Journal of Applied Mathematics and Computational Mechanics 2023, 22(3), 19-32
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2023.3.02 e-ISSN 2353-0588

AN ITERATIVE APPROACH FOR SOLVING FRACTIONAL ORDER CAUCHY REACTION-DIFFUSION EQUATIONS

Manoj Kumar

Department of Mathematics, National Defence Academy
Khadakwasla, Pune-411023, India
mkmath14@gmail.com

Received: 8 April 2023; Accepted: 31 August 2023

Abstract. Reaction-diffusion equations are vitally important due to their role in developing sturdy models in various scientific fields. In the present work, we address an algorithm of the Daftardar-Gejji and Jafari method for solving the nonlinear functional equations of the form $\psi = f + L(\psi) + N(\psi)$. Further, we employ this algorithm to solve Caputo derivative-based time-fractional Cauchy reaction-diffusion equations. We obtain solutions in a series form that converges to a closed form. Furthermore, we perform numerical simulations for the various values of the order of fractional derivatives. The computational procedure of the proposed algorithm is not burdensome. However, it is time-efficient and can easily be implemented using a computer algebra system.

MSC 2010: *34A08, 35A22, 35R10, 35R11*
Keywords: *Caputo derivative, Mittag-Leffler function, Cauchy reaction-diffusion equations, Daftardar-Gejji and Jafari method*

1. Introduction

Fractional partial differential equations (FPDEs) are an excellent tool to describe several natural phenomena. In recent decades, several models in various disciplines such as the simulation of an outbreak of dengue fever [1], newsvendor model [2], tumor growth model [3], economic growth model [4], stock model [5], modeling of viscoelastic materials [6], model for bike share systems [7], modeling of a lithium-ion battery [8], epidemic model of computer virus [9], quantum mechanical models [10], and so on have been proposed in terms of FPDEs. For a deeper understanding of the abstruse behavior and to draw conclusions from these equations for the underlying process/phenomena, we need their solutions, which is the most difficult part of the FPDEs. These equations involve fractional derivatives which are non-local in nature and depend on history. Various conventional and non-conventional methods such as finite difference methods [11], wavelet methods [12], meshless method [13], expansion techniques [14], homotopy asymptotic method [15], Adomian decomposition method (ADM) [16], Lie symmetric analysis [17], Daftardar Gejji and Jafari method

(DGJM) [18], variational iteration method [19], Kudryashov's method [20] and so on have been proposed in the past. Time-fractional Cauchy reaction-diffusion equations (CRDEs) are a type of partial differential equation that involve fractional derivatives in the time variable and describe the evolution of a system over time with reaction and diffusion processes. These equations are widely studied in various scientific fields due to their ability to model complex phenomena with memory effects and anomalous diffusion behavior. Various applications of CRDEs have been found in different fields. In biology and medicine, these equations are used to model the spreading of diseases, tumor growth, etc. CRDEs are employed to understand chemical reactions that exhibit complex kinetics and diffusion processes. In population dynamics, these equations are utilized to model the behavior of ecological and biological populations. In material science, CRDEs are expended for modeling transport phenomena in porous media whereas in finance and economics, these are used for modeling the financial markets with memory effects and long-range dependence [21]. Moreover, in the literature, CRDEs have been solved by various methods such as Laplace Adomian decomposition method [22], optimal homotopy asymptotic method [23], generalized differential transform and residual power series methods [24], homotopy perturbation method (HPM) [25], Sehu transform [26], Sumudu transform iterative method [27], fractional iteration algorithm [28], semi-analytical methods [29], homotopy analysis transform method [30], and so on.

During the literature review on time-fractional Cauchy reaction-diffusion equations, we found that DGJM is not being used for solving these equations. However, DGJM is free from discretization and does not involve any tedious computations. Thus in the present study, we implement DGJM's algorithm developed by Kumar et al. [31] to find the approximate solutions of the following form of time-fractional Cauchy reaction-diffusion equations:

$$\begin{aligned} \partial_t^{\mu}\psi(x,t) &= \delta\partial_x^2\psi(x,t) + k(x,t)\psi(x,t),\ 0 \le \mu \le 1, \\ \psi(x,0) &= \psi_0(x), (x,t) \in \Omega \subset \mathbb{R}^2, \end{aligned}$$

where the fractional derivative μ is considered in the Caputo sense, $\delta > 0$ the diffusion coefficient, ψ the concentration and k the reaction parameter. We give various illustrative examples and represent the solutions graphically. The proposed algorithm is time-efficient and does not include tedious computations as required in ADM and HPM. The presentation of this work is as follows: In Section 2, we introduce some basic definitions and notations. In Section 3, we present DGJM's algorithm for solving nonlinear functional equations. In Section 4, we solve various well-known CRDEs and hence demonstrate the applicability of the proposed algorithm. Finally, we summarize the results in Section 5.

2. Preliminaries

In this section, we present three useful definitions and one property.

Definition 2.1 [32] Time-fractional Riemann-Liouville integral of order $\mu > 0$, of a real-valued function $\psi(x,t)$ is defined as

$$I_t^{\mu}\psi(x,t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}\psi(x,s)ds.$$

Definition 2.2 [32] Time-fractional Caputo derivative operator of order $\mu > 0$, of a real-valued function $\psi(x,t)$ is defined as

$$\begin{aligned}\partial_t^{\mu}\psi(x,t) &= I_t^{n-\mu}\left[\partial_t^{n}\psi(x,t)\right],\\ &= \begin{cases} \dfrac{1}{\Gamma(n-\mu)}\displaystyle\int_0^t (t-s)^{n-\mu-1}\partial_s^{n}\psi(x,s)ds, & n-1<\mu<n,\\ \partial_t^{n}\psi(x,t), & \mu = n \in \mathbb{N}.\end{cases}\end{aligned}$$

