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SCHRAMM SPACES AND COMPOSITION OPERATORS

Małgorzata Wróbel

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
malgorzata.wrobel@pcz.pl

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Abstract. We give some properties of Schramm functions; among others, we prove that the family of all continuous piecewise linear functions defined on a real interval I is contained in the space $\Phi BV(I)$ of functions of bounded variation in the sense of Schramm. Moreover, we show that the generating function of the corresponding Nemytskij composition operator acting between Banach spaces $C\Phi BV(I)$ of continuous functions of bounded Schramm variation has to be continuous and additionally we show that a space $C\Phi BV(I)$ has the Matkowski property.

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1. Introduction

An operator H acting between two classes of functions $f : I \to \mathbb{R}$ (where I is a compact real interval) given by $H(f)(x) := h(x, f(x))$ for some function $h : I \times \mathbb{R} \to \mathbb{R}$ is said to be a *composition (Nemytskij or superposition) operator*.

The theory of a Nemytskij composition operator is closely connected with the study of solutions of nonlinear integral equations of the Hammerstein type and the Volterra-Hammerstein type [1]. The smoothness properties of Nemytskij superposition operators are used in the mathematical modelling of physical, biological and other phenomena; for instance, in traveling wave models describing nonlinear dynamics in semiconductor lasers [2], in material modelling [3], or in economy for certain problems of option pricing within the Black-Scholes model for time--dependent volatility [4].

In the present paper we will investigate the Nemytskij composition operators H acting between two Banach spaces of the type $C\Phi BV(I)$ of continuous functions of bounded variation in the sense of Schramm (see Definition 1). In Section 2, we give some properties of Schramm functions which are useful in showing that the function h, generating the corresponding operator, has to be continuous. It is worth noting that the continuity of h, contrary to expectation, is not obvious. It happens that the generating function is not adequate regular to the classes of the functions on which the corresponding operator is defined. Matkowski's example shows that a discontinuous function can generate a Nemytskij operator mapping the space of continuously differentiable functions into itself. In Section 3, under the additional assumption that the composition operators are uniformly continuous, we observe that operators of such a type must be of the form

$$H(f) = \alpha \cdot f + \beta,$$

where α and β are the elements from the range. In particular, we get that a Banach space $C\Phi BV(I)$ has the Matkowski property. There are many other spaces with the Matkowski property. More details can be found in Chapter 6 of monograph [5]. Let us note that in [6], it was shown that the Banach spaces $\Phi BV(I)$ of functions of bounded Schramm variation have the weak Matkowski property.

2. Some properties of Schramm functions

We start by recalling some very basic facts as definitions and known results concerning the space of functions of bounded variation in the sense of Schramm.

Let $\mathscr{F}$ be the set of all convex functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$ such that $\varphi(0) = 0$ and $\varphi(t) > 0$ for $t > 0$. Note that ([7], Remark 2.1) if $\varphi \in \mathscr{F}$, then φ is continuous, strictly increasing, $\lim\limits_{t\longrightarrow\infty} \varphi(t) = \infty$, superadditive, i.e.,

$$\varphi(t_1) + \varphi(t_2) \le \varphi(t_1 + t_2), \quad t_1 \ge 0,\ t_2 \ge 0,$$

and

$$\varphi(\lambda t) \le \lambda \varphi(t), \quad t \ge 0, \qquad \lambda \in [0,1]. \tag{1}$$

A sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$ of functions from $\mathscr{F}$ satisfying the following two conditions:

$$\varphi_{i+1}(t) \le \varphi_i(t), \quad t > 0,\ i \in \mathbb{N}, \tag{2}$$

and

$$\sum_{i=1}^{\infty} \varphi_i(t) \quad \text{diverges for all} \quad t > 0,$$

is said to be a *Schramm sequence.*

Let $I = [a,b]$ $(a,b \in \mathbb{R},\ a < b)$ be an interval and let $\mathbb{R}^I$ stand for the set of all real functions defined on I. Denote by $\mathbf{S}[a,b]$ the family of all finite non-ordered collections $S_j = \left\{[a_1,b_1],[a_2,b_2],...,[a_j,b_j]\right\}$, $j \in \mathbb{N}$, of non-overlapping intervals $[a_i,b_i] \subset I$, $i = 1,...,j$, i.e., $(a_i,b_i) \cap (a_l,b_l) = \emptyset$, $i,l \in \{1,...,j\}$, $i \ne l$.

