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NUMERICAL APPROXIMATION OF THE RIEMANN-LIOUVILLE FRACTIONAL INTEGRALS USING THE AKIMA SPLINE INTERPOLATION

Grzegorz Grodzki

Department of Computer Science, Czestochowa University of Technology
Czestochowa, Poland
grzegorz.grodzki@icis.pcz.pl

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Abstract. This paper presents the numerical algorithms for evaluating the values of the left- and right-sided Riemann-Liouville fractional integrals using the linear and Akima cubic spline interpolations. Sample numerical calculations have been performed based on the derived algorithms. The results are presented in two tables. Knowledge of the closed analytical expressions for sample fractional integrals makes it possible to determine the numerical errors and the experimental rates of convergence for each derived algorithm.

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1. Introduction

The fractional calculus [1, 2] is treated as a generalization of the classical calculus, where the integrals and derivatives are of convolutional type and usually have power-type kernels. These fractional operators of non-integer orders can be used to describe the various processes with memory by using them in the differential-integral equations. Applications of the fractional calculus in science and engineering are significant and continue to grow – selected works [3-7] can be mentioned. It is difficult to indicate all areas of science (without omitting any) where these operators can be used. There are different types of fractional integral and differential operators [1, 8] that are proposed by Riemann, Liouville, Weyl, Riesz, Grunwald, Letnikov, Marchaud and many other sciences, and new definitions of fractional operators appear all the time.

In this work, the main focus is on the development and investigation of numerical methods for evaluating the values of left- and right-sided Riemann-Liouville fractional integrals. The left-sided fractional integral $I^{\alpha}_{a^+}y(x)$ and the right-sided fractional integral $I^{\alpha}_{b^-}y(x)$ of order α > 0 of the given function $y(x)$ on the interval [a, b] are defined in the following ways, respectively [1, 2, 8]

$$I^{\alpha}_{a^+}y(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\frac{y(\xi)}{(x-\xi)^{1-\alpha}}d\xi,\quad \text{for } x>a \tag{1}$$

$$I^{\alpha}_{b^-}y(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\frac{y(\xi)}{(\xi-x)^{1-\alpha}}d\xi,\quad \text{for } x<b \tag{2}$$

where Γ is the Gamma function. These fractional integral operators are used in many aspects of mathematics, especially to transform the fractional ordinary/partial differential equations into fractional integral equations. Hence, there is the need to develop appropriate numerical methods for approximate evaluation of the above fractional integrals, especially in cases where the integrand function has a complicated form or the closed analytical forms of the fractional integrals have not been designated yet. Here, it is worth mentioning the book [9] written by Oldham and Spanier, in which the pioneering methods of approximating the fractional operators are described. Analysing the literature (e.g. [10-16]), it can be noticed that better (with high accuracy, fast convergence) approximation methods for the fractional integrals and derivatives are constantly being developed.

Typically, two main steps need to be used to develop a numerical method of integration. The first step is to replace the integrand function by a piecewise-polynomial interpolant on the grid of points, while the second step involves integration of the interpolant instead of the original function. Depending on the adopted type of polynomial, various numerical schemes with a specified accuracy are obtained.

Hiroshi Akima published article [17] in 1970, in which he proposed a new interpolation algorithm applicable in the successive intervals that are determined by the given points. This algorithm is based on the piecewise function composed of the set of polynomials, each of degree three, at most. In his method, the slope of the curve is locally determined at each given point, whereas each polynomial being a part of the interpolation curve between a pair of given points is determined by the coordinates of points and the slopes at these points. H. Akima improved his own interpolation method many times, e.g. [18]. In the current literature [19-21], his method is known as the Akima cubic spline interpolation.

Section 2 of this work presents approximation algorithms, in particular, the introduction to the linear and cubic spline interpolations and the presentation of the Akima spline. The study of numerical evaluations of fractional integrals (1)-(2) by applying these spline interpolation methods for the integrand function is presented in Section 3. In Section 4, the quality of the proposed numerical algorithms on two examples by presenting numerical errors and determining the Experimental Order of Convergence (EOC) is tested.

2. Interpolation of integrand function using splines

The integrand function $y(x)$ should be sufficiently smooth on the considered interval $[a, b]$. The interval $[a, b]$ is divided into N sub-intervals $[x_i, x_{i+1}]$, for $i = 0, 1, ..., N-1$, with the constant step length Δx, where the coordinates of $N+1$ nodal points are as follows

$$x_i = a + i\,\Delta x, \quad \Delta x = (b-a)/N \tag{3}$$

Moreover, at this set of points, the values of function $y(x)$ are determined as $y_i = y(x_i)$, for $i = 0, 1, ..., N$, and constitute the input data to the algorithms.

