JAM
CM
Journal of Applied Mathematics and Computational Mechanics 2023, 22(4), 53-63
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2023.4.05 e-ISSN 2353-0588

MODELLING OF THE SOLAR HEATING OF A MULTI-LAYERED SPHERICAL CONE

Urszula Siedlecka

Department of Mathematics, Czestochowa University of Technology
Czestochowa, Poland
urszula.siedlecka@pcz.pl

Received: 17 October 2023; Accepted: 11 December 2023

Abstract. In this paper, the solar heating of a multi-layered spherical body with azimuthal symmetry is considered. The mathematical model is related to the determination of the steady state of the temperature distribution in the spherical cone consisting of concentric spherical layers. The solar heating is composed of two parts of the heat flux: direct and diffusion. Also, the simultaneous cooling of the cone by its outer surface (as convective heat flow to the environment) is taken into account. The proposed system of the partial differential equations supplemented by the adequate boundary conditions is solved in the analytical way by using, among others, the Legendre functions of the first kind. The sample results of temperature distribution in the cross-section of the cone with different polar angles are also presented.

MSC 2010: *35Q79, 42C10*
Keywords: *heat conduction, solar heating, Legendre function*

1. Introduction

The heat transfer occurs in so many physical processes that it is difficult nowadays to imagine a situation in which it does not occur. The many materials that are the subject of scientists' research include composite materials that are widely used in real physical and engineering systems. The temperature distribution is examined, among others, in layered slabs, cylinders or spheres [1-7]. For example, heat transfer in a sphere is considered in one [8], two [9] or three dimensions [10].

Heat flows are classified in different ways and steady and unsteady heat flow are two important examples of them. Although the study of unsteady state seems to have more applications, many scientists still study the issue of the steady state [11, 12]. Another criterion for the classification of issues related to heat flow may be the division related to various ways of heating the body from the outside. It seems that in recent years, problems related to the heating of the body by solar radiation have attracted particular attention [13, 14].

The sun is our main source of energy. This energy, called solar energy, interacts with components of the atmosphere as it travels down to the earth. Only part of the solar radiation reaches the earth's surface without scattering and absorption, and this is the so-called direct radiation. However, radiation that does not come directly from the sun moving along a straight path, but comes from all directions of the sky, is called diffuse radiation. In summary, radiation reaching the earth's surface consists of three components: direct radiation, diffuse radiation, and radiation reflected to the surface from surrounding surfaces. The absorption of solar energy by any bodies leads to an increase in their own temperature.

In this paper, we present the mathematical model of the solar heating of a multi--layered spherical body with azimuthal symmetry. The object of the considerations is the spherical cone consisting with concentric spherical layers. Not only the solar heating (consists of two parts of the heat flux: direct and diffusion) but also the cooling of the cone by its outer surface (as convective heat flow to the environment) are taken into account. An analytical solution of the problem is derived in the form of the product of two functions, where one of them is the Legendre functions of the first kind. The temperature distribution in the cross-section of the cone for different polar angles is also presented.

2. Mathematical modelling of the problem

The starting point of our consideration is the stationary heat conduction equation in the form

$$\nabla \cdot \left(\lambda(r)\nabla T(r,\varphi)\right) = 0 \tag{1}$$

where λ is the thermal conductivity of the material, r is the radial coordinate, φ is the polar coordinate and T is the temperature. Moreover, the operator $\nabla \cdot \left(\lambda(r)\nabla\right)$ is defined in the spherical coordinates as follows

$$\nabla \cdot \left(\lambda(r)\nabla\right) = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(\lambda(r) r^2 \frac{\partial}{\partial r}\right) + \frac{\lambda(r)}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}\right) + \frac{\lambda(r)}{\sin^2\varphi}\frac{\partial^2}{\partial\theta^2}\right] \tag{2}$$

where θ is the azimuthal angle coordinate. Furthermore, we assume that the temperature distribution in the body is azimuthally invariant (azimuthal symmetry) so we can drop the third term in the bracket in the equation. What's more, we can simplify the operator $\nabla \cdot \left(\lambda(r)\nabla\right)$ by introducing a new variable μ, related to the polar angle φ, by the following relationship

$$\mu = \cos(\varphi) \tag{3}$$

Taking into consideration the azimuthal symmetry, the operator (2) can be written now in the simplified form

$$\nabla\cdot\left(\lambda(r)\nabla\right)=\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(\lambda(r)r^2\frac{\partial}{\partial r}\right)+\lambda(r)\frac{\partial}{\partial\mu}\left(\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\right)\right] \tag{4}$$