Definition 2.3 [33] Mittag-Leffler function with one parameter μ is defined as

$$E_{\mu}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\mu k+1)}, \quad Re(\mu) > 0, z \in \mathbb{C}.$$

Theorem 2.1 [33] Let $\psi(x,t) \in C^n[0,T]$ and $n-1 < \mu < n, n \in N$ then

$$I_t^{\mu}\left(\partial_t^{\mu}\psi(x,t)\right) = \psi(x,t) - \sum_{k=0}^{n-1}\frac{\psi^k(x,0)}{k!}t^k, \ \ t > 0.$$

3. Daftardar Gejji and Jafari Method's Algorithm

Consider the following nonlinear functional equation:

$$\psi(x,t) = f + L(\psi(x,t)) + N(\psi(x,t)), \tag{1}$$

where f is a known function, L – a linear operator and N – a known nonlinear operator from a Banach space $B \to B$. For solving the Eq. (1), Daftardar-Gejji and Jafari introduced an iterative method [18], which has been widely used for solving a variety of equations in the literature successfully. Recently, Kumar et al. [31] proposed an algorithm for DGJM, which is discussed below. In DGJM, a solution of the Eq. (1)

is assumed as

$$\psi = \sum_{i=0}^{\infty} \psi_i, \tag{2}$$

where the iterative terms $\psi_i's$ are calculated as follows:

$$\psi_0 = S_0 = f, \tag{3}$$
$$\psi_1 = L(\psi_0) + N(S_0), \tag{4}$$
$$\psi_2 = L(\psi_1) + N(S_1) - N(S_0), \tag{5}$$
$$\psi_3 = L(\psi_2) + N(S_2) - N(S_1), \tag{6}$$
$$\vdots$$
$$\psi_n = L(\psi_{n-1}) + N(S_{n-1}) - N(S_{n-2}), \tag{7}$$

where $S_n = \psi_0 + \psi_1 + \cdots \psi_n,\ \ n = 0,1,2\cdots$. On adding the equations (3)-(7), we get

$$\psi_0 + \psi_1 + \cdots + \psi_n = f + L(\psi_0 + \psi_1 + \cdots + \cdots \psi_{n-1}) + N(S_{n-1}),$$

which is equivalent to

$$S_n = f + L(S_{n-1}) + N(S_{n-1}). \tag{8}$$

(Note that as $n \to \infty$, $S_n \to \psi$ *i.e.* $\lim_{n\to\infty} S_n = \psi$ and hence eqn (8) converges to (1)). Thus we get the following recursive formula for calculating $S_n's$:

$$\left.\begin{array}{l} S_0 = f, \\ S_n = S_0 + L(S_{n-1}) + N(S_{n-1}),\ \ n = 1,2,\cdots, \end{array}\right\} \tag{9}$$

The formula defined in (9) is an algorithm for DGJM. Note that we denote the $(n+1)$ term solution of (1) by S_n.

Remarks: Let N be a non-linear operator from a Banach space $B \to B$ such that $\|N(\psi)\| \leq k\|\psi\|, 0 < k < 1$. Then $\|S_{j+1} - S_j\| = \|N(S_j) - N(S_{j-1})\| = \|\psi_{j+1}\|$ $= \|L(\psi_j) + N(S_j) - N(S_{j-1})\| \leq \|L(\psi_j)\| + k\|S_j - S_{j-1}\| \leq \|L(\psi_j) + kL(\psi_{j-1})\| +$ $k^2\|S_{j-1} - S_{j-2}\| .... \leq \|L\Big(\sum_{m=0}^{j-i} k^m \psi_{j-m}\Big)\| + k^j\|S_1 - S_0\|$. In view of the Weierstrass test, S_n converges to the solution of (1).

4. Illustrative examples

In this section, we solve six examples of linear and nonlinear time-fractional Cauchy reaction-diffusion equations using the DGJM's algorithm defined in (9). Note that we use Mathematica 10.0 for doing calculations and graphical simulations.

Example 4.1 Consider the following time-fractional Cauchy reaction-diffusion equation

$$\partial_t^\mu \psi(x,t) = \partial_x^2 \psi(x,t) - \psi(x,t), \ \ 0 < \mu \leq 1, \tag{10}$$

$$\psi(x,0) = e^{-x} + x. \tag{11}$$

Integrating the Eq. (10), we get

$$\psi(x,t) = \psi(x,0) + I_t^\mu[\partial_x^2 \psi(x,t) - \psi(x,t)] = \psi(x,0) + L[\psi], \tag{12}$$

where $L(\psi) = I_t^\mu[\partial_x^2 \psi(x,t) - \psi(x,t)]$. Using the algorithm (9), we get

$$\begin{aligned}
S_0 &= \psi(x,0) = e^{-x} + x,\\
S_1 &= S_0 + L(S_0) = S_0 + I_t^\mu[\partial_x^2 S_0(x,t) - S_0(x,t)] = e^{-x} + x - \frac{xt^\mu}{\Gamma(\mu+1)},\\
S_2 &= S_0 + L(S_1) = S_0 + I_t^\mu[\partial_x^2 S_1(x,t) - S_1(x,t)]\\
&= e^{-x} + x - \frac{xt^\mu}{\Gamma(\mu+1)} + \frac{xt^{2\mu}}{\Gamma(2\mu+1)},\\
&\vdots\\
S_n &= e^{-x} + x - \frac{xt^\mu}{\Gamma(\mu+1)} + \frac{xt^{2\mu}}{\Gamma(2\mu+1)} + \cdots + (-1)^n \frac{xt^{n\mu}}{\Gamma(n\mu+1)}.
\end{aligned}$$

Hence as $n \to \infty$, S_n converges to the following closed-form solution

$$\psi(x,t) = \lim_{n\to\infty} S_n = e^{-x} + xE_\mu(-t^\mu). \tag{13}$$

For $\mu = 1$, the solution (13) turns to the exact solution of the classical Cauchy reaction-diffusion equation [23]. Moreover, we obtain the same result as obtained by the Laplace Adomian decomposition method in [22] and the Sumudu transform iterative method in [27]. The five-term approximate solutions of (10)-(11) are presented graphically in Figures 1 and 2. Moreover, the absolute errors for $x = 1$ and $\mu = 0.7, 0.8, 0.9, 1.0$ are computed in Table 1. We observe that as the value of the fractional derivative is increased, the absolute error is reduced.