Here, function $y(x)$ is replaced by the interpolation formula which can be a polynomial of arbitrary degree or a spline. The high-degree polynomials may cause larger oscillations between interpolation points and may be a poor predictor of the interpolation function between points. Whereas the spline curves are usually at most third degree polynomials and are only piecewise continuous. The piecewise function $s(x)$ can be expressed as

$$y(x) \cong s(x) = \begin{cases} s_0(x), & \text{if } x \in [x_0, x_1] \\ s_1(x), & \text{if } x \in [x_1, x_2] \\ \cdots & \\ s_{N-1}(x), & \text{if } x \in [x_{N-1}, x_N] \end{cases} \tag{4}$$

where $s_i(x)$, for $i = 0, 1, ..., N-1$, are polynomials of degree p in each sub-interval as

$$s_i(x) = \sum_{k=0}^{p} c_{k,i} (x - x_i)^k, \quad \text{for } x \in [x_i, x_{i+1}] \tag{5}$$

where $c_{k,i}$, for $k = 0, 1, ..., p$, are the coefficients of polynomial $s_i(x)$ in i-th sub-interval $[x_i, x_{i+1}]$. The important feature of the spline $s(x)$ that interpolates the set of the data points (x_0, y_0), (x_1, y_1), ..., (x_N, y_N) is the relationship $s(x_i) = y_i$, for $i = 0, 1, ..., N$, which means that the piecewise polynomials are linked in the set of points. The determination way of coefficients $c_{k,i}$ depends on the kind of spline interpolation being used.

2.1. Linear spline interpolation

This kind of spline is mainly used to compare the obtained numerical results and is briefly presented. The linear spline used for numerical integration is known as the trapezoidal rule [22]. Here, the set of the data points is approximated by the piecewise first degree polynomial (linear function), between the adjacent data points (x_i, y_i) and (x_{i+1}, y_{i+1}), for $i = 0, 1, ..., N-1$. The first degree polynomial ($p = 1$) in i-th sub-interval is constructed as

$$s_i(x) = \sum_{k=0}^{p} c_{k,i}(x - x_i)^k = c_{0,i} + c_{1,i}(x - x_i), \quad \text{for } x \in [x_i, x_{i+1}] \tag{6}$$

where the polynomial coefficients have the forms

$$\begin{aligned} c_{0,i} &= y_i \\ c_{1,i} &= \frac{y_{i+1} - y_i}{\Delta x} \end{aligned} \tag{7}$$

2.2. Akima cubic spline interpolation

The cubic splines produce a curve that appears to be seamless and has smooth characteristics. A piecewise continuous curve passes through each of the data points (x_i, y_i), for $i = 0, 1, ..., N$, wherein $N \geq 4$, in the given order and the separate polynomial of third degree ($p = 3$) (so-called the cubic polynomial segment) in each sub-interval has own set of coefficients, i.e.

$$s_i(x) = c_{0,i} + c_{1,i}(x - x_i) + c_{2,i}(x - x_i)^2 + c_{3,i}(x - x_i)^3 \tag{8}$$

where $x \in [x_i, x_{i+1}]$, for $i = 0, 1, ..., N-1$. These four coefficients $c_{0,i}$, $c_{1,i}$, $c_{2,i}$ and $c_{3,i}$ for i-th polynomial, in each of the N sub-intervals should be determined, and hence, in order to define the whole spline, a total of $4N$ independent dependencies imposed on the spline are required.

The Akima interpolation spline [19, 21] needs only be once continuously differentiable in contrast to the natural cubic spline that is constructed to be twice continuously differentiable everywhere. The Akima spline in the selected sub-interval is built on the basis of the given values of function $y(x)$ and its first derivatives (slopes) at the ends of the sub-interval. In order to find the coefficients of the Akima spline $c_{0,i}$, $c_{1,i}$, $c_{2,i}$ and $c_{3,i}$, for $i = 0, 1, ..., N-1$, in Eq. (8), first, the slopes of the line segment in each sub-interval $[x_i, x_{i+1}]$, for $i = 0, 1, ..., N-1$, are calculated as

$$m_i = \frac{y_{i+1} - y_i}{\Delta x} \tag{9}$$

and next, the spline slopes w_i are determined on the basis of values of m_i using formula

$$w_i = \begin{cases} m_0 & \text{if } i = 0 \\ \dfrac{m_0 + m_1}{2} & \text{if } i = 1 \\ \hat{w}_i & \text{if } i = 2, ..., N-2 \\ \dfrac{m_{N-2} + m_{N-1}}{2} & \text{if } i = N-1 \\ m_{N-1} & \text{if } i = N \end{cases} \tag{10}$$

where

$$\hat{w}_i = \begin{cases} \dfrac{|m_{i+1} - m_i| m_{i-1} + |m_{i-1} - m_{i-2}| m_i}{|m_{i+1} - m_i| + |m_{i-1} - m_{i-2}|}, & \text{if} |m_{i+1} - m_i| + |m_{i-1} - m_{i-2}| \neq 0 \\ \dfrac{m_{i-1} + m_i}{2}, & \text{otherwise} \end{cases} \tag{11}$$