We consider the stationary heat conduction in a spherical medium which consists with n concentric spherical layers which are defined by the radius interval: $r_{i-1}\le r\le r_i$ $(i=1,...,n)$ and $0\le\varphi\le\varphi_0$, where $0<\varphi_0\le\pi/2$. The medium is a spherical cone (Fig. 1) for $0<\varphi_0<\pi/2$ and a hemisphere for $\varphi_0=\pi/2$.

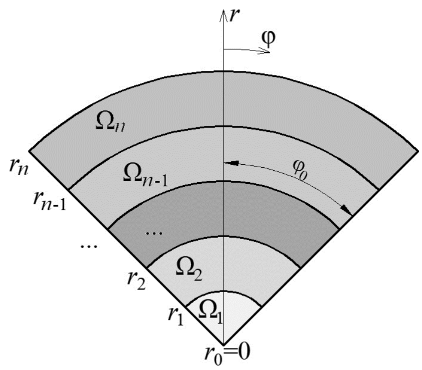

Fig. 1. The main cross-section of a multi-layered spherical cone under considerations

The differential equation governing the temperature in the i-th spherical layer has the following form

$$\nabla\cdot\left(\lambda_i(r)\nabla T_i(r,\mu)\right)=0 \tag{5}$$

where T is the temperature, λ is the thermal conductivity, $\mu\in[\mu_0,1]$ and $\mu_0=\cos(\varphi_0)$.

The boundary conditions (which consist of, among others, the perfect thermal contacts between the neighbouring layers) are as follows:

$$\left|T_1(0,\mu)\right|<\infty \tag{6}$$

$$-\lambda_i\left.\frac{\partial T_i(r,\mu)}{\partial\mu}\right|_{\mu=\mu_0}=q_{\mu_0}=0,\quad i=1,...,n \tag{7}$$

$$T_i(r_i,\mu)=T_{i+1}(r_i,\mu),\quad i=1,...,n-1 \tag{8}$$

$$\lambda_i(r)\frac{\partial T_i(r,\mu)}{\partial r}\bigg|_{r=r_i} = \lambda_{i+1}(r)\frac{\partial T_{i+1}(r,\mu)}{\partial r}\bigg|_{r=r_i}, \quad i=1,...,n-1 \tag{9}$$

$$-\lambda_n(r)\frac{\partial T_n(r,\mu)}{\partial r}\bigg|_{r=r_n} = \alpha_{conv}\left(T_n(r_n,\mu)-T_{amb}\right)-q_{direct}\,\mu-q_{diffusion} \tag{10}$$

where q_{μ_0} is the heat flux on lateral surface of the cone, α_{conv} is the convective heat transfer coefficient, T_{amb} is the ambient temperature, q_{direct} is the direct solar radiation and $q_{diffusion}$ is the diffuse solar radiation, moreover, the term $\alpha_{conv}\left(T_n(r_n,\mu)-T_{amb}\right)$ is known as the convective heat transfer q_{conv} (Fig. 2).

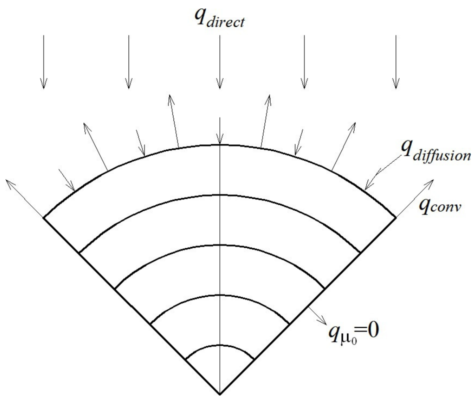

Fig. 2. Boundary conditions for the considered problem

3. Solution of the problem under considerations

An analytical solution to the boundary problem (5)-(10) can be presented in the form of a product, and introducing functions $M(\mu)$ and $R_i(r)$, we write the functions T_i as

$$T_i(r,\mu)=R_i(r)M(\mu), \quad i=1,...,n \tag{11}$$

Next, we substitute the functions T_i into the Equation (5). After separating the variables and assuming the separation constant as $\beta(\beta+1)$, where β is a real number, we get two homogenous differential equations – the Lagrange equation and the Euler equation:

$$\frac{d}{d\mu}\left(\left(1-\mu^2\right)\frac{d}{d\mu}\right)M(\mu)+\beta(\beta+1)M(\mu)=0, \quad \mu_0\le\mu\le 1 \tag{12}$$

$$\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R_i(r)-\beta(\beta+1)R_i(r)=0,\quad r_{i-1}\le r\le r_i,\quad i=1,\dots,n \tag{13}$$