Given a map $f \in \mathbb{R}^I$, an appropriate collection $S_j \in \mathbf{S}[a,b]$, $j \in \mathbb{N}$, and a Schramm sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$, we set

$$var_\Phi(f,S_j) := \sum_{i=1}^{j} \varphi_i(|f(b_i) - f(a_i)|), \qquad j \in \mathbb{N}. \tag{3}$$

Definition 1. Let $I = [a,b] \subset \mathbb{R}$ be a compact interval, and let $(\varphi_i)_{i=1}^{\infty}$ be a Schramm sequence. We say that a function $f \in \mathbb{R}^I$ is of *bounded* Φ*-variation in the sense of Schramm in I (or bounded Schramm variation)*, if the Φ*-variation of f on I*, defined by

$$Var_\Phi(f) = Var_\Phi(f,I) := \sup\{var_\Phi(f,S_j) : S_j \in \mathbf{S}[a,b]\}, \tag{4}$$

is finite.

In what follows, we denote by $(\Phi BV(I), \|\cdot\|_\Phi)$ the Banach space ([8], Theorem 2.3) of all functions $f \in \mathbb{R}^I$ such that $Var_\Phi\left(\frac{f}{\lambda}\right) < \infty$ for some constant $\lambda > 0$ with the norm

$$\|f\|_\Phi := |f(a)| + p_\Phi(f), \quad f \in \Phi BV(I),$$

where the Luxemburg-Nakano-Orlicz seminorm p_Φ is defined as

$$p_\Phi(f) = p_\Phi(f,I) := \inf\left\{\varepsilon > 0 : Var_\Phi\left(\frac{f}{\varepsilon}\right) \leq 1\right\}, \;\; f \in \Phi BV(I).$$

Various spaces of the functions of generalized bounded variation may be obtained by making special choices of functions φ_i, $i \in \mathbb{N}$, for instance for $\varphi_i(u) = u$, $i \in \mathbb{N}$, the condition (4) coincides with the classical concept of variation in the sense of Jordan; for $\varphi_i(u) = u^p$, $p > 1$, $i \in \mathbb{N}$, in the sense of Wiener, for $\varphi_i(u) = \lambda_i u$, $i \in \mathbb{N}$, where a nonincreasing sequence of positive reals $(\lambda_i)_{i \in \mathbb{N}}$ is a Waterman sequence (i.e., if $\lambda_i \to 0$ as $i \to \infty$ and $\sum_{i=1}^{\infty} \lambda_i = \infty$) in the sense of Waterman.

Certain properties of the functional Var_Φ are listed in the following lemma.

Lemma 1. Given a Schramm sequence $\Phi = (\varphi_i)_{i=1}^{\infty}$, the functional Var_Φ is:

(i) nondecreasing, that is if J_1, J_2 are sub-intervals of I and $J_1 \subset J_2$, then

$$Var_\Phi(f,J_1) \leq Var_\Phi(f,J_2);$$

(ii) superadditive (with respect of the intervals), that is if J_1, J_2 are sub-intervals of I such that $J_1 \cap J_2$ is a singleton, then

$$Var_\Phi(f,J_1) + Var_\Phi(f,J_2) \leq Var_\Phi(f,J_1 \cup J_2);$$

(iii) sequentially lower semicontinuous, that is

$$Var_{\Phi}(f,I) \leq \liminf_{n\to\infty} Var_{\Phi}(f_n,I),$$

if $f_n \in \mathbb{R}^I$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in I$.

Proof. Since the conditions *(i)* and *(ii)* follow directly from Definition 1 and the superadditivity of an arbitrary φ_i, $i \in \mathbb{N}$, the proofs will be omitted.

To prove *(iii)*, fix an arbitrary collection $S_j = \{[a_1,b_1],\ [a_2,b_2],...,[a_j,b_j]\} \in \mathbf{S}[a,b]$, $j \in \mathbb{N}$, and take a sequence $f_n \in \mathbb{R}^I$, $n \in \mathbb{N}$, pointwise convergent to f. Taking into account (3), the continuity of φ_i, $i \in \{1,..,j\}$, and the pointwise convergence of f_n to f, we get

$$\lim_{n\to\infty} (var_{\Phi}(f_n,S_j) - var_{\Phi}(f,S_j))$$

$$= \lim_{n\to\infty} \left(\sum_{i=1}^{j} (\varphi_i(|f_n(b_i) - f_n(a_i)|) - \varphi_i(|f(b_i) - f(a_i)|)) \right) = 0. \tag{5}$$