The Akima spline must satisfy four conditions of continuity of the spline function together with its first derivative: $s_i(x_i) = y_i$, $s_i(x_{i+1}) = y_{i+1}$, $s_i'(x_i) = w_i$ and $s_i'(x_{i+1}) = w_{i+1}$. Hence, the final forms of the coefficients in Eq. (8) are as follows:

$$\begin{aligned} c_{0,i} &= y_i \\ c_{1,i} &= w_i \\ c_{2,i} &= \frac{1}{\Delta x}\left(3m_i - 2w_i - w_{i+1}\right) \\ c_{3,i} &= \frac{1}{\left(\Delta x\right)^2}\left(w_i + w_{i+1} - 2m_i\right) \end{aligned} \tag{12}$$

One can note that in the case of the Akima spline, no linear system of equations must be solved to determine these above coefficients, and an essential advantage is that this kind of spline requires relatively small computational effort. This spline is based on the local interpolation by the polynomial of third degree, meaning that the polynomial coefficients in the inner sub-interval $[x_i, x_{i+1}]$ depend on the values of $y_{i-2}, y_{i-1}, y_i, y_{i+1}, y_{i+2}, y_{i+3}$ only, and generally, no fewer than 5 points are needed to construct this spline. Here, it is possible to determine parts of the spline function without knowing all the data points (x_i, y_i), for $i = 0, 1, ..., N$, and the variation/ /perturbation of any data point only has an effect on the spline coefficients in its immediate neighbourhood. The interpolation error in the inner sub-intervals has order $O((\Delta x)^2)$ [21].

3. Numerical fractional integration

Once the form of the spline is known, then the integrand function $y(x)$ is replaced by the spline $s(x)$ and is integrated analytically at each sub-interval. So, the approximate values of the left- and right-sided fractional integrals (1)-(2) of function $y(x)$ are calculated as

$$I_{a^+}^{\alpha} y(x) \cong I_{a^+}^{\alpha} s(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{s(\xi)}{(x-\xi)^{1-\alpha}} d\xi \tag{13}$$

$$I_{b^-}^{\alpha} y(x) \cong I_{b^-}^{\alpha} s(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{s(\xi)}{(\xi-x)^{1-\alpha}} d\xi \tag{14}$$

These values are determined in the set of data points $x \in \{x_M\}$, $M = 1, ..., N$ for integral (1), and $M = 0, ..., N-1$ for integral (2), respectively, in the following ways

$$\begin{aligned} I^{\alpha}_{a^+} s(x)\Big|_{x=x_M} &= \sum_{i=0}^{M-1} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{s_i(\xi)}{(x_M-\xi)^{1-\alpha}} d\xi \\ &= \sum_{i=0}^{M-1} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{1}{(x_M-\xi)^{1-\alpha}} \sum_{k=0}^{p} c_{k,i} (\xi - x_i)^k d\xi \\ &= \sum_{i=0}^{M-1} \sum_{k=0}^{p} c_{k,i} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{(\xi - x_i)^k}{(x_M-\xi)^{1-\alpha}} d\xi = \sum_{i=0}^{M-1} \sum_{k=0}^{p} c_{k,i} J^{\alpha,k,i,M}_{a^+} \end{aligned} \tag{15}$$

$$\begin{aligned} I^{\alpha}_{b^-} s(x)\Big|_{x=x_M} &= \sum_{i=M}^{N-1} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{s_i(\xi)}{(\xi - x_M)^{1-\alpha}} d\xi \\ &= \sum_{i=M}^{N-1} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{1}{(\xi - x_M)^{1-\alpha}} \sum_{k=0}^{p} c_{k,i} (\xi - x_i)^k d\xi \\ &= \sum_{i=M}^{N-1} \sum_{k=0}^{p} c_{k,i} \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{(\xi - x_i)^k}{(\xi - x_M)^{1-\alpha}} d\xi = \sum_{i=M}^{N-1} \sum_{k=0}^{p} c_{k,i} J^{\alpha,k,i,M}_{b^-} \end{aligned} \tag{16}$$