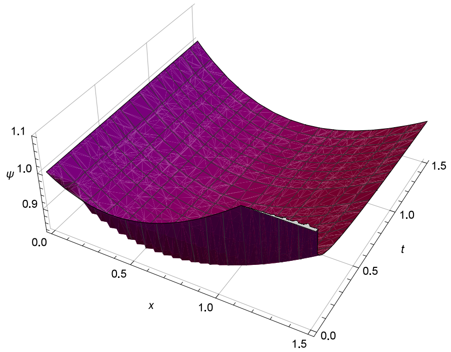
Example 4.2 Consider the following time-fractional Cauchy reaction-diffusion equation:

$$\partial_t^\mu \psi(x,t) = \partial_x^2 \psi(x,t) - (1+4x^2)\psi(x,t), \ \ 0 < \mu \leq 1, \tag{14}$$

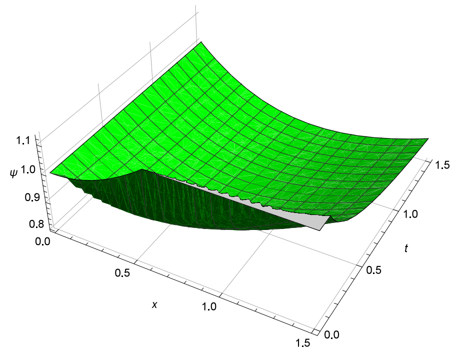
$$\psi(x,0) = e^{x^2}. \tag{15}$$

On integrating the Eq. (14), we get

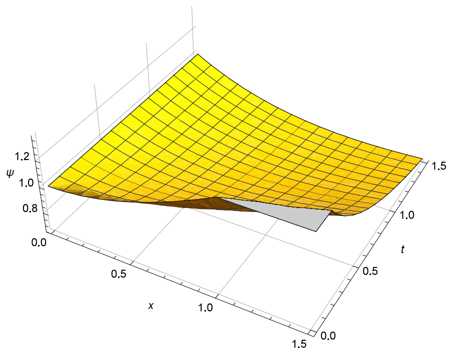
$$\psi(x,t) = \psi(x,0) + I_t^\mu[\partial_x^2 \psi(x,t) - (1+4x^2)\psi(x,t)] = \psi(x,0) + L[\psi],$$



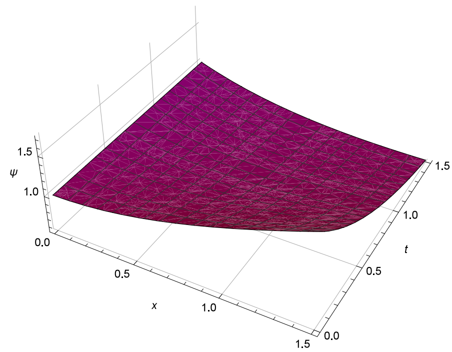
(a)



(b)



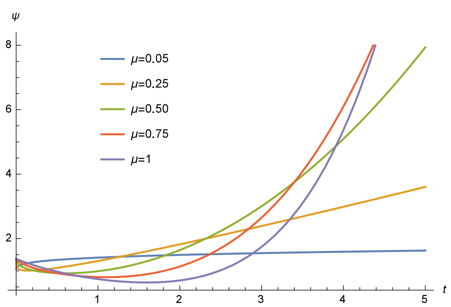
(c)



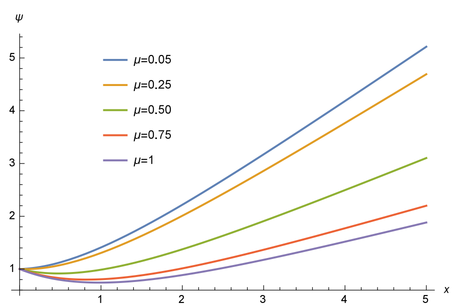
(d)

Fig. 1. For: a) $\mu = 0.025$, b) $\mu = 0.25$, c) $\mu = 0.75$, d) $\mu = 1$, the approximate solutions of (10) and (11)

.



(a)



(b)

Fig. 2. At: a) $x = 1$, b) $t = 1$, the five-term approximate solutions of (10) and (11)

.

Table 1. For $x = 1$, the absolute errors in the five-term approximate solutions of (10) and (11)

t	Error ($\mu = 0.7$)	Error ($\mu = 0.8$)	Error ($\mu = 0.9$)	Error ($\mu = 1.0$)
0.01	8.47652×10^{-9}	4.13751×10^{-10}	1.90392×10^{-11}	8.32001×10^{-13}
0.02	9.50658×10^{-8}	6.58582×10^{-9}	4.2953×10^{-10}	2.65779×10^{-11}
0.03	3.90052×10^{-7}	3.31862×10^{-8}	2.65572×10^{-9}	2.01492×10^{-10}
0.04	1.06051×10^{-6}	1.04432×10^{-7}	9.66556×10^{-9}	8.47677×10^{-10}
0.05	2.30162×10^{-6}	2.53919×10^{-7}	2.63133×10^{-8}	2.58262×10^{-9}
0.06	4.33182×10^{-6}	5.24465×10^{-7}	5.96167×10^{-8}	6.41575×10^{-9}
0.07	7.38958×10^{-6}	9.67969×10^{-7}	1.18995×10^{-7}	1.38441×10^{-8}
0.08	0.000011731	1.64527×10^{-6}	2.16475×10^{-7}	2.69467×10^{-8}
0.09	0.0000176282	2.62603×10^{-6}	3.66881×10^{-7}	4.84788×10^{-8}
0.10	0.0000253673	3.9886×10^{-6}	5.88003×10^{-7}	8.1964×10^{-8}

