\lambda) \sin(\lambda t)\right) \\
&= QB_{L,\alpha}(\lambda)(A_{L,\alpha}(\lambda) \cos(\lambda(t-L)) - B_{L,\alpha}(\lambda) \sin(\lambda(t-L))) \\
&- QA_{L,\alpha}(\lambda)(B_{L,\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,\alpha}(\lambda) \sin(\lambda(t-L))) \\
&= [(A_{L,\alpha}(\lambda))^2 + (B_{L,\alpha}(\lambda))^2] \sin(\lambda t),
\end{aligned}$$

where, in addition, we applied the fact that the sine function is an odd function, and the cosine function is an even function. We omit the calculations for the cosine function as they are analogous to these presented above.
Let us note that as the $\mathscr{L}$-operator is a linear one we also get that an arbitrary linear combination of sine and cosine functions is its eigenfunction as well:

$$\mathscr{L}\ (C_1 \sin(\lambda t) + C_2 \cos(\lambda t)) = \rho_{L,\alpha}(\lambda)(C_1 \sin(\lambda t) + C_2 \cos(\lambda t)), \tag{27}$$

where $\lambda \in R$ and eigenvalues of the $\mathscr{L}$-operator are given by the formula below:

$$\rho_{L,\alpha}(\lambda) = (A_{L,\alpha}(\lambda))^2 + (B_{L,\alpha}(\lambda))^2 \tag{28}$$

with $A_{L,\alpha}$ and $B_{L,\alpha}$ defined in (12), (13). It is easy to check that

$$\rho_{0,1}(\lambda) = \lambda^2, \quad \lim_{L\to\infty} \rho_{L,\alpha}(\lambda) = \lambda^{2\alpha}, \tag{29}$$

where the first relation corresponds to the classical oscillator operator and the second one to the fractional operator with infinite memory length. We shall return to these two limit cases in the study of the eigenvalue problem subject to homogeneous Dirichlet conditions.
Now, we apply Proposition 4 and calculate the product of functions f and $\mathscr{L}g$ in interval $[-ML, ML]$.

Proposition 5 *Let assumptions of Proposition 3 be fulfilled, fractional order $\alpha \in (0,1)$ and functions ${}_{t-L}D_t^{\alpha} f$ and ${}_{t-L}D_t^{\alpha} g$ be continuous in $[-ML, ML]$. Then, the following formula holds*

$$\int_{-ML}^{ML} f(t)\ \mathscr{L} g(t) dt = \int_{-ML}^{ML} (\mathscr{L} f(t))\, g(t) dt - f(t)\, {}_tI_{t+L}^{1-\alpha}\ {}_{t-L}D_t^{\alpha}\ g(t)|_{t=-ML}^{ML}$$

$$+ g(t)\, {}_tI_{t+L}^{1-\alpha}\ {}_{t-L}D_t^{\alpha}\ f(t)|_{t=-ML}^{ML} = \int_{-ML}^{ML} (\mathscr{L} f(t))\, g(t) dt. \tag{30}$$

Let us check the value of the boundary terms for sine and cosine functions. They are summed up in the following property.

PROPERTY 1 Let $L > 0$, $\alpha \in (0,1)$ and $\lambda \in R$. Then, the following formulas are valid for any $t \in R$

$${}_tI_{t+L}^{1-\alpha}\ {}_{t-L}D_t^{\alpha} \sin(\lambda t) = \rho_{L,1-\alpha}(\lambda) \cos(\lambda t)/\lambda, \tag{31}$$

$${}_tI_{t+L}^{1-\alpha}\ {}_{t-L}D_t^{\alpha} \cos(\lambda t) = -\rho_{L,1-\alpha}(\lambda) \sin(\lambda t)/\lambda. \tag{32}$$