Taking into account the function (11) in conditions (6)-(10), we receive the boundary conditions for the functions M and R_i. The function $M(\mu)$ satisfies the conditions:

$$|M(\mu)|<\infty,\quad \mu\in[\mu_0,1) \tag{14}$$

$$M'(\mu_0)=0 \tag{15}$$

The functions $R_i(r)$ satisfy the following conditions at $r=0$ and at interfaces $r=r_i$, $i=1,2,\dots,n-1$:

$$|R_1(0)|<\infty, \tag{16}$$

$$R_i(r_i)=R_{i+1}(r_i),\quad i=1,\dots,n-1, \tag{17}$$

$$\lambda_i\left.\frac{dR_i(r)}{dr}\right|_{r=r_i}=\lambda_{i+1}\left.\frac{dR_{i+1}(r)}{dr}\right|_{r=r_i},\quad i=1,\dots,n-1. \tag{18}$$

Moreover, the function $R_n(r)$, as the radial part of the function (11), satisfies the condition (10).

We can present the solution to the Equation (12), which satisfies the condition (14), in the form

$$M(\mu)=c\cdot P_\beta(\mu) \tag{19}$$

where c is a constant and $P_\beta(\mu)$ is the Legendre function of the first kind. Using the derivative of the Legendre function [15] and the boundary condition (15), we obtain the following equation

$$\mu_0P_\beta(\mu_0)-P_{\beta+1}(\mu_0)=0 \tag{20}$$

The roots of the Equation (20) for $\mu_0=0$ (i.e. for hemisphere), are equal to $\beta_m=2m,\ m=0,1,2,\dots$. In this case, the eigenfunctions $P_{2m}(\mu)$, where m is a positive integer number, are the Legendre polynomials. The roots of this equation for $\mu_0\in[0,1)$ are determined numerically, and some of them are given in Table 1.

Table 1. Numerical values of roots β_m of Eq. (20) for selected φ_0, $\mu_0 = \cos(\varphi_0)$

φ_0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
μ_0	0.9659258	0.8660254	0.7071068	0.5000000	0.2588190	0.0000000
β_0	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000	0.0000000
β_1	14.1446160	6.8353981	4.4053292	3.1956912	2.4750556	2.0000000
β_2	26.3022528	12.9082841	8.4471126	6.2195292	4.8859074	4.0000000
β_3	38.3630172	18.9364458	12.4633288	9.2288494	7.2902007	6.0000000
β_4	50.3952217	24.9513811	16.4719397	12.2338091	9.6924933	8.0000000
β_5	62.4151685	30.9606343	20.4772774	15.2368863	12.0939179	10.0000000
β_6	74.4287362	36.9669292	24.4809096	18.2389812	14.4948886	12.0000000
β_7	86.4385628	42.9714887	28.4835410	21.2404994	16.8955923	14.0000000
β_8	98.4460081	48.9749435	32.4855350	24.2416499	19.2961258	16.0000000
β_9	110.4518440	54.9776516	36.4870981	27.2425520	21.6965441	18.0000000
β_{10}	122.4565415	60.9798315	40.4883564	30.2432783	24.0968810	20.0000000
β_{11}	134.4604040	66.9816239	44.4893911	33.2438755	26.4971580	22.0000000
β_{12}	146.4636360	72.9831238	48.4902569	36.2443752	28.8973899	24.0000000
β_{13}	158.4663802	78.9843973	52.4909921	39.2447996	31.2975867	26.0000000
β_{14}	170.4687393	84.9854921	56.4916241	42.2451644	33.6977560	28.0000000
β_{15}	182.4707891	90.9864434	60.4921733	45.2454815	36.0979031	30.0000000
β_{16}	194.4725866	96.9872775	64.4926549	48.2457595	38.4980321	32.0000000
β_{17}	206.4741757	102.9880150	68.4930806	51.2460052	40.8981461	34.0000000
β_{18}	218.4755906	108.9886717	72.4934597	54.2462241	43.2982477	36.0000000
β_{19}	230.4768586	114.9892601	76.4937994	57.2464202	45.6983387	38.0000000
β_{20}	242.4780013	120.9897905	80.4941056	60.2465970	48.0984207	40.0000000
β_{100}	1202.4955563	600.9979377	400.4988093	300.2493126	240.0996810	200.0000000
β_{1000}	12002.4995546	6000.9997933	4000.4998807	3000.2499311	2400.0999680	2000.0000000