where $L(\psi) = I_t^{\mu}[\partial_x^2 \psi(x,t) - (1+4x^2)\psi(x,t)]$. Using the recurrence relation (9), we get

$$S_0 = \psi(x,0) = e^{x^2},$$

$$S_1 = S_0 + L(S_0) = e^{x^2} + I_t^{\mu}[\partial_x^2 S_0 - (1+4x^2)S_0] = e^{x^2}\left(1 + \frac{t^{\mu}}{\Gamma(\mu+1)}\right),$$

$$S_2 = S_1 + L(S_1) = e^{x^2} + I_t^{\mu}[\partial_x^2 S_1 - (1+4x^2)S_1] = e^{x^2}\left(1 + \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)}\right),$$

$$\vdots$$

$$S_n = e^{x^2}\left(1 + \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)} + \cdots + \frac{t^{n\mu}}{\Gamma(n\mu+1)}\right).$$

Thus, as $n \to \infty$, S_n converges to the following closed form solution:

$$\psi(x,t) = \lim_{n\to\infty} S_n = e^{x^2} E_{\mu}(t^{\mu}).$$

At $x = 1$ and $t = 1$, the five-term approximate solutions of (14) and (15) for various values of μ are plotted in Figure 3.

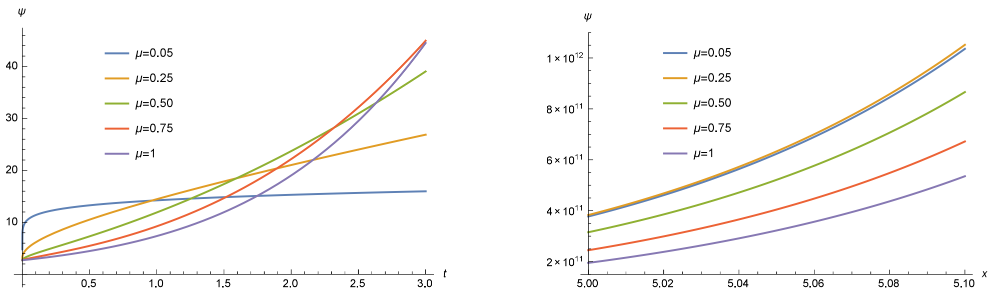
Example 4.3 Consider the following nonlinear time-fractional Cauchy reaction--diffusion equation:

$$\partial_t^{\mu}\psi(x,t) = \partial_x^2\psi(x,t) - \partial_x\psi(x,t) + \psi(x,t)\partial_x^2\psi(x,t) - \psi^2(x,t) + \psi(x,t), \quad (16)$$

$$\psi(x,0) = e^x,\ 0 < \mu \le 1. \quad (17)$$

Integrating the Eq. (16), we get

$$\psi(x,t) = \psi(x,0) + I_t^{\mu}[\partial_x^2\psi - \partial_x\psi + \psi\partial_x^2\psi - \psi^2 + \psi] = \psi(x,0) + L[\psi] + N[\psi],$$



(a) (b)

Fig. 3. At a) $x = 1$, b) $t = 1$ the five-term approximate solutions of (14) and (15)

.

where $L[\psi] = I_t^{\mu}[\partial_x^2\psi - \partial_x\psi + \psi]$ and $N[\psi] = I_t^{\mu}[\psi\partial_x^2\psi - \psi^2]$. Using the algorithm (9), we get

$$
\begin{aligned}
S_0 &= \psi(x,0) = e^x,\\
S_1 &= S_0 + L[S_0] + N[S_0] = e^x + I_t^{\mu}[\partial_x^2 S_0 - \partial_x S_0 + S_0] + I_t^{\mu}[S_0\partial_x^2 S_0 - S_0^2]\\
&= e^x\Big(1 + \frac{t^{\mu}}{\Gamma(\mu+1)}\Big),\\
S_2 &= S_0 + L[S_1] + N[S_1] = e^x + I_t^{\mu}[\partial_x^2 S_1 - \partial_x S_1 + S_1] + I_t^{\mu}[S_1\partial_x^2 S_1 - S_1^2]\\
&= e^x\Big(1 + \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)}\Big),\\
&\vdots\\
S_n &= e^x\Big(1 + \frac{t^{\mu}}{\Gamma(\mu+1)} + \frac{t^{2\mu}}{\Gamma(2\mu+1)} + \cdots + \frac{t^{n\mu}}{\Gamma(n\mu+1)}\Big).
\end{aligned}
$$

Therefore, as $n \to \infty$, we get the following closed form solution of (16) and (17)

$$\psi(x,t) = \lim_{n\to\infty} S_n = e^x E_{\mu}(t^{\mu}).$$

Example 4.4 Consider the following time-fractional CRDE

$$\partial_t^{\mu}\psi(x,t) = \partial_x^2\psi(x,t) - (2 + 4x^2 - 2t)\psi(x,t),\ t > 0,\ 0 < \mu \le 1, \tag{18}$$

$$\psi(x,0) = e^{x^2}. \tag{19}$$

Integrating on both sides of Eq. (18), we get

$$\psi(x,t) = \psi(x,0) + I_t^{\mu}[\partial_x^2\psi(x,t) - (2 + 4x^2 - 2t)\psi(x,t)] = \psi(x,0) + L[\psi].$$

