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STABILITY ANALYSIS AND NUMERICAL IMPLEMENTATION OF THE THIRD-ORDER FRACTIONAL PARTIAL DIFFERENTIAL EQUATION BASED ON THE CAPUTO FRACTIONAL DERIVATIVE

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Abstract. This paper examines a third-order fractional partial differential equation (FPDE) in the Caputo sense. The Theta difference method (TDM) is utilized to investigate the problem, and a first-order difference scheme is developed. Stability estimates are obtained by applying the Von Neumann analysis method. A test problem is presented as an application, and numerical results are obtained using Matlab software. Error estimates, as well as exact and approximate solutions are presented in a data analysis table. The simulation results are shown through error analysis tables and figures.

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1. Introduction

Fractional Analysis (FA) is a field of mathematics that extends classical calculus by dealing with non-integer fractional derivative and integration operations. The expression of fractional derivative operators was developed simultaneously with classical ones, and the theory of fractional calculus has rapidly evolved since the 19th century, encompassing fractional differential equations, fractional dynamics, and fractional geometry. Presently, FA has a wide range of applications across various fields of modern engineering and science, including electrical, mechanical, bioengineering, viscoelasticity, control theory, optics, chemical physics, and more. It is generally accepted that fractional calculus tools and procedures are used in almost every area of modern engineering and science. Fractional-order systems are considered to be more descriptive of real-world processes than their integer-order counterparts. These novel fractional-order models are a crucial factor in the success of fractional calculus applications, as they are often more accurate and have more degrees of freedom than the corresponding classical models [1-7].

As finding the exact solution for fractional partial differential equations is rare in literature, approximations and numerical methods have been developed by researchers. Various numerical and analytical methods have been proposed for solving these classifications of equations. For instance, the authors in [8] used the explicit finite difference method to obtain an approximate solution for the fractional order pseudo-hyperbolic partial differential equation. In [9], a novel hybrid strategy based on the Laplace transform and optimal decomposition method was presented to develop an approximate solution for a nonlinear system of fractional partial differential equations in the Caputo sense. The conformable fractional power series approach was proposed by the authors in [10] to obtain numerical solutions for a coupled system of nonlinear fractional partial differential equations. Additionally, the authors in [11] applied the Crank-Nicholson difference technique to produce approximate solutions for the mobile-immobile advection-dispersion model based on the Caputo and Atangana-Baleanu Caputo fractional derivatives. The replicating kernel approach was developed by the authors in [12] to handle classes of time-fractional partial differential equations. Results on the numerical simulation for time-fractional partial differential equations emerging in transonic multiphase flows, which are described by the Tricomi and Keldysh equations of Robin function types, were considered by the authors in [13]. The Volterra integrodifferential equation of order in the Atangana-Beleanu-Caputo (ABC) sense was studied by the authors in [14] in both linear and nonlinear forms. Finally, the Laplace residual power series approach was introduced in [15] to provide the approximate solutions of the fractional partial differential equations in the Caputo sense. This study aims to investigate the theta difference method for a third-order partial differential equation with a Caputo derivative of fractional order. The study also includes stability analyses of the proposed problem using the Von-Neumann approach.

To achieve this objective, we will focus on the following fractional partial differential equation, utilizing the Caputo fractional derivative:

$$\begin{cases} z_{ttt}(t,x) + z_{tt}(t,x) + {}_0^C D_t^\alpha z(t,x) + z(t,x) = z_{xx}(t,x) + f(t,x), \\ \qquad 0 < x < L,\ 0 < t < T,\ \ 0 \le \alpha < 1, \\ z(0,x) = \sigma_1(x),\ \ z_t(0,x) = \sigma_2(x),\ \ z_{tt}(0,x) = \sigma_3(x),\ \ 0 \le t \le T, \\ \qquad z(t,0) = z(t,L) = 0,\ \ 0 \le x \le L. \end{cases} \tag{1}$$