Proof: For the sine functions, we get formula (31) by applying Proposition 1, formulas (10), (11) and properties of trigonometric functions

$$\begin{aligned}
&{}_tI_{t+L}^{1-\alpha}\ {}_{t-L}D_t^{\alpha} \sin(\lambda t) = \lambda\ {}_tI_{t+L}^{1-\alpha}\ {}_{t-L}I_t^{1-\alpha}(\cos(\lambda t)) \\
&= {}_tI_{t+L}^{1-\alpha}\, (B_{L,1-\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,1-\alpha}(\lambda) \sin(\lambda(t-L))) \\
&= Q\ {}_{t-L}I_t^{1-\alpha}\, (B_{L,1-\alpha}(\lambda) \cos(\lambda t) - A_{L,1-\alpha}(\lambda) \sin(\lambda t)) \\
&= Q\, B_{L,1-\alpha}(\lambda) \left(\frac{B_{L,1-\alpha}(\lambda) \cos(\lambda(t-L)) + A_{L,1-\alpha}(\lambda) \sin(\lambda(t-L)}{\lambda} \right) \\
&- Q\, A_{L,1-\alpha}(\lambda) \left(\frac{-A_{L,1-\alpha}(\lambda) \cos(\lambda(t-L)) + B_{L,1-\alpha}(\lambda) \sin(\lambda(t-L)}{\lambda} \right) \\
&= B_{L,1-\alpha}(\lambda) \cdot \frac{B_{L,1-\alpha}(\lambda) \cos(\lambda(t)) - A_{L,1-\alpha}(\lambda) \sin(\lambda(t))}{\lambda} \\
&+ A_{L,1-\alpha}(\lambda) \cdot \frac{A_{L,1-\alpha}(\lambda) \cos(\lambda(t)) + B_{L,1-\alpha}(\lambda) \sin(\lambda(t))}{\lambda} = \frac{\rho_{L,1-\alpha}(\lambda)}{\lambda} \cos(\lambda t).
\end{aligned}$$

3.1. Eigenfunction problem with homogeneous Dirichlet boundary conditions

In this part of the paper we shall consider the fractional oscillator eigenfunction problem subjected to Dirichlet boundary conditions in interval $[-ML, ML]$

$$\mathscr{L} Y_{\Lambda}(t) = \rho_{L,\alpha}(\Lambda) Y_{\Lambda}(t), \quad t \in [-ML, ML], \quad \alpha \in (0,1), \tag{33}$$

$$Y_{\Lambda}(-ML) = Y_{\Lambda}(ML) = 0. \tag{34}$$

The above homogeneous boundary conditions lead to the following two discrete sets of eigenfunctions. The first of them corresponds to condition $C_2 = 0$ in (27) and is indexed by parameters Λ_k, $k \in N$

$$\Lambda_k = \frac{k\pi}{ML}, \quad Y_{\Lambda_k}(t) = \sin\left(\frac{k\pi t}{ML}\right). \tag{35}$$

We observe that the derived eigenfunctions obey periodicity conditions (19). Now, we shall show that these eigenfunctions are orthogonal provided they are indexed by distinct values of Λ_k:

$$\int_{-ML}^{ML} Y_{\Lambda_k}(t)\, Y_{\Lambda_m}(t)dt = 0, \quad \Lambda_k \neq \Lambda_m. \tag{36}$$

In the considered case, the proof of orthogonality is based on the following relation resulting from Proposition 5

$$\int_{-ML}^{ML} (\mathscr{L} Y_{\Lambda_k}(t))\, Y_{\Lambda_m}(t)dt = \int_{-ML}^{ML} Y_{\Lambda_k}(t)\, \mathscr{L} Y_{\Lambda_m}(t)dt$$

which yields the equality below

$$\rho_{L,\alpha}(\Lambda_k) \int_{-ML}^{ML} Y_{\Lambda_k}(t)\, Y_{\Lambda_m}(t)dt = \rho_{L,\alpha}(\Lambda_m) \int_{-ML}^{ML} Y_{\Lambda_m}(t)\, Y_{\Lambda_m}(t)dt.$$

We compare the left and right side of the above equality and we obtain the the orthogonality relation (36).

The second subset of eigenfunctions, obtained for constant $C_1 = 0$ in (27), is indexed by $\tilde{\Lambda}_k$, $k \in N_0$

$$\tilde{\Lambda}_k = \frac{(k+1/2)\pi}{ML}, \quad Y_{\tilde{\Lambda}_k}(t) = \cos\left(\frac{(k+1/2)\pi t}{ML}\right). \tag{37}$$

We note that the derived eigenfunctions from the second subset are antiperiodic (20). Now, the proof of orthogonality of these eigenfunctions is analogous to the one presented for set of functions Y_{Λ_k}, $k \in N$, so we omit the calculations and quote the respective orthogonality rule:

$$\int_{-ML}^{ML} Y_{\tilde{\Lambda}_k}(t)\, Y_{\tilde{\Lambda}_m}(t)dt = 0, \quad \tilde{\Lambda}_k \neq \tilde{\Lambda}_m. \tag{38}$$

Finally, we add the orthogonality relation for eigenfunctions from the two subsets

$$\int_{-ML}^{ML} Y_{\Lambda_k}(t)\, Y_{\tilde{\Lambda}_m}(t)dt = 0 \tag{39}$$

which results from the fact that functions Y_{Λ_k} are odd in interval $[-ML, ML]$ whereas $Y_{\tilde{\Lambda}_m}$ are even ones.