In view of the recurrence relation (9), we get

$$
\begin{aligned}
S_0 &= \psi(x,0) = e^{x^2},\\
S_1 &= S_0 + L[S_0] = I_t^{\mu}[\partial_x^2 S_0 - (2+4x^2-2t)S_0] = \frac{2e^{x^2}t^{\mu+1}}{\Gamma(\mu+1)} + e^{x^2},\\
S_2 &= S_1 + L[S_1] = \frac{2e^{x^2}t^{\mu+1}}{\Gamma(\mu+2)} + \frac{4e^{x^2}\Gamma(\mu+3)t^{2\mu+2}}{\Gamma(\mu+2)\Gamma(2\mu+3)} + e^{x^2},\\
S_3 &= \frac{2e^{x^2}t^{\mu+1}}{\Gamma(\mu+2)} + \frac{4e^{x^2}\Gamma(\mu+3)t^{2\mu+2}}{\Gamma(\mu+2)\Gamma(2\mu+3)} + \frac{8e^{x^2}\Gamma(\mu+3)\Gamma(2\mu+4)t^{3\mu+3}}{\Gamma(\mu+2)\Gamma(2\mu+3)\Gamma(3\mu+4)} + e^{x^2},\\
S_4 &= \frac{2e^{x^2}t^{\mu+1}}{\Gamma(\mu+2)} + \frac{4e^{x^2}\Gamma(\mu+3)t^{2\mu+2}}{\Gamma(\mu+2)\Gamma(2\mu+3)} + \frac{8e^{x^2}\Gamma(\mu+3)\Gamma(2\mu+4)t^{3\mu+3}}{\Gamma(\mu+2)\Gamma(2\mu+3)\Gamma(3\mu+4)}\\
&\quad + \frac{16e^{x^2}\Gamma(\mu+3)\Gamma(2\mu+4)\Gamma(3\mu+5)t^{4\mu+4}}{\Gamma(\mu+2)\Gamma(2\mu+3)\Gamma(3\mu+4)\Gamma(4\mu+5)} + e^{x^2}
\end{aligned}
$$

It is clear that for $\mu = 1$, the series solution S_4 of (18) and (19) turns to

$$S_4 = e^{x^2} + t^2 e^{x^2} + \frac{1}{2}t^4 e^{x^2} + + \frac{1}{6}t^6 e^{x^2} + \frac{1}{24}t^8 e^{x^2},$$

which converges to $\psi(x,t) = e^{x^2+t^2}$. Five-term approximate solutions of (18) and (19) are depicted in Figures 4 and 5. It is observed that the behavior of the solutions depends on the value of the fractional derivative operator μ. Further, for $\mu = 1$, the exact, approximate solutions and absolute errors are given in Table 2.

Example 4.5 Consider the following time-fractional Cauchy reaction-diffusion equation

$$\partial_t^{\mu}\psi(x,t) = \partial_x^2\psi(x,t) + (\cos x - \sin^2 x - 1)\psi(x,t),\ 0 < \mu \le 1, \tag{20}$$

$$\psi(x,0) = \frac{1}{10}e^{\cos x - 11}. \tag{21}$$

Integral equation corresponding to (20) and (21) is

$$\psi(x,t) = \psi(x,0) + I_t^{\mu}[\partial_x^2\psi(x,t) + (\cos x - \sin^2 x - 1)\psi(x,t)] = \psi(x,0) + L[\psi].$$

In view of the algorithm (9), we get

$$
\begin{aligned}
S_0 &= \psi(x,0) = \frac{1}{10}e^{\cos x - 11},\\
S_1 &= S_0 + L[S_0] = I_t^{\mu}[\partial_x^2 S_0 + (\cos x - \sin^2 x - 1)S_0] = \frac{1}{10}e^{\cos(x)-11} - \frac{t^{\mu}e^{\cos(x)-11}}{10\Gamma(\mu+1)},\\
S_2 &= S_0 + L[S_1] = I_t^{\mu}[\partial_x^2 S_1 + (\cos x - \sin^2 x - 1)S_1]
\end{aligned}
$$

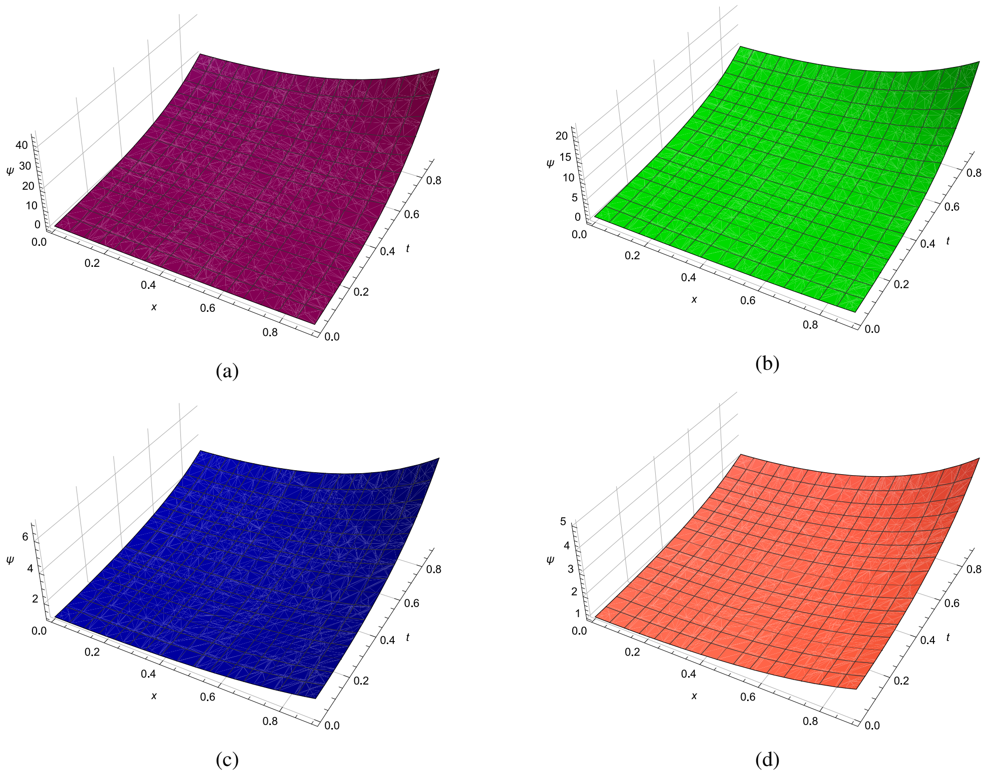


Fig. 4. For: a) $\mu = 0.025$, b) $\mu = 0.25$, c) $\mu = 0.75$, d) $\mu = 1$, the five-term approximate solutions of (18) and (19)

.