Since, by Definition 1,

$$var_{\Phi}(f_n,S_j) \leq Var_{\Phi}(f_n,I), \qquad n \in \mathbb{N},$$

taking the limit inferior in both sides of the above inequality and applying (5), we obtain

$$var_{\Phi}(f,S_j) \leq \liminf_{n\to\infty} Var_{\Phi}(f_n,I),$$

whence, the arbitrariness of collection S_j completes the proof. □

Lemma 2. Let $[a,b] \subset \mathbb{R}$ $(a,b \in \mathbb{R},\ a < b)$ be a closed interval and $\Phi = (\varphi_i)_{i=1}^{\infty}$ be a Schramm sequence. If a function $f : [a,b] \to \mathbb{R}$ is a monotonic function, then $f \in \Phi BV(I)$ and

$$Var_{\Phi}(f) = \varphi_1(|f(b) - f(a)|).$$

Proof. Since $S = \{[a,b]\} \in \mathbf{S}[a,b]$, we have

$$Var_{\Phi}(f) \geq \varphi_1(|f(b) - f(a)|).$$

To prove the inverse inequality, assume that $S_j = \{[a_1,b_1],\ [a_2,b_2],...,[a_j,b_j]\}$ is a finite (non-ordered) collection of non-overlapping subintervals of $[a,b]$. Making use of (2), (3) and the superadditivity of φ_1, we get

$$\begin{aligned} Var_{\Phi}(f,S) &= \sum_{i=1}^{j} \varphi_i \left(\left|f(b_i)-f(a_i)\right|\right) \leq \sum_{i=1}^{j} \varphi_1 \left(\left|f(b_i)-f(a_i)\right|\right) \\ &\leq \varphi_1 \left(\sum_{i=1}^{j} \left|f(b_i)-f(a_i)\right| \right) \leq \varphi_1 \left(\left|f(b)-f(a)\right|\right), \end{aligned}$$

which completes the proof. □

Let Δ_s stand for the set of all permutations of the set $\{1,...,s\}$.

Lemma 3. *Let* $I=[a,b]\subset\mathbb{R}$ $(a,b\in\mathbb{R},\ a<b)$, $n\in\mathbb{N}$, $(x_i,y_i)\in I\times\mathbb{R},\ i=0,\ldots,n$, *such that*

$$a=x_0<x_1<\ldots<x_{n-1}<x_n=b$$

be fixed and let $\Phi=(\varphi_i)_{i=1}^{\infty}$ *be a Schramm sequence. Then the continuous function* $f:I\to\mathbb{R}$ *defined by*

$$f(t)=\begin{cases} y_i \quad \text{if} \quad t=x_i, \quad i=0,\ldots,n, \\ \text{affine otherwise} \end{cases}$$

is of bounded Schramm variation on I and

$$Var_{\Phi}(f)=\max\left\{ \sum_{i=1}^{s} \varphi_{\sigma_s(i)} \left(\left|y_{k_i}-y_{k_{i-1}}\right|\right) : \sigma_s\in\Delta_s\ ,\ s\in\{1,...,n\}\ ,\ y_{k_0}=y_0,\ y_{k_s}=y_n \right\}.$$

Proof. Fix arbitrarily $s\in\{1,...,n\}$. Since

$$S_s=\left\{ \left[x_{k_{i-1}},x_{k_i}\right] : x_{k_0}=x_0,\ \ x_{k_s}=x_n,\ \ i=1,\ldots,s \right\} \in \mathbf{S}[a,b],$$

putting $y_{k_0}=y_0$ and $y_{k_s}=y_n$ we get

$$Var_{\Phi}(f,S_s) \geq \sum_{i=1}^{s} \varphi_{\sigma_s(i)} \left(\left|y_{k_i}-y_{k_{i-1}}\right|\right)$$

for all permutations $\sigma_s:\{1,...,s\}\to\{1,...,s\}$ and, consequently, by (4), we obtain

$$Var_{\Phi}(f)\geq\max\left\{ \sum_{i=1}^{s} \varphi_{\sigma_s(i)} \left(\left|y_{k_i}-y_{k_{i-1}}\right|\right) : \sigma_s\in\Delta_s\ ,\ \ s\in\{1,...,n\}\ ,\ \ y_{k_0}=y_0,\ \ y_{k_s}=y_n \right\}. \tag{6}$$