Remark 1 Let us point out that the derived eigenfunctions given in (35), (37) solve the classical oscillator problem subject to Dirichlet boundary conditions in interval $[-ML, ML]$ as well. Eigenvalues, corresponding to sine and cosine sets of eigenfunctions, are given by the equations below (here $k \in N$, $m \in N_0$):

$$\rho_{0,1}(\Lambda_k) = \left(\frac{k\pi}{ML}\right)^2, \quad \rho_{0,1}(\tilde{\Lambda}_m) = \left(\frac{(m+1/2)\pi}{ML}\right)^2. \tag{40}$$

This property of solutions implies that the orthogonality properties (36), (38) can be obtained using the classical oscillator operator and the above eigenvalues. □

Remark 2 Functions (35) and (37) also solve the oscillator problem formulated by Rivero and collaborators in [28] on line $(-\infty, \infty)$. In the construction, they applied Liouville derivatives of order $\alpha \in (1/2, 1)$

$$D_+^\alpha f(t) = \frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-s)^{-\alpha} f(s)ds \tag{41}$$

$$D_-^\alpha f(t) = -\frac{d}{dt}\frac{1}{\Gamma(1-\alpha)}\int_{t}^{\infty}(s-t)^{-\alpha} f(s)ds \tag{42}$$

$$D_-^\alpha D_+^\alpha Y_\Lambda(t) = \rho(\Lambda) Y_\Lambda(t) \tag{43}$$

When we assume that solutions fulfill the homogeneous Dirichlet midpoint boundary conditions (34) and solve eigenvalues equation (43), we get the following explicit formulas for eigenvalues (here $k \in N$, $m \in N_0$):

$$\rho_{\infty,\alpha}(\Lambda_k) = \left(\frac{k\pi}{ML}\right)^{2\alpha}, \quad \rho_{\infty,\alpha}(\tilde{\Lambda}_m) = \left(\frac{(m+1/2)\pi}{ML}\right)^{2\alpha}. \tag{44}$$

3.2. Oscillator eigenfunction problem with homogeneous fractional von Neumann boundary conditions

The next example is the study of fractional oscillator problem:

$$\mathscr{L} Y_\Lambda(t) = \rho_{L,\alpha}(\Lambda) Y_\Lambda(t), \quad t \in [-ML, ML], \quad \alpha \in (0,1), \tag{45}$$

with homogeneous fractional Neumann boundary conditions that appear as follows:

$$I_t^{1-\alpha}\ {}_tD_{t+L}^\alpha\, Y_\Lambda(t)|_{t=-ML} = I_t^{1-\alpha}\ {}_tD_{t+L}^\alpha\, Y_\Lambda(t)|_{t=ML} = 0. \tag{46}$$

Taking into account the general form of solution given in (27), we observe that eigenfunctions' set can be split into two subsets similar to the problem subject to homogeneous Dirichlet boundary conditions. The first subset corresponds to value of constant $C_1 = 0$ and consists of cosine functions

$$Y_\Lambda(t) = \cos(\Lambda t),$$

which leads to discrete set of cosine eigenfunctions with indices provided by boundary conditions:

$$I_t^{1-\alpha}\, {}_tD_{t+L}^{\alpha} \cos(\Lambda t)|_{t=\pm ML} = -\rho_{L,1-\alpha}(\Lambda) \sin(\pm\Lambda ML)/\Lambda = 0,$$
$$\sin(-\Lambda ML) = \sin(\Lambda ML) = 0 = \sin(k\pi),$$

and appear as follows for $k \in N_0$

$$\Lambda_k = \frac{k\pi}{ML}, \quad Y_{\Lambda_k}(t) = \cos\left(\frac{k\pi t}{ML}\right). \tag{47}$$