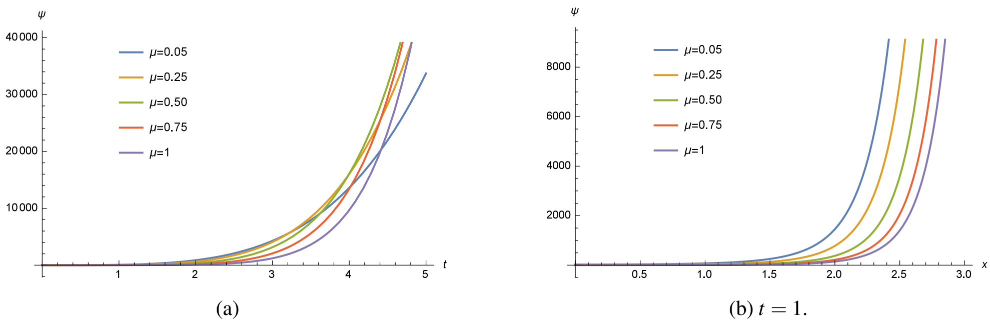


Fig. 5. At: a) $x = 1$, b) $t = 1$, the five-term DGJM approximate solutions of (18) and (19)

.

Table 2. For $\mu = 1$, the absolute errors in the five-term solutions of (18) and (19)

x	t	ψ_{exact}	ψ_{approx}	$\mid \psi_{exact}-\psi_{approx} \mid$
0.1	0.2	1.051271096376024	1.051271095508336	8.67689×10^{-10}
0.3	0.4	1.284025416687741	1.284024434492921	9.82195×10^{-7}
0.5	0.6	1.840431398781637	1.840362607684473	0.0000687911
0.7	0.8	3.095656500124712	3.094024663852053	0.00163184
0.9	1.0	6.11044743223061	6.088084130582111	0.0223633

$$= \frac{1}{10}e^{\cos(x)-11} - \frac{t^{\mu}e^{\cos(x)-11}}{10\Gamma(\mu+1)} + \frac{t^{2\mu}e^{\cos(x)-11}}{10\Gamma(2\mu+1)},$$

$$\vdots$$

$$S_n = \frac{1}{10}e^{\cos(x)-11} - \frac{t^{\mu}e^{\cos(x)-11}}{10\Gamma(\mu+1)} + \cdots + \frac{t^{n\mu}e^{\cos(x)-11}}{10\Gamma(n\mu+1)}.$$

As $n \to \infty$, the series solution S_n converges to following closed form solution:

$$\psi(x,t) = \lim_{n\to\infty} S_n = \frac{1}{10}e^{\cos(x)-11}E_{\mu}(-t^{\mu}).$$

Note that the same solution is obtained by the homotopy analysis transform method in [30].

Example 4.6 Consider the following time-fractional Cauchy reaction-diffusion equation

$$\partial_t^{\mu}\psi(x,t) = \partial_x^2\psi(x,t) + 2t\psi(x,t),\ 0 < \mu \leq 1, \tag{22}$$

$$\psi(x,0) = e^x. \tag{23}$$

Integrating the Eq. (22), we get

$$\psi(x,t) = \psi(x,0) + I_t^{\mu}[\partial_x^2\psi(x,t) + 2t\psi(x,t)] = \psi(x,0) + L[\psi].$$

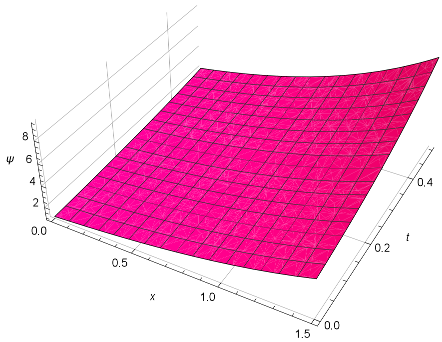
Using the algorithm defined in (9), we get

$$S_0 = \psi(x,0) = e^x$$

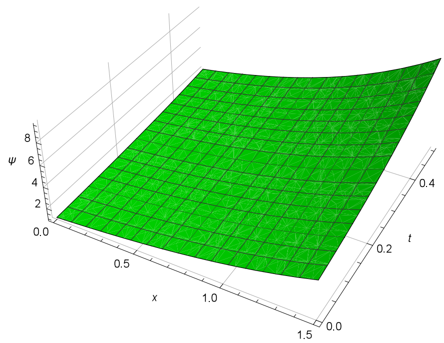
$$S_1 = S_0 + L[S_0] = e^x + I_t^{\mu}[\partial_x^2 S_0 + 2tS_0] = e^x + \frac{2e^x t^{\mu+1}}{\Gamma(\mu+2)} + \frac{e^x t^{\mu}}{\Gamma(\mu+1)},$$