The functions $P_{\beta_m}(\mu)$, $m = 0, 1, 2, \dots$, create an orthogonal set of functions [16], and the orthogonality condition of these functions can be written as

$$\int_{\mu_0}^{1} P_{\beta_m}(\mu) P_{\beta_n}(\mu)\, d\mu = N_m^{\mu}\, \delta_{m,n}, \quad m, n = 0, 1, 2, \dots \tag{21}$$

where $\delta_{m,n}$ is the Kronecker delta and $N_m^{\mu} = \int_{\mu_0}^{1} \left(P_{\beta_m}(\mu)\right)^2 d\mu$.

We give the general solution to the Euler equation (13) for $\beta = \beta_m$, $m = 0, 1, 2, \ldots$, by

$$R_{i,m}(r) = B_{1,i}\left(\frac{r}{r_i}\right)^{\beta_m} + B_{2,i}\left(\frac{r}{r_i}\right)^{-(\beta_m+1)}, \quad r_{i-1} \le r \le r_i, \quad i = 1, \ldots, n \tag{22}$$

where $B_{1,i}$, $B_{2,i}$ are arbitrary constants. Next, using boundary conditions (17)-(18), we obtain a set of $2n-2$ equations with unknowns: $B_{1,i}$, $B_{2,i}$, $i = 1, \ldots, n-1$. The received equations are the following

$$r_i^{\beta_m+1} r_{i+1}^{\beta_m}\left(B_{1,i} + B_{2,i}\right) - r_i^{2\beta_m+1} B_{1,i+1} - r_{i+1}^{2\beta_m+1} B_{2,i+1} = 0 \tag{23}$$

$$\lambda_i r_i^{\beta_m+1} r_{i+1}^{\beta_m}\left[\beta_m B_{1,i} - \left(\beta_m + 1\right) B_{2,i}\right] - \lambda_{i+1}\left[\beta_m r_i^{2\beta_m+1} B_{1,i+1} - \left(\beta_m + 1\right) r_{i+1}^{2\beta_m+1} B_{2,i+1}\right] = 0 \tag{24}$$

Taking into account condition (16), we assume $B_{2,1} = 0$ in the Equations (22)-(24).

We complete the system of the Equations (23)-(24) by an equation obtained on the basis of the boundary condition (10). Let's remind that the functions T_i are presented by the Equation (11) as a product of two appropriate functions. To satisfy the condition (10), we assume that

$$T_i(r,\mu) = \sum_{m=0}^{\infty} R_{i,m}(r) P_{\beta_m}(\mu), \quad i = 1, 2, \ldots, n \tag{25}$$

Next, we substitute the functions T_i given by the Equation (25), for $i = n$, into the condition (10), multiply the received equation by $P_{\beta_{m'}}(\mu)$ and integrate it with respect to μ in the interval $[\mu_0, 1]$. Finally, we use the orthogonality condition (21), and we get the condition for the function $R_{n,m}(r)$

$$\lambda_n \left.\frac{dR_{n,m}(r)}{dr}\right|_{r=r_n} + \alpha_{conv} R_{n,m}(r_n) = \frac{1}{N_m^{\mu}} \int_{\mu_0}^{1} \left(\alpha_{conv} T_{amb} + q_{direct}\,\mu + q_{diffusion}\right) P_{\beta_m}(\mu)\, d\mu \tag{26}$$