It is easy to check that the periodicity condition (19) is valid. Therefore, we can apply Proposition 5, formula (30) and obtain orthogonality rule (36) valid for cosine eigenfunctions indexed by Λ_k, $k \in N_0$. We omit the proof of orthogonality of cosine eigenfunctions, as it is analogous to that presented in previous section. Now, we derive the second subset of eigenfunctions corresponding to value of constant $C_2 = 0$ in (27). It now contains sine functions

$$Y_{\tilde{\Lambda}}(t) = \sin(\tilde{\Lambda} t),$$

and homogeneous Neumann boundary conditions lead again to discrete set of sine eigenfunctions with indices provided by boundary conditions:

$$I_t^{1-\alpha}\, {}_tD_{t+L}^{\alpha} \sin(\tilde{\Lambda} t)|_{t=\pm ML} = \rho_{L,1-\alpha}(\tilde{\Lambda}) \cos(\pm\tilde{\Lambda} ML)/\tilde{\Lambda} = 0,$$
$$\cos(-\tilde{\Lambda} ML) = \cos(\tilde{\Lambda} ML) = 0 = \cos((k+1/2)\pi).$$

We obtain the following set of indices $\tilde{\Lambda}_k$, $k \in N_0$ and eigenfunctions

$$\tilde{\Lambda}_k = \frac{(k+1/2)\pi}{ML}, \quad Y_{\tilde{\Lambda}_k}(t) = \sin\left(\frac{(k+1/2)\pi t}{ML}\right). \tag{48}$$

For the sine eigenfunctions' subset, antiperiodicity condition (20) is fulfilled. Therefore, we again state that applying Proposition 5 and proceeding similarly to the orthogonality proof presented in the previous section we arrive at the orthogonality rule (38) valid for sine eigenfunctions indexed by $\tilde{\Lambda}_k$, $k \in N_0$. The remark on orthogonality of functions from the two subsets also holds.

The orthogonality result can also be inferred from the fact that eigenfunctions (47), (48) solve the classical oscillator equation subject to standard Neumann boundary conditions in interval $[-ML, ML]$.

4. Conclusions

In this paper, we studied fractional oscillator eigenproblem with fixed memory length subject to homogeneous Dirichlet and Neumann boundary conditions. A number of useful and important properties of the considered operators were established including connection between the left and and right operators via reflection transformation, product rule for fractional integrals and fractional integration by the parts rule for periodic / antiperiodic functions. All these derived relations were fundamental in the study of eigenproblem of fractional oscillator in the interval $[-ML, ML]$. The general solution of the discussed eigenproblem is identical with general solutions of the classical oscillator equation and the oscillator equation (43). The same remark applies to solutions fulfilling homogeneous Dirichlet boundary conditions in the interval $[-ML, ML]$. They were derived in the form of sets of sine / cosine eigenfunctions determined by discrete sets of parameters Λ.

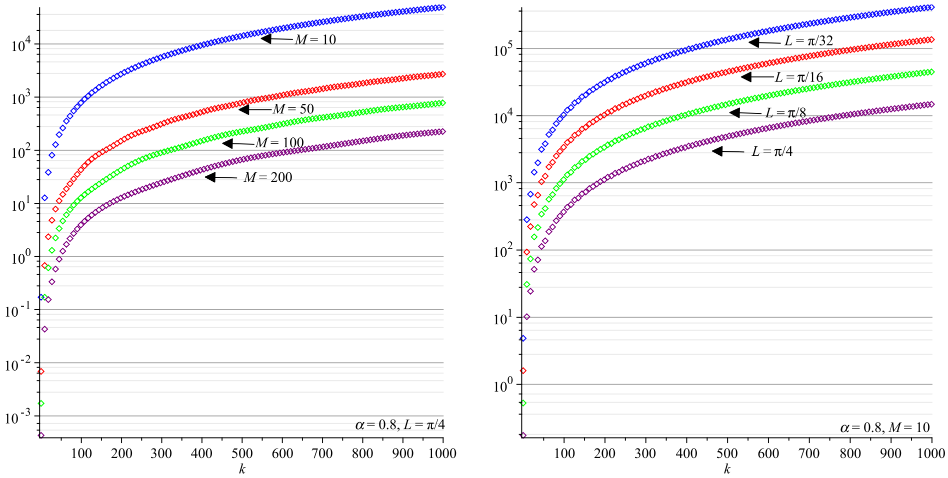

Fig. 1. Semi-log plot illustrating the impact of M (the left-hand side) and L (the right-hand side) on $\rho_{L,\alpha}(\lambda)$ for Dirichlet and Neumann boundary conditions for the interval $[-ML, ML]$