$$S_2 = S_0 + L[S_1] = I_t^{\mu}[\partial_x^2 S_1 + 2tS_1] = e^x + \frac{e^x t^{2\mu}}{\Gamma(2\mu+1)} + \frac{2e^x t^{\mu+1}}{\Gamma(\mu+2)} + \frac{2e^x t^{2\mu+1}}{\Gamma(2\mu+2)}$$
$$+ \frac{2e^x\Gamma(\mu+2)t^{2\mu+1}}{\Gamma(\mu+1)\Gamma(2\mu+2)} + \frac{4e^x\Gamma(\mu+3)t^{2\mu+2}}{\Gamma(\mu+2)\Gamma(2\mu+3)} + \frac{e^x t^{\mu}}{\Gamma(\mu+1)},$$

$$S_3 = S_0 + L[S_2] = e^x + I_t^{\mu}[\partial_x^2 S_2 + 2tS_2] = e^x + \frac{e^x t^{2\mu}}{\Gamma(2\mu+1)} + \frac{e^x t^{3\mu}}{\Gamma(3\mu+1)} + \frac{2e^x t^{\mu+1}}{\Gamma(\mu+2)}$$



(a)



(b)

Fig. 6. For $\mu = 1$: a) four-term approximate solution, b) exact solution of (22) and (23)

.

$$
\begin{aligned}
&+\frac{2e^x t^{2\mu+1}}{\Gamma(2\mu+2)} + \frac{2e^x\Gamma(\mu+2)t^{2\mu+1}}{\Gamma(\mu+1)\Gamma(2\mu+2)} + \frac{4e^x\Gamma(\mu+3)t^{2\mu+2}}{\Gamma(\mu+2)\Gamma(2\mu+3)} + \frac{2e^x t^{3\mu+1}}{\Gamma(3\mu+2)}\\
&+\frac{2e^x\Gamma(\mu+2)t^{3\mu+1}}{\Gamma(\mu+1)\Gamma(3\mu+2)} + \frac{2e^x\Gamma(2\mu+2)t^{3\mu+1}}{\Gamma(2\mu+1)\Gamma(3\mu+2)} + \frac{4e^x\Gamma(\mu+3)t^{3\mu+2}}{\Gamma(\mu+2)\Gamma(3\mu+3)}\\
&+\frac{4e^x\Gamma(2\mu+3)t^{3\mu+2}}{\Gamma(2\mu+2)\Gamma(3\mu+3)} + \frac{4e^x\Gamma(\mu+2)\Gamma(2\mu+3)t^{3\mu+2}}{\Gamma(\mu+1)\Gamma(2\mu+2)\Gamma(3\mu+3)} + \frac{e^x t^{\mu}}{\Gamma(\mu+1)}\\
&+\frac{8e^x\Gamma(\mu+3)\Gamma(2\mu+4)t^{3\mu+3}}{\Gamma(\mu+2)\Gamma(2\mu+3)\Gamma(3\mu+4)}.
\end{aligned} \tag{24}
$$

Note that for $\mu = 1$, the DGJM solution converges to the exact solution $\psi(x,t) = e^{x+t+t^2}$ of (22) and (23). The four-term approximate solution of (22) and (23) is S_3 which is given in Eq. (24). For $\mu = 1$, the four-term approximate solution and the exact solution of (22) and (23) are represented graphically in Figure 6. We observe that the approximate and the exact solutions are in good agreement. Moreover, our solution is the same as that obtained by Sehu transform in [34].

5. Conclusion

We presented an algorithm for DGJM to solve the functional equations of the form $\psi = f + L(\psi) + N(\psi)$. We solved various examples of linear and nonlinear time-fractional Cauchy reaction-diffusion equations using the proposed algorithm. Further, the obtained approximate solutions are represented graphically. We observed that the behavior of the obtained solutions changes as the value of the fractional derivative operator change and hence depends on μ. Moreover, for some examples, we calculated the absolute errors in their solutions obtained by the proposed algorithm. We obtain the solutions in a series form that converges to a closed-form solution. Note that we used Mathematica 10.0 for calculations and graphical simulations. The present method is robust, time-efficient, free from tedious calculations, and can easily be implemented using a computer or symbolic algebra system. Hence the proposed method is appropriate for solving time-fractional Cauchy reaction-diffusion equations.

References

[1] Diethelm, K. (2013). A fractional calculus based model for the simulation of an outbreak of dengue fever. *Nonlinear Dynamics*, *71*, 613-619.

[2] Chen, J., Zhang, T., & Zhou, Y. (2020). Dynamics of a risk-averse newsvendor model with continuous-time delay in supply chain financing. *Mathematics and Computers in Simulation*, *169*, 133-148.

[3] Valentim Jr, C.A., Oliveira, N.A., Rabi, J.A., & David, S.A. (2020). Can fractional calculus help improve tumor growth models? *Journal of Computational and Applied Mathematics*, *379*, 112964.

[4] Johansyah, M.D., Supriatna, A.K., Rusyaman, E., & Saputra, J. (2021). Application of fractional differential equation in economic growth model: A systematic review approach. *AIMS Mathematics*, *6*(9), 10266-10280.

[5] Lu, Z., Yan, H., & Zhu, Y. (2019). European option pricing model based on uncertain fractional differential equation. *Fuzzy Optimization and Decision Making*, *18*, 199-217.

[6] Diethelm, K. (2021). Fast solution methods for fractional differential equations in the modeling of viscoelastic materials. In *2021 9th International Conference on Systems and Control (ICSC)* (pp. 455-460). IEEE.

[7] Graef, J.R., Ho, S.S., Kong, L., & Wang, M. (2019). A fractional differential equation model for bike share systems. *Journal of Nonlinear Functional Analysis*, *2019*(1).

[8] Jiang, Y., Xia, B., Zhao, X., Nguyen, T., Mi, C., & de Callafon, R.A. (2017). Data-based fractional differential models for non-linear dynamic modeling of a lithium-ion battery. *Energy*, *135*, 171-181.