After that, we substitute the functions $R_{i,m}(r)$, given by the Equation (22), for $i = n$, into the Equation (26), and we obtain the following equation

$$\left(\alpha_{conv} + \frac{\beta_m \lambda_n}{r_n}\right) B_{1,n} + \left(\alpha_{conv} - \frac{(\beta_m + 1)\lambda_n}{r_n}\right) B_{2,n} = \frac{I_m}{N_m^{\mu}} \tag{27}$$

where $I_m = \int_{\mu_0}^{1} \left(\alpha_{conv} T_{amb} + q_{direct}\,\mu + q_{diffusion}\right) P_{\beta_m}(\mu)\, d\mu$. The integral I_m may be expressed in the analytical form as

$$I_m = \left(\alpha_{conv} T_{amb} + q_{diffusion}\right) I_{0m} + q_{direct}\, I_{1m} \tag{28}$$

where

$$I_{0m} = \int_{\mu_0}^{1} P_{\beta_m}(\mu)\, d\mu = \left. \frac{P_{\beta_m+1}(\mu) - P_{\beta_m-1}(\mu)}{2\beta_m + 1} \right|_{\mu=\mu_0}^{\mu=1} \tag{29}$$

$$\begin{aligned} I_{1m} &= \int_{\mu_0}^{1} \mu\, P_{\beta_m}(\mu)\, d\mu = \\ &\left. \frac{1}{2\beta_m + 1} \left(\beta_m \frac{P_{\beta_m}(\mu) - P_{\beta_m-2}(\mu)}{2\beta_m - 1} + \left(\beta_m + 1\right) \frac{P_{\beta_m+2}(\mu) - P_{\beta_m}(\mu)}{2\beta_m + 3} \right) \right|_{\mu=\mu_0}^{\mu=1} \end{aligned} \tag{30}$$

The Equations (27) and (23)-(24) create a system of $2n-1$ equations, solved for $\beta = \beta_m$, $m = 0, 1, 2, \ldots$, with respect to $B_{1,1}, B_{1,2}, B_{2,2}, \ldots, B_{1,n}, B_{2,n}$. Therefore, the functions T_i, $i = 1, \ldots, n$, are finally defined by the Equation (25), while the functions $R_{i,m}$ are given by the Equation (22).

4. Numerical example

The presented example concerns the heat conduction in a spherical cone without an internal heat source. The cone consists of $n = 5$ layers with the following radii: $r_1 = 0.3$ m, $r_2 = 0.5$ m, $r_3 = 0.7$ m, $r_4 = 0.9$ m and $r_5 = 1$ m. The physical data assumed in the computation are the following: the thermal conductivities in the appropriate layers are $\lambda_1 = 80\,\mathrm{W/(mK)}$, $\lambda_2 = 40\,\mathrm{W/(mK)}$, $\lambda_3 = 20\,\mathrm{W/(mK)}$, $\lambda_4 = 10\,\mathrm{W/(mK)}$ and $\lambda_5 = 5\,\mathrm{W/(mK)}$, respectively; the direct solar radiation $q_{direct} = 0.9 \cdot q_{solar}$, the diffuse solar radiation $q_{diffusion} = 0.1 \cdot q_{solar}$ for the solar radiation $q_{solar} = 850\,\mathrm{W/m^2}$; the ambient temperature $T_{amb} = 20^{\circ}\mathrm{C}$; the convective heat transfer coefficient $\alpha_{conv} = 10\,\mathrm{W/(m^2K)}$. In Figure 3, contour plots of the function $T(r,\mu)$ illustrating the temperature distribution in the cross-section of the considered cone for different values of the apex angles $\varphi_0 = \pi/12;\ \pi/6;\ \pi/4;\ \pi/3;\ 5\pi/12;\ \pi/2$ are presented. The following notations are used in the Figure: $T_{\min} = T(r_n, \mu_0)$, $T_{\max} = T(r_n, 1)$, and $T_c = T(0,\mu)$ is the temperature in the center of the cone.