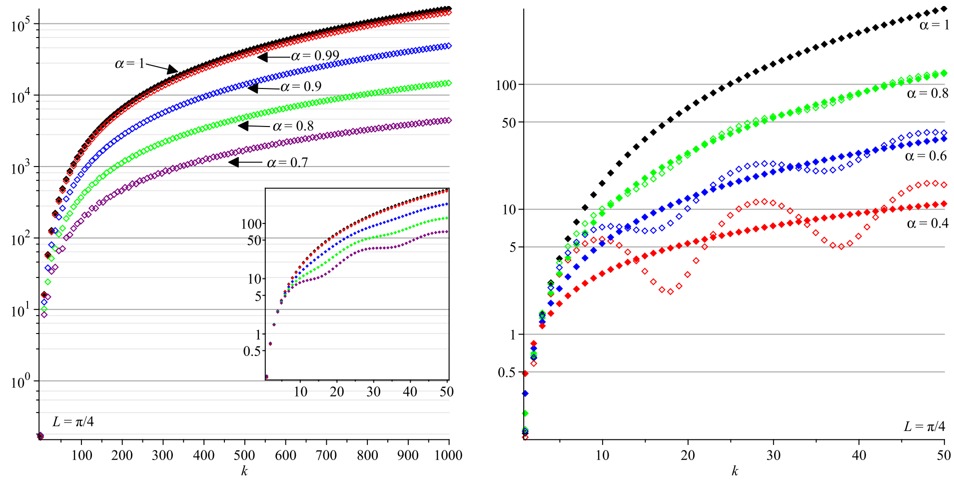

Fig. 2. Semi-log plot illustrating the impact of α on $\rho_{L,\alpha}(\lambda)$ (the left-hand side). The comparison of eigenvalues $\rho_{L,\alpha}(\lambda)$ and $\rho_{\infty,\alpha}(\lambda)$ (the right-hand side) for the interval $[-10L, 10L]$

The eigenvalues of the studied fractional operator depend on the order $\alpha \in (0,1)$, memory length $L > 0$ and the respective size of the interval $M > 0$. They differ from spectra obtained in both cases: the classical result and eigenvalues of the fractional oscillator with infinite memory length. In Figures 1 and 2, we show the behaviour of eigenvalues $\rho_{L,\alpha}$ determined by parameters Λ given in (35), (37) and (47), (48). Figure 1 includes two graphs: first, we fix values $\alpha = 0,8$ and $L = \pi/4$ analyzing the influence of increasing value of size parameter M and observe that eigenvalues decrease. Next, keeping $\alpha = 0,8$ and $M = 10$, we get a similar result – when the length of memory increases, the eigenvalues decrease. Figure 2 also comprises two graphs. In the first one, we present the influence of fractional order α for fixed values of memory length $L = \pi/4$ and size parameter $M = 10$. The range is $m,k = 1000$, and we note that eigenvalues are increasing functions of fractional order and tend to $\rho_{0,1}$ (denoting eigenvalues of classical oscillator corresponding to $\alpha = 1$) when fractional order tends to 1. In the same figure the graph in the range $m,k = 50$ is added (a kind of the zoom picture of the full graph). We observe an interesting feature here, namely concave semi-log plot – when order $\alpha = 0,7$ changes its behaviour including concave and convex parts. In the second graph, we present the comparison of eigenvalues of oscillator with fixed memory length $\rho_{L,\alpha}$ and infinite memory length $\rho_{\infty,\alpha}$ in the range $m,k = 50$. Here, we observe oscillations of the $\rho_{\pi/4,\,0.6}$ and $\rho_{\pi/4,\,0.4}$.

In conclusion, we derived and discussed new results on eigenfunctions / eigenvalues problem for fractional oscillator equation with fixed memory length. The study of one of the simplest eigenproblems within the framework of fractional calculus with fixed memory length will be the first step in further discussion on Sturm-Liouville problems with these type of operators. We also expect that the presented systems of orthogonal eigenfunctions will be useful in study of fractional diffusion problems.

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