Here, $\sigma_1(x), \sigma_2(x), \sigma_3(x)$ and $f(t,x)$ are known functions, and $z(t,x)$ is an unknown function, $L, T > 0$. Additionally, ${}_0^C D_t^\alpha z(t,x)$ is the Caputo fractional derivative and is defined in [16], as

$$\frac{D^\alpha z(t,x)}{\partial t^\alpha} = D_t^\alpha z(t,x) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{n-\beta-1}} \frac{\partial^n w(p,x)}{\partial p^n} dp,\ \ (n-1 < \beta < n), \tag{2}$$

here $\Gamma(.)$ denotes the Gamma function, and for $\alpha = n \in N$, it is defined as:

$$D_t^{\alpha} z(t,x) = \frac{D^{\alpha} z(t,x)}{\partial t^{\alpha}} = \frac{D^{n} z(t,x)}{\partial t^{n}} \tag{3}$$

The Caputo fractional derivative is frequently referenced in the literature on fractional calculus since it is a logical extension of the classical result and has several advantageous properties. The Caputo derivative, a fractional derivative, first considers the history of a function. As implied by the name of the Caputo derivative, it may thus capture the long-term behavior of a system and has a memory effect. Second, it is known that the value of the Caputo derivative for constant functions is zero. This condition is essential because it ensures the zeroness of the result of a constant signal, which is a desired property in many applications.

Therefore, in the proposed model (1), the Caputo fractional derivative [16], is used because it is a natural extension of the classical derivative with a memory effect, and it has the desirable property of having a well-defined derivative for constant functions. The fractional derivative has been used to model several real-world events [17, 18].

The Theta difference method was first proposed in [19], and then Modanli and Akgul implemented it for solving fractional order differential equations in [20]. The theta difference method is the most general when compared to finite difference methods. Specifically, this method is called the forward (Implicit) difference scheme for $\theta = 0$, the Crank-Nicholson difference scheme method for $\theta = 1/2$, and the backward (Explicit) difference scheme for $\theta = 1$. Therefore, instead of using one of the other finite difference methods, applying the Theta difference method and also one could analyze all three other above-mentioned techniques in this paper makes this study different from previous studies.

The structure of this work is as follows: in the 2nd section, we constructed a finite difference scheme and provided its stability for the given problem; in the 3rd section, the numerical result is provided; and the final section summarizes the findings of our investigation.

2. Theta finite difference method and stability analysis

In this section, we will develop a finite difference method using the theta difference scheme for the proposed model (1) and analyze its stability. To achieve this, we will use a rectangular domain of $w^h \times w^\tau$, which will be divided into a grid mesh with intervals that $h = \frac{r}{M}$ and $\tau = \frac{v}{N}$, for the x-axis and t-axis, respectively. And $x_n = nh, n = 1,2, \dots, M,\ t_k = k\tau, k = 1,2, \dots, N$.

At the points $(t_k, x_n) \in w^h \times w^\tau$, we can write the model (1) as

$$z_{ttt}(t_k, x_n) + z_{tt}(t_k, x_n) + {}_0^C D_t^{\alpha} z(t_k, x_n) + z(t_k, x_n) = z_{xx}(t_k, x_n) + f(t_k, x_n). \tag{4}$$

We will now create a technique for the first-order difference scheme for equation (4) using the following formulas:

The first-order difference technique for the Caputo derivative ${}_0^C D_t^\alpha z(t_k, x_n)$ is defined in [21], as

$$ {}_0^C D_t^\alpha z(t_k, x_n) \cong s_{\alpha,\tau} \sum_{j=0}^{k} \psi_j{}^{(\alpha)} (z_n^{k-j+1} - z_n^{k-j}), \tag{5} $$

Here, $\psi_j{}^{(\alpha)} = (j+1)^{1-\alpha} - j^{1-\alpha}$, $s_{\alpha,\tau} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}$.