[9] Iqbal, Z., Rehman, M.A.U., Imran, M., Ahmed, N., Fatima, U., Akgül, A., & Jarad, F. (2023). A finite difference scheme to solve a fractional order epidemic model of computer virus. *AIMS Mathematics*, *8*, 2337-2359.

[10] Liaqat, M.I., Akgül, A., De la Sen, M., & Bayram, M. (2023). Approximate and exact solutions in the sense of conformable derivatives of quantum mechanics models using a novel algorithm. *Symmetry*, *15*(3), 744.

[11] Zhang, Y. (2009). A finite difference method for fractional partial differential equation. *Applied Mathematics and Computation*, *215*(2), 524-529.

[12] Ray, S.S., & Gupta, A.K. (2018). *Wavelet Methods for Solving Partial Differential Equations and Fractional Differential Equations*. CRC Press.

[13] Shakeel, M., Hussain, I., Ahmad, H., Ahmad, I., Thounthong, P., & Zhang, Y.F. (2020). Meshless technique for the solution of time-fractional partial differential equations having real-world applications. *Journal of Function Spaces*, *2020*, 1-17.

[14] Yokus, A., Durur, H., Ahmad, H., Thounthong, P., & Zhang, Y.F. (2020). Construction of exact traveling wave solutions of the Bogoyavlenskii equation by $(G'/G, 1/G)$-expansion and $(1/G')$-expansion techniques. *Results in Physics*, *19*, 103409.

[15] Wang, F., Ali, S.N., Ahmad, I., Ahmad, H., Alam, K.M., & Thounthong, P. (2022). Solution of Burgers' equation appears in fluid mechanics by multistage optimal homotopy asymptotic method. *Thermal Science*, *26*(1 Part B), 815-821.

[16] Li, C., & Wang, Y. (2009). Numerical algorithm based on Adomian decomposition for fractional differential equations. *Computers & Mathematics with Applications*, *57*(10), 1672-1681.

[17] Bakkyaraj, T., & Thomas, R. (2022). Lie symmetry analysis and exact solution of (2+1)-dimensional nonlinear time-fractional differential-difference equations. *Pramana*, *96*(4), 225.

[18] Daftardar-Gejji, V., & Jafari, H. (2006). An iterative method for solving nonlinear functional equations. *Journal of Mathematical Analysis and Applications*, *316*(2), 753-763.

[19] Ahmad, H., Khan, T.A., & Yao, S.W. (2020). An efficient approach for the numerical solution of fifth-order KdV equations. *Open Mathematics*, *18*(1), 738-748.

[20] Yusuf, A., Sulaiman, T.A., Khalil, E.M., Bayram, M., & Ahmad, H. (2021). Construction of multi-wave complexiton solutions of the Kadomtsev-Petviashvili equation via two efficient analyzing techniques. *Results in Physics*, *21*, 103775.

[21] Brikaa, M.G. (2015). An analytic algorithm for the space-time fractional reaction-diffusion equation. *Journal of Interpolation and Approximation in Scientific Computing*, *2015*(2), 112-127.

[22] Gul, H., Alrabaiah, H., Ali, S., Shah, K., & Muhammad, S. (2020). Computation of solution to fractional order partial reaction diffusion equations. *Journal of Advanced Research*, *25*, 31-38.

[23] Ali, S., Bushnaq, S., Shah, K., & Arif, M. (2017). Numerical treatment of fractional order Cauchy reaction diffusion equations. *Chaos, Solitons & Fractals*, *103*, 578-587.

[24] Kumar, S., Ghosh, S., Jleli, M., & Araci, S. (2022). A fractional system of Cauchy-reaction diffusion equations by adopting Robotnov function. *Numerical Methods for Partial Differential Equations*, *38*(3), 470-489.

[25] Yildirim, A. (2009). Application of He's homotopy perturbation method for solving the Cauchy reaction-diffusion problem. *Computers & Mathematics with Applications*, *57*(4), 612-618.

[26] Rezapour, S., Liaqat, M.I., & Etemad, S. (2022). An effective new iterative method to solve conformable Cauchy reaction-diffusion equation via the Shehu transform. *Journal of Mathematics*, *2022*.

[27] Wang, K., & Liu, S. (2016). A new Sumudu transform iterative method for time-fractional Cauchy reaction-diffusion equation. *SpringerPlus*, *5*, 1-20.

[28] Ahmad, H., Khan, T.A., Ahmad, I., Stanimirović, P.S., & Chu, Y.M. (2020). A new analyzing technique for nonlinear time fractional Cauchy reaction-diffusion model equations. *Results in Physics*, *19*, 103462.

[29] Chu, Y.M., Shah, N.A., Ahmad, H., Chung, J.D., & Khaled, S.M. (2021). A comparative study of semi-analytical methods for solving fractional-order Cauchy reaction-diffusion equation. *Fractals*, *29*(06), 2150143.

[30] Kumar, S., Kumar, A., Abbas, S., Al Qurashi, M., & Baleanu, D. (2020). A modified analytical approach with existence and uniqueness for fractional Cauchy reaction-diffusion equations. *Advances in Difference Equations*, *2020*(1), 1-18.

[31] Kumar, M., Jhinga, A., & Daftardar-Gejji, V. (2020). New algorithm for solving non-linear functional equations. *International Journal of Applied and Computational Mathematics*, *6*(2), 26.

[32] Kilbas, A.A., Marichev, O.I., & Samko, S.G. (1993). *Fractional Integrals and Derivatives (Theory and Applications)*. CRS Press.

[33] Daftardar-Gejji, V. (Ed.). (2013). *Fractional Calculus*. Alpha Science International Limited.

[34] Areshi, M., Zidan, A.M., Shah, R., & Nonlaopon, K. (2021). A modified technique of fractional-order Cauchy-reaction diffusion equation via Shehu transform. *Journal of Function Spaces*, *2021*, 1-15.