The particular integrals $J^{\alpha,k,i,M}_{a^+}$ and $J^{\alpha,k,i,M}_{b^-}$, for $k = 0, 1, ..., 3$, with regard to data point x_M in i-th sub-interval, are computed analytically. In the case of the first one, they are as follows

$$\begin{aligned} J^{\alpha,k,i,M}_{a^+} &= \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{(\xi - x_i)^k}{(x_M-\xi)^{1-\alpha}} d\xi \overset{\xi = x_i + u\Delta x}{=} \frac{(\Delta x)^{\alpha+k}}{\Gamma(\alpha)} \int_0^1 \frac{u^k}{(M-i-u)^{1-\alpha}} du \\ &= \frac{(\Delta x)^{\alpha+k}}{\Gamma(\alpha+k+1)} \left(k!(M-i)^{\alpha+k} - \theta^{\alpha,k,i,M}_{a^+} (M-i-1)^{\alpha} \right) \end{aligned} \tag{17}$$

for $i < M$, where

$$\theta^{\alpha,k,i,M}_{a^+} = \begin{cases} 1 & \text{if } k = 0 \\ M - i + \alpha & \text{if } k = 1 \\ 2(M-i)^2 + 2\alpha(M-i) + \alpha^2 + \alpha & \text{if } k = 2 \\ 6(M-i)^3 + 6\alpha(M-i)^2 + 3\alpha(\alpha+1)(M-i) & \\ \quad + \alpha^3 + 3\alpha^2 + 2\alpha & \text{if } k = 3 \end{cases} \tag{18}$$

While, in the case of the second one, the integrals $J_{b^-}^{\alpha,k,i,M}$ take the following analytical forms

$$\begin{aligned} J_{b^-}^{\alpha,k,i,M} &= \frac{1}{\Gamma(\alpha)} \int_{x_i}^{x_{i+1}} \frac{(\xi - x_i)^k}{(\xi - x_M)^{1-\alpha}} d\xi \overset{\xi = x_i + u\Delta x}{=} \frac{(\Delta x)^{\alpha+k}}{\Gamma(\alpha)} \int_0^1 \frac{u^k}{(u - M + i)^{1-\alpha}} du \\ &= \frac{(\Delta x)^{\alpha+k}}{\Gamma(\alpha + k + 1)} \left((-1)^{k+1} k! (i - M)^{\alpha+k} + \theta_{b^-}^{\alpha,k,i,M} (i - M + 1)^{\alpha} \right) \end{aligned} \tag{19}$$

for $i \geq M$, where

$$\theta_{b^-}^{\alpha,k,i,M} = \begin{cases} 1 & \text{if } k = 0 \\ -(i - M) + \alpha & \text{if } k = 1 \\ 2(i - M)^2 - 2\alpha(i - M) + \alpha^2 + \alpha & \text{if } k = 2 \\ -6(i - M)^3 + 6\alpha(i - M)^2 - 3\alpha(\alpha + 1)(i - M) \\ \quad + \alpha^3 + 3\alpha^2 + 2\alpha & \text{if } k = 3 \end{cases} \tag{20}$$

4. Examples of computations

The correctness of the proposed numerical schemes can be verified based on sample calculations and their analysis. The important feature of numerical integration method is the determination of the Experimental Order of Convergence (EOC) [11, 15] for the developed scheme. If the exact solutions of the fractional integrals are known, then the computational error of the numerical integration scheme obtained on the size grid N is determined by the formula

$$err_N = I_{a^+}^{\alpha} y(x) - \Psi_N \tag{21}$$

where Ψ_N is the numerical value of $I_{a^+}^{\alpha} y(x)$ and is simultaneously the exact value of $I_{a^+}^{\alpha} s(x)$ at the node point x_M. Similarly, the err_N can be calculated for the right--sided fractional integral. Knowledge of the error values calculated for the size grid N and $N/2$ allows one to determine the EOC in the form

$$EOC_N = \log_2 \frac{|err_{N/2}|}{|err_N|} \tag{22}$$

which should be computed over a range of different grid sizes.

Example 1

In this example, let's consider the integrand function $y(x)$ that is the seventh-degree polynomial in the form

$$y(x) = 2x^7 - 14x^6 + 17x^5 + 50x^4 - 66x^3 - 84x^2 + 50x + 100 \tag{23}$$

in the interval $[a, b]$, for $a = -1$ and $b = 3$. For the assumed interval endpoints, the values of function $y(x)$ and its derivatives are the following: $y(a) = 49$, $y(b) = 61$, $y'(a) = 3$, $y'(b) = -157$, $y''(a) = -16$, $y''(b) = -384$. For polynomial functions, one can easily find the analytical forms of the left- and right-sided fractional integrals of order $\alpha > 0$. For this purpose, function $y(x)$ must be transformed to contain the expressions $(x - a)$ and $(b - x)$, hence one obtains

$$\begin{aligned} y(x) = {} & 2(x+1)^7 - 28(x+1)^6 + 143(x+1)^5 - 315(x+1)^4 + 254(x+1)^3 \\ & - 8(x+1)^2 + 3(x+1) + 49 \end{aligned} \tag{24}$$

$$\begin{aligned} y(x) = {} & -2(3-x)^7 + 28(3-x)^6 - 143(3-x)^5 + 305(3-x)^4 - 174(3-x)^3 \\ & - 192(3-x)^2 + 157(3-x) + 61 \end{aligned} \tag{25}$$