To create a technique for the remaining parts of equation (4), we will utilize the Taylor expansion series concerning t and x, resulting in a first-order difference method, as follows

$$ z_{txx}(t_k, x_n) = \frac{1}{\tau}\left(\frac{z_{n+1}^{k} - 2z_n^{k+1} + z_{n-1}^{k}}{h^2} - \frac{z_{n+1}^{k-1} - 2z_n^{k} + z_{n-1}^{k-1}}{h^2} \right), \tag{6} $$

$$ z_{xx}(t_k, x_n) \cong \frac{1}{h^2}\left\{ \left[\theta [z_{n+1}^{k+1} - 2z_n^{k+1} + z_{n-1}^{k+1}\right] + (1-\theta)\left[z_{n+1}^{k} - 2z_n^{k} + z_{n-1}^{k}\right] \right\} \tag{7} $$

Since the finite difference of the formula (7) depends on θ, then it is known as the theta difference scheme method. And according to [21], with respect of t, we can write the difference scheme formulas for other parts, as follows

$$ z_{tt}(t_k, x_n) \cong \frac{z_n^{k+1} - 2z_n^{k} + z_n^{k-1}}{\tau^2}, \tag{8} $$

$$ z_{ttt}(t_k, x_n) \cong \frac{z_n^{k+2} - 3z_n^{k+1} + 3z_n^{k} - z_n^{k-1}}{\tau^3} \tag{9} $$

The equations (5)-(9) can be utilized to create a method of difference scheme for the model (1), which can be expressed as follows:

$$ \begin{cases} \frac{z_n^{k+2} - 3z_n^{k+1} + 3z_n^{k} - z_n^{k-1}}{\tau^3} + \frac{z_n^{k+1} - 2z_n^{k} + z_n^{k-1}}{\tau^2} + s_{\alpha,\tau} \sum_{j=0}^{k} \psi_j{}^{(\alpha)} (z_n^{k-j+1} - z_n^{k-j}) \\ \quad -\frac{1}{h^2}\left\{ \left[\theta [z_{n+1}^{k+1} - 2z_n^{k+1} + z_{n-1}^{k+1}\right] + (1-\theta)\left[z_{n+1}^{k} - 2z_n^{k} + z_{n-1}^{k}\right] \right\} \\ \quad +\theta z_n^{k+1} + (1-\theta) z_n^{k} = \theta f_n^{k+1} + (1-\theta) f_n^{k}, \\ z_n^0 = \sigma_1(x_n), \frac{z_n^1 - z_n^0}{\tau} = \sigma_2(x_n), \frac{z_n^2 - 2z_n^1 + z_n^0}{\tau^2} = \sigma_3(x_n), z_0^k = z_M^k = 0, 0 \le k \le N. \end{cases} \tag{10} $$

Theorem 2.1. The formula (10) is satisfies the stability estimates.

Proof. We will use the Von-Neumann analysis method to showcase the stability estimates for formula (10), which are presented below

$$ z_n^k = \lambda^k e^{in\varphi}, \tag{11} $$

using the initial conditions ($f_0^1, f_0^2 \to 0$), and writing the formula (10) according to the formula (11), we get the following formula

$$\begin{cases} \frac{\lambda^{k+2}e^{in\varphi}-3\lambda^{k+1}e^{in\varphi}+3\lambda^{k}e^{in\varphi}-\lambda^{k-1}e^{in\varphi}}{\tau^3}+\frac{\lambda^{k+1}e^{in\varphi}-2\lambda^{k}e^{in\varphi}+\lambda^{k-1}e^{in\varphi}}{\tau^2} \\ +s_{\alpha,\tau}\sum_{j=0}^{k}\psi_j{}^{(\alpha)}\left(\lambda^{k-j+1}e^{in\varphi}-\lambda^{k-j}e^{in\varphi}\right) \\ -\frac{1}{h^2}\{\theta\lambda^{k+1}\left(e^{i(n+1)\varphi}-2e^{in\varphi}+e^{i(n-1)\varphi}\right) \\ +\theta\lambda^{k+1}e^{in\varphi}+(1-\theta)\lambda^{k}e^{in\varphi} \\ +(1-\theta)\lambda^{k}\left(\left(e^{i(n+1)\varphi}-2e^{in\varphi}+e^{i(n-1)\varphi}\right)\right\}=f_n^k. \end{cases} \tag{12}$$