To prove the inverse inequality, take an arbitrary collection

$$S_j=\left\{ [a_1,b_1]\ ,\ [a_2,b_2]\ ,...,[a_j,b_j] \right\} \in \mathbf{S}[a,b]\ , \quad j\in\mathbb{N}.$$

Let us observe, that the set of points defining the collection S_j can be completed to finite partition

$$P=(t_0,t_1,\ldots,t_{m-1},t_m),\ a=t_0<t_1<\ldots<t_{m-1}<t_m=b,$$

such that

$$x_0=x_{k_0}=t_0=t_{i_0}<...<t_{i_1}\le x_{k_1}\le x_{k_2}\le t_{i_1+1}<...<t_{i_2}\le x_{k_2+1}$$
$$\le ...\le x_{k_{s-1}}\le t_{i_{s-2}+1}<...<t_{i_{r-1}}\le x_{k_{s-1}+1}\le x_{k_s}=t_m=x_n,$$

and, in view of (3), we find

$$\sum_{i=1}^{j}\varphi_i\left(\left|f(b_i)-f(a_i)\right|\right)\le\sum_{i=1}^{m}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|), \tag{7}$$

for any permutation $\sigma_m\in\Delta_m$. Suppose, without loss of generality, that $f(t_0)\le f(t_1)$ and denote by r_1 the larger of the numbers $1,\ldots,m$ such that

$$y_0=f(t_0)\le f(t_1)\le\ldots\le f(t_{r_1}). \tag{8}$$

Putting

$$\overline{\sigma}_m(r_1-j):=\min\{\sigma_m(i):i=1,...,r_1-j\},\quad j\in\{0,1\}, \tag{9}$$

in view of (2), (9) and the superadditivity of $\varphi_{\overline{\sigma}_m(r_1)}$, we get

$$\begin{aligned}\sum_{i=1}^{r_1}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|) &\le \sum_{i=1}^{r_1}\varphi_{\overline{\sigma}_m(r_1)}(|f(t_i)-f(t_{i-1})|)\\ &\le \varphi_{\overline{\sigma}_m(r_1)}\left(\sum_{i=1}^{r_1}(f(t_i)-f(t_{i-1}))\right),\end{aligned}$$

and, consequently, by (8),

$$\sum_{i=1}^{r_1}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)\le\varphi_{\overline{\sigma}_m(r_1)}\left(f(t_{r_1})-f(t_0)\right), \tag{10}$$

whence,

$$\sum_{i=1}^{m}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)\le\varphi_1(y_n-y_0)=\varphi_1(|y_n-y_0|),$$

in the case where $r_1=m$.

Now, if $r_1<m$, then $f(t_{r_1})>f(t_{r_1+1})$. As in the first step, denote by r_2 the larger of numbers $r_1+1,\ldots,m$ such that

$$f(t_{r_1})>f(t_{r_1+1})>\ldots>f(t_{r_2}). \tag{11}$$

The following observation plays a key role in further considerations. There exists $k_{r_1}\in\{0,...,n-1\}$ such that either (A):

$$t_{r_1} = \max\left\{t_i:\ t_i \in \left(x_{k_{r_1}}, x_{k_{r_1}+1}\right]\right\}$$

or (B):

$$t_{r_1} = \min\left\{t_i:\ t_i \in \left(x_{k_{r_1+1}}, x_{k_{r_1+1}+1}\right]\right\}.$$

Hence, if (A), then

$$|f(t_{r_1+1}) - f(t_{r_1})| \le \overline{y}_{r_1} - f(t_{r_1+1}), \tag{12}$$

where

$$\overline{y}_{r_1} := \max\left\{f\left(x_{k_{r_1}}\right), f\left(x_{k_{r_1+1}}\right)\right\} = \max\left\{y_{k_{r_1}}, y_{k_{r_1+1}}\right\}$$

and if (B), then

$$|f(t_{r_1}) - f(t_{r_1-1})| \le \overline{y}_{r_1+1} - f(t_{r_1-1}), \tag{13}$$

where

$$\overline{y}_{r_1+1} := \max\left\{f\left(x_{k_{r_1+1}}\right), f\left(x_{k_{r_1+1}+1}\right)\right\} = \max\left\{y_{k_{r_1+1}}, y_{k_{r_1+1}+1}\right\}.$$