Then we can directly apply the properties of the left- and right-sided fractional integration of the power functions $(x - a)^\beta$ and $(b - x)^\beta$, for $\beta > -1$ and $\alpha > 0$ [1, 2]

$$I^{\alpha}_{a^+} (x-a)^{\beta} = \mu^{\alpha}_{\beta} (x-a)^{\beta+\alpha}, \quad I^{\alpha}_{b^-} (b-x)^{\beta} = \mu^{\alpha}_{\beta} (b-x)^{\beta+\alpha} \tag{26}$$

where $\mu^{\alpha}_{\beta} = \Gamma(\beta+1)/\Gamma(\beta+\alpha+1)$, in particular $\Gamma(k) = (k - 1)!$, for $k = 1, 2, \ldots$.

The analytical forms of both fractional integrals for function (23) (for $a = -1$ and $b = 3$, respectively) are expressed by the formulas

$$\begin{aligned} I^{\alpha}_{-1^+} y(x) = {} & 2\mu^{\alpha}_{7} (x+1)^{7+\alpha} - 28\mu^{\alpha}_{6} (x+1)^{6+\alpha} + 143\mu^{\alpha}_{5} (x+1)^{5+\alpha} - 315\mu^{\alpha}_{4} (x+1)^{4+\alpha} \\ & + 254\mu^{\alpha}_{3} (x+1)^{3+\alpha} - 8\mu^{\alpha}_{2} (x+1)^{2+\alpha} + 3\mu^{\alpha}_{1} (x+1)^{1+\alpha} + 49\mu^{\alpha}_{0} (x+1)^{\alpha} \end{aligned} \tag{27}$$

$$\begin{aligned} I^{\alpha}_{3^-} y(x) = {} & -2\mu^{\alpha}_{7} (3-x)^{7+\alpha} + 28\mu^{\alpha}_{6} (3-x)^{6+\alpha} - 143\mu^{\alpha}_{5} (3-x)^{5+\alpha} + 305\mu^{\alpha}_{4} (3-x)^{4+\alpha} \\ & - 174\mu^{\alpha}_{3} (3-x)^{3+\alpha} - 192\mu^{\alpha}_{2} (3-x)^{2+\alpha} + 157\mu^{\alpha}_{1} (3-x)^{1+\alpha} + 61\mu^{\alpha}_{0} (3-x)^{\alpha} \end{aligned} \tag{28}$$

The numerical values of $\left. I^{\alpha}_{-1^+} y(x) \right|_{x=3}$ and $\left. I^{\alpha}_{3^-} y(x) \right|_{x=-1}$ obtained for two derived methods for different values of α and N are calculated and investigated. In Table 1, the numerical errors err_N (21) and the EOC (22) for the sets of α ∈ {0.4, 0.7, 1.0, 1.4, 1.7, 2.0, 2.4} and N = 100, 200, 400, 800, 1600, 3200, 6400 are shown. The numerical errors err_N are determined on the basis of the analytical values:

$$I_{-1^+}^{0.4} y(x)\Big|_{x=3} = 130.3665287985740172 \quad I_{3^-}^{0.4} y(x)\Big|_{x=-1} = 123.0668888183057899$$
$$I_{-1^+}^{0.7} y(x)\Big|_{x=3} = 191.1396208343589412 \quad I_{3^-}^{0.7} y(x)\Big|_{x=-1} = 190.1213105630270004$$
$$I_{-1^+}^{1.0} y(x)\Big|_{x=3} = 262.6666666666666667 \quad I_{3^-}^{1.0} y(x)\Big|_{x=-1} = 262.6666666666666667$$
$$I_{-1^+}^{1.4} y(x)\Big|_{x=3} = 372.7230746639883467 \quad I_{3^-}^{1.4} y(x)\Big|_{x=-1} = 362.1596030472979745$$
$$I_{-1^+}^{1.7} y(x)\Big|_{x=3} = 460.9207653841978629 \quad I_{3^-}^{1.7} y(x)\Big|_{x=-1} = 434.9813917147052662$$
$$I_{-1^+}^{2.0} y(x)\Big|_{x=3} = 547.5301587301587302 \quad I_{3^-}^{2.0} y(x)\Big|_{x=-1} = 503.1365079365079365$$
$$I_{-1^+}^{2.4} y(x)\Big|_{x=3} = 650.0634681532517990 \quad I_{3^-}^{2.4} y(x)\Big|_{x=-1} = 582.1439742715791388$$

It should be noted that $I_{-1^+}^{1.0} y(x)\Big|_{x=3} = I_{3^-}^{1.0} y(x)\Big|_{x=-1}$, which corresponds to the classical integration of function on the same interval [a,b].