Taking $k = 1$, and $n = 0$, we then obtain

$$\begin{cases} \frac{\lambda^3-3\lambda^2+3\lambda-1}{\tau^3}+\frac{\lambda^2-2\lambda+1}{\tau^2}+s_{\alpha,\tau}[(\lambda-1)(2^{1-\alpha}-1) \\ +(\lambda^2-\lambda)]-\frac{\theta\lambda^2}{h^2}\left(e^{i\varphi}-2+e^{-i\varphi}\right)+\theta\lambda^2 \\ +(1-\theta)\lambda-\frac{(1-\theta)\lambda}{h^2}\left(e^{i\varphi}-2+e^{-i\varphi}\right)=0 \end{cases} \tag{13}$$

We can rewrite the formula (13) as follows

$$P(\lambda)=\lambda^3+b\lambda^2+c\lambda+d. \tag{14}$$

Here,

$$\begin{cases} a=1, \\ b=\tau^3\left(-\frac{3}{\tau^3}+\frac{1}{\tau^2}+s_{\alpha,\tau}+4\,\frac{\theta}{h^2}sin^2\frac{\varphi}{2}+\theta\right), \\ c=\tau^3\left(\frac{3}{\tau^3}-\frac{2}{\tau^2}+2(2^{-\alpha}-1)s_{\alpha,\tau}+4\,\frac{1-\theta}{h^2}sin^2\frac{\varphi}{2}+1-\theta\right), \\ d=\tau^3\left(-\frac{1}{\tau^3}+\frac{1}{\tau^2}-(2^{1-\alpha}-1)s_{\alpha,\tau}\right). \end{cases} \tag{15}$$

From the formulas (15), taking as following:

$$p=\sqrt{27d^2+4b^3d-18bdc+4c^3-(bc)^2},$$

$$s=\frac{p}{6\sqrt{3}}-\frac{27d-9bc+2b^3}{54},$$

we have the following roots of the equation (14)

$$\lambda_1=\sqrt[3]{s}+\frac{b^2-3c}{9\sqrt{s}}-\frac{b}{3} \tag{16}$$

$$\lambda_2=r_1\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \tag{17}$$

$$\lambda_3 = r_1\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right). \tag{18}$$

If the norm of the equations (16), (17) and (18) is taken, it seen that

$$|\lambda_1| = |\lambda_2| = |\lambda_3|.$$

To prove Theorem 2.1, we used a Matlab program to compute equation (16) and obtain the maximum values of the equation's roots. The resulting table provides an estimation of stability.

Table 1. Results of stability analysis

$\tau, h, \varphi, \alpha, \theta$	$\lvert\lambda_1\rvert = \lvert\lambda_2\rvert = \lvert\lambda_3\rvert$
$\tau = h = 0.5, \varphi = \pi, \alpha = 1, \theta = 0$	0.1619
$\tau = h = 0.5, \varphi = \pi, \alpha = 0, \theta = 0$	0.2132
$\tau = h = 0.5, \varphi = \pi, \alpha = 1, \theta = 0.5$	0.0586
$\tau = h = 0.5, \varphi = \pi, \alpha = 0, \theta = 0.5$	0.0030
$\tau = h = 0.5, \varphi = \pi, \alpha = 1, \theta = 1$	0.1619
$\tau = h = 0.5, \varphi = \pi, \alpha = 0, \theta = 1$	0.2132

The results obtained in Table 1 are calculated for the maximum of τ, h and φ values and all possible values of θ. Although α is in the range $0 < \alpha \leq 1$, it is calculated for both 0 and 1 values to test the roots of equation (16). Since the obtained roots of the equation (16) for these two values are still less than 1, it is seen that $|\lambda_1| \leq 1, |\lambda_2| \leq 1, |\lambda_3| \leq 1$ are provided. Therefore, since the finite difference scheme equation (10) is stable for all values, it can be said that it is unconditionally stable.

3. Numerical implementation

In this section, we aim to obtain an approximate solution for a third-order fractional partial differential equation that employs the Caputo derivative. To achieve this, we will use the Theta difference approach and apply a first-order accuracy difference scheme. We will then use a Matlab program to conduct various simulations for different values of N and M.

Now we will give the algorithm of the solution:

Algorithm

Step 1. Input time increment $\tau = \frac{v}{N}$ and space increment $h = \frac{r}{M}$, α fractional order, and θ.