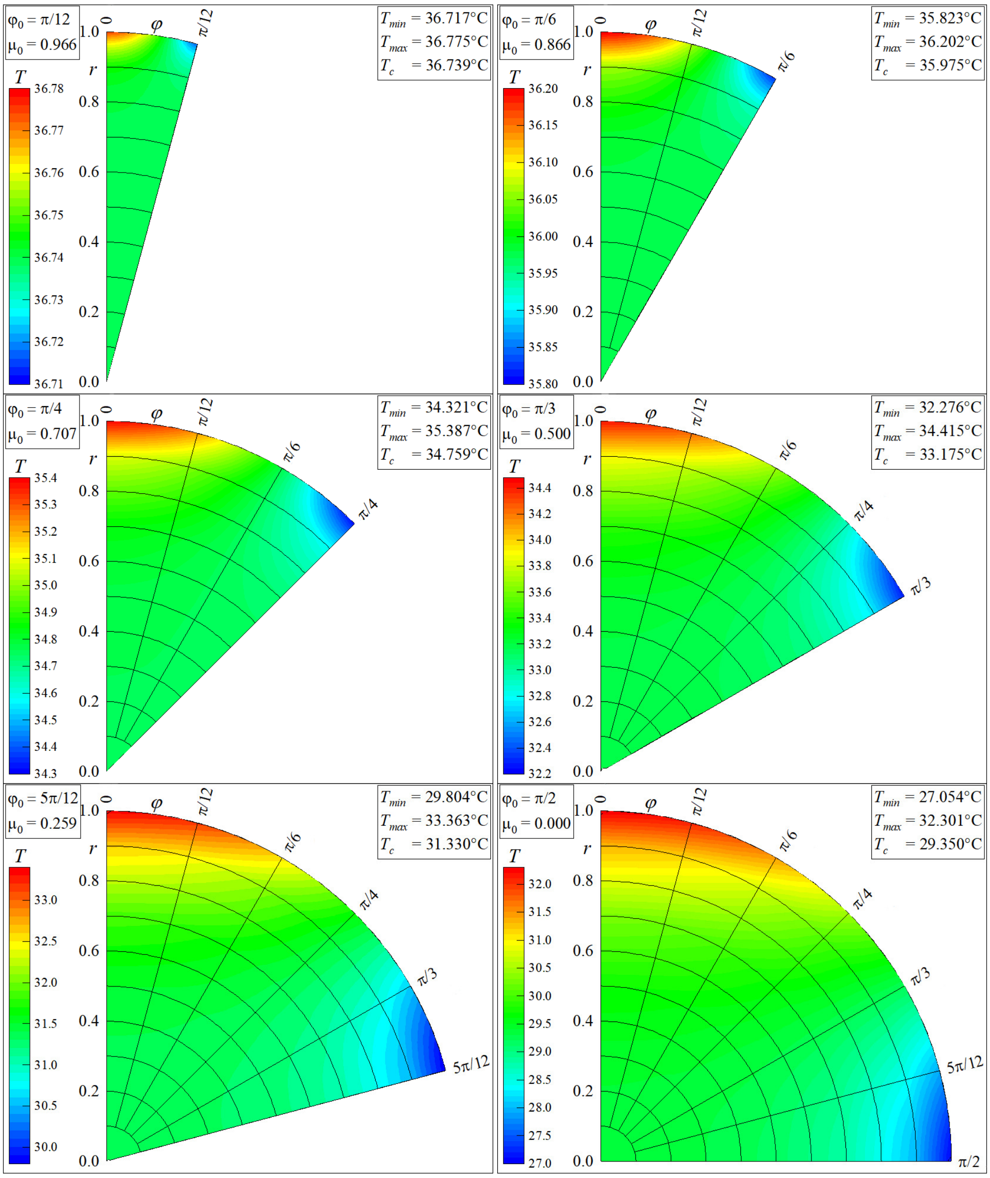

Fig. 3. Distributions of the temperature field for the steady state heat transfer in the cross section of the spherical cones of apex angles: $\varphi_0 = \{\pi/12;\, \pi/6;\, \pi/4;\, \pi/3;\, 5\pi/12;\, \pi/2\}$

We assume that the sun's rays are consistent with the symmetry axis of the cone, i.e. they fall at an angle $\varphi = 0$. Then the highest temperature (on the outer surface $r = r_n$) is for $\mu = 1$ and the lowest for $\mu = \mu_0$. The temperature difference between the extreme points of the cone is greater, for a higher value of φ_0. Moreover, if φ_0 is greater, then the temperature of the center of the cone is smaller.

5. Conclusions

The analytical solution of the heat conduction problem in a spherical cone with azimuthal symmetry is presented. The cone consists with n concentric spherical layers. The mathematical model is related to the determination of the steady state of the temperature distribution in the considered medium. Additionally, the considerations assume that the heating of the cone by solar energy consists of two parts of the heat flow: direct and diffusion, with the cone being cooled by its outer surface as a result of convective heat flow to the surroundings. A solution is derived in the form of a product of three functions, among others, using Legendre functions of the first kind. From the computational point of view, the derived analytical solution is partially supported by numerical methods, among others, to determine the roots β_m of the Equation (20), to solve the linear system of equations (in order to determine the coefficients), to numerically calculate the values of the Legendre function of the first of non-integer degree. Also, the infinite sum that appears in the Equation (25) is limited to 50 terms.