Example 2

Here, the nonlinear integrand function of the form

$$y(x) = 2\exp\left(-\frac{3}{2}x\right) + \frac{5}{x^{1+\alpha}}\exp\left(-\frac{1}{x}\right) + \sqrt{x}I_1\left(\frac{1}{2}\sqrt{x}\right) + 1 \tag{29}$$

is considered. The function $I_1(.)$ is the modified Bessel function of the first kind of order 1. The approximated values of the left-sided fractional integral $I_{0^+}^{\alpha} y(x)\Big|_{x=1.5}$ are calculated. For $x = a \to 0$, the second term of (29) takes the value 0.

We refer to the following properties [1, 23] and we recall

$$\begin{aligned}
&I_{0^+}^{\alpha} \exp(cx) = x^{\alpha} E_{1,1+\alpha}(cx) \\
&I_{0^+}^{\alpha} \left(\frac{1}{x^{1+\alpha}}\exp\left(-\frac{c}{x}\right)\right) = \frac{1}{c^{\alpha} x^{1-\alpha}}\exp\left(-\frac{c}{x}\right) \\
&I_{0^+}^{\alpha} \left(\sqrt{x}\, I_1\left(c\sqrt{x}\right)\right) = \frac{2^{\alpha}}{c^{\alpha}} x^{\frac{\alpha+1}{2}} I_{1+\alpha}\left(c\sqrt{x}\right) \\
&I_{0^+}^{\alpha} 1 = \frac{x^{\alpha}}{\Gamma(1+\alpha)}
\end{aligned} \tag{30}$$

where $E_{1,1+\alpha}(.)$ is a two-parameter Mittag-Leffler function, and c is any constant.

Table 1. Results related to Example 1

α	N	left-sided fractional integral				right-sided fractional integral			
		linear spline		Akima spline		linear spline		Akima spline	
		err_N	EOC	err_N	EOC	err_N	EOC	err_N	EOC
0.4	100	4.093e-02	–	6.873e-03	–	–9.281e-04	–	1.712e-04	–
	200	1.050e-02	1.963	1.288e-03	2.416	–2.004e-04	2.211	4.526e-05	1.919
	400	2.676e-03	1.972	2.424e-04	2.410	–4.582e-05	2.129	9.887e-06	2.195
	800	6.790e-04	1.979	4.574e-05	2.406	–1.083e-05	2.081	2.026e-06	2.287
	1600	1.716e-04	1.984	8.646e-06	2.403	–2.608e-06	2.054	3.774e-07	2.425
	3200	4.326e-05	1.988	1.636e-06	2.402	–6.349e-07	2.038	7.100e-08	2.410
	6400	1.088e-05	1.991	3.097e-07	2.401	–1.557e-07	2.028	1.335e-08	2.411
0.7	100	3.305e-02	–	3.006e-03	–	7.901e-03	–	5.115e-04	–
	200	8.308e-03	1.992	4.591e-04	2.711	1.984e-03	1.994	7.210e-05	2.827
	400	2.085e-03	1.995	7.024e-05	2.709	4.968e-04	1.997	1.003e-05	2.846
	800	5.223e-04	1.997	1.076e-05	2.706	1.243e-04	1.999	1.391e-06	2.850
	1600	1.308e-04	1.998	1.652e-06	2.704	3.109e-05	1.999	1.888e-07	2.881
	3200	3.272e-05	1.999	2.538e-07	2.703	7.773e-06	2.000	2.584e-08	2.869
	6400	8.183e-06	1.999	3.901e-08	2.702	1.943e-06	2.000	3.559e-09	2.860
1.0	100	2.133e-02	–	1.048e-03	–	2.133e-02	–	1.048e-03	–
	200	5.333e-03	2.000	1.322e-04	2.987	5.333e-03	2.000	1.322e-04	2.987
	400	1.333e-03	2.000	1.659e-05	2.994	1.333e-03	2.000	1.659e-05	2.994
	800	3.333e-04	2.000	2.079e-06	2.997	3.333e-04	2.000	2.079e-06	2.997
	1600	8.333e-05	2.000	2.601e-07	2.998	8.333e-05	2.000	2.601e-07	2.998
	3200	2.083e-05	2.000	3.253e-08	2.999	2.083e-05	2.000	3.253e-08	2.999
	6400	5.208e-06	2.000	4.068e-09	3.000	5.208e-06	2.000	4.068e-09	3.000
1.4	100	8.939e-03	–	2.097e-04	–	4.385e-02	–	2.072e-03	–
	200	2.234e-03	2.001	2.471e-05	3.085	1.096e-02	2.000	2.561e-04	3.017
	400	5.583e-04	2.000	2.825e-06	3.129	2.740e-03	2.000	3.176e-05	3.011
	800	1.396e-04	2.000	3.187e-07	3.148	6.850e-04	2.000	3.951e-06	3.007
	1600	3.489e-05	2.000	3.592e-08	3.150	1.712e-04	2.000	4.926e-07	3.004
	3200	8.722e-06	2.000	4.072e-09	3.141	4.281e-05	2.000	6.147e-08	3.002
	6400	2.181e-06	2.000	4.664e-10	3.126	1.070e-05	2.000	7.677e-09	3.001
1.7	100	3.239e-03	–	4.416e-05	–	6.353e-02	–	3.062e-03	–
	200	8.113e-04	1.997	1.091e-05	2.017	1.588e-02	2.000	3.779e-04	3.019
	400	2.029e-04	1.999	1.700e-06	2.683	3.970e-03	2.000	4.687e-05	3.011
	800	5.073e-05	2.000	2.318e-07	2.875	9.925e-04	2.000	5.835e-06	3.006
	1600	1.268e-05	2.000	2.998e-08	2.951	2.481e-04	2.000	7.278e-07	3.003
	3200	3.171e-06	2.000	3.795e-09	2.982	6.203e-05	2.000	9.087e-08	3.002
	6400	7.927e-07	2.000	4.758e-10	2.995	1.551e-05	2.000	1.135e-08	3.001
2.0	100	–1.773e–05	–	–1.552e-05	–	8.534e-02	–	4.207e-03	–
	200	–1.109e-06	3.999	9.195e-06	0.755	2.133e-02	2.000	5.195e-04	3.018
	400	–6.933e-08	4.000	1.892e-06	2.281	5.333e-03	2.000	6.448e-05	3.010
	800	–4.333e-09	4.000	2.845e-07	2.734	1.333e-03	2.000	8.031e-06	3.005
	1600	–2.708e-10	4.000	3.859e-08	2.882	3.333e-04	2.000	1.002e-06	3.003
	3200	–1.693e-11	4.000	5.015e-09	2.944	8.333e-05	2.000	1.251e-07	3.001
	6400	–1.058e-12	4.000	6.390e-10	2.973	2.083e-05	2.000	1.563e-08	3.001
2.4	100	–2.347e-03	–	–5.417e-05	–	1.169e-01	–	5.880e-03	–
	200	–5.815e-04	2.013	1.083e-05	2.323	2.923e-02	2.000	7.268e-04	3.016
	400	–1.450e-04	2.003	2.523e-06	2.102	7.307e-03	2.000	9.030e-05	3.009
	800	–3.624e-05	2.001	3.907e-07	2.691	1.827e-03	2.000	1.125e-05	3.005
	1600	–9.059e-06	2.000	5.359e-08	2.866	4.567e-04	2.000	1.404e-06	3.002
	3200	–2.265e-06	2.000	6.998e-09	2.937	1.142e-04	2.000	1.754e-07	3.001
	6400	–5.662e-07	2.000	8.936e-10	2.969	2.854e-05	2.000	2.191e-08	3.001