Step 2. Use the first-order of accuracy difference scheme and then write the equation (10) in matrices form, as

$$AZ_{n+1} + BZ_n + CZ_{n-1} = D\varphi_n, \;\; 1 \leq n \leq M - 1.$$

Step 3. Determine the entries of the matrices A, B, C and D.
Step 4. Find α_1, β_1.
Step 5. Compute $\alpha_{n+1}, \beta_{n+1}$.
Step 6. Compute $Z_{n-s}(n = M-1, \dots, 2.1)$, $M_n = 0$, using the following formula

$$Z_n = \alpha_{n+1} Z_{n+1} + \beta_{n+1}.$$

Example 3.1. We investigate the following third order fractional partial differential equation based on Caputo derivative, as follows

$$\begin{cases} z_{ttt}(t,x) + z_{tt}(t,x) + {}_0^C D_t^\alpha z(t,x) + z(t,x) = z_{xx}(t,x) + f(t,x), \\ f(t,x) = (6t + 4 + 6\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - 2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + (\pi^2+1) \\ \times (t^3 - t^2 + t - 1))sin\pi x, 0 < x < 1, 0 < t < 1, 0 < \alpha \leq 1, \\ z(0,x) = -sin\pi x, z_t(0,x) = sin\pi x, z_{tt}(0,x) = -2sin\pi x, 0 \leq t \leq 1, \\ z(t,0) = z(t,\pi) = 0, \ \ 0 \leq x \leq \pi. \end{cases} \quad (19)$$

By utilizing the Laplace transform method, we successfully identified the exact solution for equation (17) as $(z(t,x) = (t^3 - t^2 + t - 1)sin\pi x)$. We can now utilize the following formula to calculate the maximum norm of error estimates for numerical analysis using the modified Gauss elimination technique for equation (10):

$$\varepsilon = \max_{\substack{n=0,1,\cdots,M \\ k=0,1,\cdots,N}} |z(x,t) - z(t_k, x_n)|,$$

the above formula calculates the error estimate (ε) between the exact solution $z(t,x)$ and numerical solutions $z(t_k, x_n)$.

Table 2. Approximate numerical results for equation (19)

N, M, α, θ	α, θ	Exact solutions	Approximate solution	(ε) Absolute error
$N = 100, M = 10$	$\alpha = 0.5, \theta = 0$	0	0.0242736667	0.0769416336
$N = 400, \ M = 20$	$\alpha = 0.5, \theta = 0$	0	0.0061143516	0.0614197650
$N = 900, \ M = 30$	$\alpha = 0.5, \theta = 0$	0	0.0027294978	0.0585788858
$N = 100, \ M = 10$	$\alpha = 0.5, \theta = 0.5$	0	0.0156509551	0.0669353393
$N = 400, \ M = 20$	$\alpha = 0.5, \theta = 0.5$	0	0.0039449046	0.0589269402
$N = 900, \ M = 30$	$\alpha = 0.5, \theta = 0.5$	0	0.0017638898	0.0574714334
$N = 100, \ M = 10$	$\alpha = 0.5, \theta = 1$	0	0.0068457815	0.0568273795
$N = 400, \ M = 20$	$\alpha = 0.5, \theta = 1$	0	0.0017636257	0.0564302376
$N = 900, \ M = 30$	$\alpha = 0.5, \theta = 1$	0	0.0079594244	0.0563633917

Table 2 provides the approximate numerical results for the proposed model (19), which was obtained using the theta difference approach for different values of each N, M, θ and α, when $0 < t < 1$, $0 < x < \pi$.