The mathematical model of the considered problem is supported by an example that shows the temperature distributions in the cone for different values of the polar angles. In the presented plots, it can be seen that the temperature difference between the extreme points of the cone is greater for a higher value of the apex angle. Furthermore, as the apex angle of the cone increases, the temperature in its center decreases. In the example considered, it is assumed that the direct solar radiation is consistent with the symmetry axis of the cone.

The next step of the research will be the developing of the mathematical model that describes the unsteady state of the temperature distribution in the considered medium. Therefore, the movement of the sun over time could be taken into account in the considerations.

References

[1] Haji-Sheikh, A., & Beck, J.V. (2002). Temperature solution in multi-dimensional multi-layer bodies. *International Journal of Heat and Mass Transfer*, *45*, 1865-1877.

[2] Lü, X., Tervola, P., & Viljanen, M. (2005). A new analytical method to solve the heat equation for a multi-dimensional composite slab. *Journal of Physics A: General Physics*, *38*(13), 2873.

[3] Lu, X., Tervola, P., & Viljanen, M. (2006). Transient analytical solution to heat conduction in composite circular cylinder. *International Journal of Heat and Mass Transfer*, *49*, 341-348.

[4] Siedlecka, U., & Kukla, S. (2013). Application of a Green's function method to heat conduction problems in multi-layered cylinders. *Journal of Applied Mathematics and Computational Mechanics*, *12*(3), 105-113.

[5] Kukla, S., & Siedlecka, U. (2014). Green's function for heat conduction problems in a multi--layered hollow cylinder. *Journal of Applied Mathematics and Computational Mechanics*, *13*(3), 115-122.

[6] Lu, X., & Viljanen, M. (2006). An analytical method to solve heat conduction in layered spheres with time-dependent boundary conditions. *Physics Letters A*, *351*, 274-282.

[7] Siedlecka, U. (2014). Radial heat conduction in a multilayered sphere. *Journal of Applied Mathematics and Computational Mechanics*, *13*(4), 111-118.

[8] Pawar, S.P., Deshmukh, K.C., & Kedar, G.D. (2013). Effect of generation on quasi-static thermal stresses in a solid sphere. *IOSR Journal of Mathematics*, *7*(5), 21-29.

[9] Jain, P.K., Singh, S., & Rizwan-uddin (2010). An exact solution for two dimensional, unsteady, multilayer heat conduction in spherical coordinates. *International Journal of Heat Mass Transfer*, *53*, 2133-2142.

[10] Suneet, S., Prashant, K.J., & Rizwan-uddin (2016). Analytical solution for three-dimensional, unsteady heat conduction in a multilayer sphere. *ASME Journal of Heat and Mass Transfer*, *138*(10), 101301.

[11] Kumar, S., Kar, V.R., & Khudayarov B.A. (2022). *Analytical Solution for the Steady-State Heat Transfer Analysis of Porous Nonhomogeneous Material Structures*. In: *Advanced Composite Materials and Structures*. CRC Press.

[12] du Toit, J., & Pretorius, Ch. (2021). Steady-state heat transfer analysis in a spherical domain revisited. *MATEC Web of Conferences*, *347*, 00006.

[13] Conroy, T., Collins, M.N., & Grimes, R. (2020). A review of steady-state thermal and mechanical modelling on tubular solar receivers. *Renewable and Sustainable Energy Reviews*, *119*, 109591.

[14] Klimenta, D., Minić, D., Pantić-Ranđelović, L., Radonjić-Mitić, I., & Premović-Zečević M. (2023). Modeling of steady-state heat transfer through various photovoltaic floor laminates. *Applied Thermal Engineering*, *229*, 120589.

[15] Szmytkowski, R. (2006). On the derivative of the Legendre function of the first kind with respect to its degree. *Journal of Physics A: Mathematical and General*, *39*, 15147-15172.

[16] Kukla, S., Siedlecka, U., & Ciesielski, M. (2022). Fractional order dual-Phase-lag model of heat conduction in a composite spherical medium. *Materials*, *15*(20), 7251.