Table 2. Results related to Example 2

α	N	linear spline		Akima spline	
		err_N	EOC	err_N	EOC
0.4	100	–1.589e-05	–	–1.682e-06	–
	200	–4.033e-06	1.978	–3.046e-07	2.465
	400	–1.020e-06	1.984	–5.590e-08	2.446
	800	–2.571e-07	1.988	–1.036e-08	2.431
	1600	–6.469e-08	1.991	–1.935e-09	2.421
	3200	–1.625e-08	1.993	–3.631e-10	2.414
	6400	–4.078e-09	1.995	–6.835e-11	2.409
0.7	100	–2.470e-05	–	–1.138e-06	–
	200	–6.187e-06	1.997	–1.631e-07	2.803
	400	–1.549e-06	1.998	–2.355e-08	2.792
	800	–3.874e-07	1.999	–3.428e-09	2.780
	1600	–9.690e-08	1.999	–5.031e-10	2.769
	3200	–2.423e-08	2.000	–7.435e-11	2.758
	6400	–6.059e-09	2.000	–1.106e-11	2.749
1.0	100	–3.176e-05	–	–8.750e-07	–
	200	–7.939e-06	2.000	–1.096e-07	2.997
	400	–1.985e-06	2.000	–1.371e-08	2.999
	800	–4.962e-07	2.000	–1.715e-09	2.999
	1600	–1.240e-07	2.000	–2.144e-10	3.000
	3200	–3.101e-08	2.000	–2.680e-11	3.000
	6400	–7.753e-09	2.000	–3.350e-12	3.000
1.4	100	–4.086e-05	–	–9.016e-07	–
	200	–1.021e-05	2.000	–1.100e-07	3.035
	400	–2.554e-06	2.000	–1.355e-08	3.022
	800	–6.384e-07	2.000	–1.681e-09	3.011
	1600	–1.596e-07	2.000	–2.090e-10	3.008
	3200	–3.990e-08	2.000	–2.602e-11	3.006
	6400	–9.975e-09	2.000	–3.242e-12	3.004
1.7	100	–5.030e-05	–	–9.840e-07	–
	200	–1.257e-05	2.000	–1.194e-07	3.043
	400	–3.143e-06	2.000	–1.467e-08	3.024
	800	–7.859e-07	2.000	–1.819e-09	3.012
	1600	–1.965e-07	2.000	–2.270e-10	3.003
	3200	–4.912e-08	2.000	–2.835e-11	3.001
	6400	–1.228e-08	2.000	–3.543e-12	3.000
2.0	100	–6.542e-05	–	–1.113e-06	–
	200	–1.636e-05	2.000	–1.272e-07	3.129
	400	–4.089e-06	2.000	–1.523e-08	3.063
	800	–1.022e-06	2.000	–1.867e-09	3.028
	1600	–2.556e-07	2.000	–2.327e-10	3.004
	3200	–6.389e-08	2.000	–2.907e-11	3.001
	6400	–1.597e-08	2.000	–3.633e-12	3.000
2.4	100	–1.012e-04	–	–1.529e-06	–
	200	–2.531e-05	2.000	–1.402e-07	3.447
	400	–6.327e-06	2.000	–1.478e-08	3.245
	800	–1.582e-06	2.000	–1.790e-09	3.046
	1600	–3.955e-07	2.000	–2.212e-10	3.016
	3200	–9.887e-08	2.000	–2.755e-11	3.005
	6400	–2.472e-08	2.000	–3.440e-12	3.001