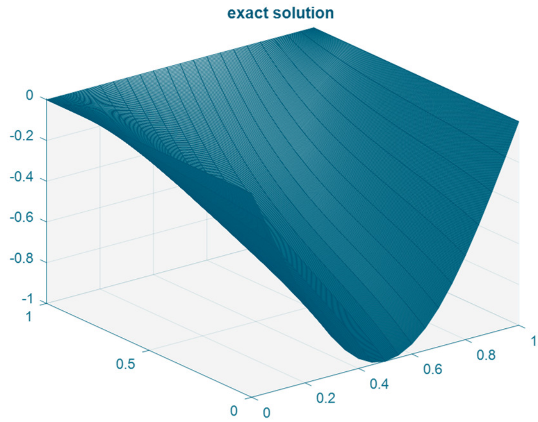

Fig. 1. Exact solution of problem (19), for $0 \le x \le 1, 0 \le t \le 1$

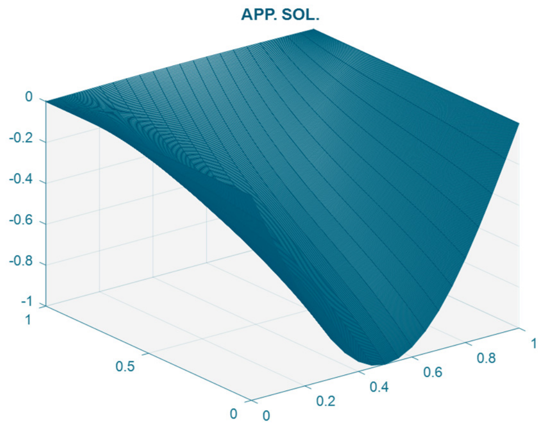

Fig. 2. Approximate solutions of problem (19), for $N = 400, M = 20$, $\alpha = 0.5, \theta = 0, 0 \le x \le 1$, and $0 \le t \le 1$

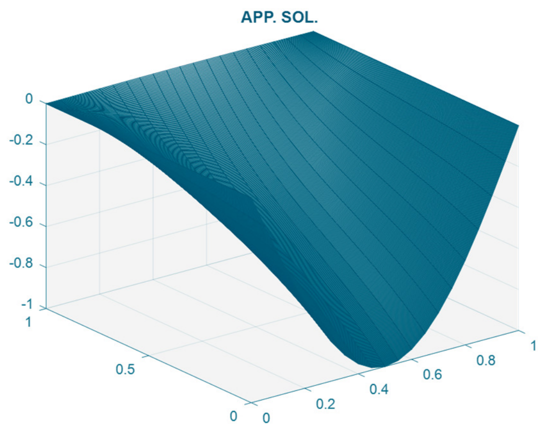

Fig. 3. Approximate solutions of problem (19), for $N = 400, M = 20$, $\alpha = 0.5, \theta = 0.5, 0 \le x \le 1$, and $0 \le t \le 1$

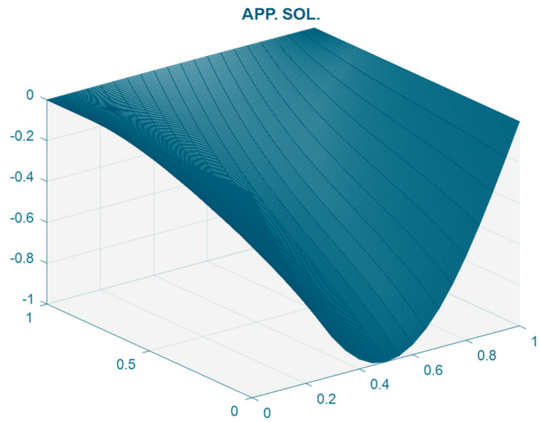

Fig. 4. Approximate solutions of problem (19), for $N = 400, M = 20$, $\alpha = 0.5, \theta = 1, 0 \leq x \leq 1$, and $0 \leq t \leq 1$

4. Conclusion

This study considered a third-order fractional partial differential equation based on the Caputo derivative. The research introduced a theta difference scheme method to solve this equation and established a finite difference scheme for the theta method. The study shows that the theta method that uses these difference schemes is stable and more general than the finite difference scheme. The Van-Newman technique provided stability estimates for the problem. To verify the effectiveness of the proposed method, a numerical implementation was performed using Matlab software. The results, including numerical findings and figures, are presented in this paper. The numerical approximate results of the proposed problem was presented in Table 2, and the Van-Newman method was provided for the theoretical stability analysis of the proposed problem and the stability results are given in Table 1. Finally, the exact and approximate solutions to the proposed problem were graphically presented in Figures 1-4 for different values.

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