The analytic form of the left-sided fractional integral of function (29) is as follows

$$I^{\alpha}_{0^+} y(x) = 2x^{\alpha} E_{1,1+\alpha}\left(-\frac{3}{2}x\right) + \frac{5}{x^{1-\alpha}} \exp\left(-\frac{1}{x}\right) + 4^{\alpha} x^{\frac{\alpha+1}{2}} I_{1+\alpha}\left(\frac{\sqrt{x}}{2}\right) + \frac{x^{\alpha}}{\Gamma(1+\alpha)} \quad (31)$$

and for the sets of α ∈ {0.4, 0.7, 1.0, 1.4, 1.7, 2.0, 2.4} for $x = 1.5$, the analytical values are equal to

$$I^{0.4}_{0^+} y(x)\Big|_{x=1.5} = 4.388047972361219323, \quad I^{0.7}_{0^+} y(x)\Big|_{x=1.5} = 5.050230318186603821$$
$$I^{1.0}_{0^+} y(x)\Big|_{x=1.5} = 5.550029334658129390, \quad I^{1.4}_{0^+} y(x)\Big|_{x=1.5} = 5.980928744710674248$$
$$I^{1.7}_{0^+} y(x)\Big|_{x=1.5} = 6.177683964194226564, \quad I^{2.0}_{0^+} y(x)\Big|_{x=1.5} = 6.324379662424091572$$
$$I^{2.4}_{0^+} y(x)\Big|_{x=1.5} = 6.534153710966576377$$

In Table 2, the errors err_N and the EOC values for numerical values of $I^{\alpha}_{0^+} y(x)\Big|_{x=1.5}$ obtained on the grid sizes N = 100, 200, 400, 800, 1600, 3200, 6400 for the above the sets of α are presented.

5. Conclusion

The numerical integration formulas for calculation of the left- and right-sided Riemann-Liouville fractional integrals, that are based on the interpolation of the integrand function using the Akima cubic spline, have been derived. Also, the integration method using the linear spline has been added to this work, in order to compare the obtained computational results by the Akima cubic splines method with this method.

Analysis of the results presented in tables: the numerical errors tend to zero as the grid size N increases for each derived numerical integration method; the calculated numerical values of the fractional integrals have good agreement with the known analytical solutions. It can be noticed that as the number N increases, the EOC values stabilize and reach the specified constant values. For the method that uses the linear spline, the EOC = 2 for α > 0 is obtained. Whereas, for the Akima spline, the values of the EOC are as follows: EOC = 2 + α for α < 1 and EOC = 3 for α ≥ 1. Hence, it can be concluded that the method using the linear spline gives worse results than the method using the Akima cubic spline.

Usually, for many kinds of cubic splines, it is necessary to solve a system of linear equations to determine the spline coefficients and from a computational point of view, it can be computationally time consuming. This problem does not occur in the case of the Akima cubic spline (and for the linear spline, of course) where the coefficients of spline segments are determined locally. Additionally, the Akima interpolation algorithm can be easily parallelized.

In conclusion, the obtained numerical results for the derived integration method that uses the Akima cubic spline gives higher values of the EOC than for the method that uses the linear spline. I think that the efficiency and the applicability of the derived numerical algorithm for the fractional integration has been confirmed. In the future, I plan to apply this method to the fractional derivatives and improve the obtained numerical methods through more accurate interpolations of the integrand function.

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