Setting

$$\overline{\sigma}_m(r_1 + j) := \min\{\sigma_m(i) : i = r_1 + j, ..., r_2\},\ \ j \in \{1, 2\}, \tag{14}$$

it follows from (2), (9)-(14) and the superadditivity of $\varphi_{\sigma_m(i)}$, $i \in \{1, ..., r_2\}$, that

$$\begin{aligned}
&\sum_{i=1}^{r_2} \varphi_{\sigma_m(i)}(|f(t_i) - f(t_{i-1})|)\\
=\ &\sum_{i=1}^{r_1} \varphi_{\sigma_m(i)}(|f(t_i) - f(t_{i-1})|)\\
&+\left(\varphi_{\sigma_m(r_1+1)}(|f(t_{r_1+1}) - f(t_{r_1})|) + \sum_{i=r_1+2}^{r_2} \varphi_{\sigma_m(i)}(|f(t_i) - f(t_{i-1})|)\right)\\
\le\ &\varphi_{\overline{\sigma}_m(r_1)}(f(t_{r_1}) - f(t_0))\\
&+\left(\varphi_{\sigma_m(r_1+1)}\left(\overline{y}_{r_1} - f(t_{r_1+1})\right) + \varphi_{\overline{\sigma}_m(r_1+2)}\left(f(t_{r_1+1}) - f(t_{r_2})\right)\right)
\end{aligned}$$

in the case (A) and

$$
\begin{aligned}
&\sum_{i=1}^{r_2} \varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)\\
= &\left(\sum_{i=1}^{r_1-1} \varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)+\varphi_{\sigma_m(r_1)}\left(|f(t_{r_1})-f(t_{i-1})|\right)\right)\\
&+\sum_{i=r_1+1}^{m} \varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|))\\
\leq &\left(\varphi_{\overline{\sigma}_m(r_1-1)}\left(f(t_{r_1-1})-f(t_0)\right)+\varphi_{\sigma_m(r_1)}\left(\overline{y}_{r_1+1}-f(t_{r_1-1})\right)\right)\\
&+\varphi_{\overline{\sigma}_m(r_1+1)}(f(t_{r_1+1}-f(t_{r_2})|)
\end{aligned}
$$

in the case (B). Since, by (2) and the superadditivity of $\varphi_{\overline{\sigma}_m(r_1+1)}$,

$$
\varphi_{\sigma_m(r_1+1)}\left(\overline{y}_{r_1}-f(t_{r_1+1})\right)+\varphi_{\overline{\sigma}_m(r_1+2)}(f(t_{r_1+1}-f(t_{r_2})|)\leq \varphi_{\overline{\sigma}_m(r_1+1)}\left(\overline{y}_{r_1}-f(t_{r_2})\right) \tag{15}
$$

and

$$
\varphi_{\overline{\sigma}_m(r_1-1)}\left(f(t_{r_1-1})-f(t_0)\right)+\varphi_{\sigma_m(r_1)}\left(\overline{y}_{r_1+1}-f(t_{r_1-1})\right)\leq \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1+1}-f(t_0)\right), \tag{16}
$$

therefore, if $r_2=m$, then we get the estimates

$$
\begin{aligned}
\sum_{i=1}^{m} \varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|) &\leq \varphi_{\overline{\sigma}_m(r_1)}\left(f(t_{r_1})-f(t_0)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}\left(\overline{y}_{r_1}-f(t_m)\right)\\
&\leq \varphi_{\overline{\sigma}_m(r_1)}\left(f(t_{r_1})-f(t_0)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}\left(\overline{y}_{r_1}-f(t_m)\right)\\
&\leq \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1}-f(t_0)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}\left(\overline{y}_{r_1}-f(t_m)\right)\\
&= \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1}-f(a)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}\left(\overline{y}_{r_1}-f(b)\right)\\
&= \varphi_{\overline{\sigma}_m(r_1)}\left(\left|\overline{y}_{r_1}-f(a)\right|\right)+\varphi_{\overline{\sigma}_m(r_1+1)}\left(\left|\overline{y}_{r_1}-f(b)\right|\right),
\end{aligned}
$$

by (15), or

$$
\begin{aligned}
\sum_{i=1}^{m} \varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|) &\leq \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1+1}-f(t_0)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}(f(t_{r_1+1}-f(t_m))\\
&\leq \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1+1}-f(t_0)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}(f(\overline{y}_{r_1+1}-f(t_m))\\
&= \varphi_{\overline{\sigma}_m(r_1)}\left(\overline{y}_{r_1+1}-f(a)\right)+\varphi_{\overline{\sigma}_m(r_1+1)}(f(\overline{y}_{r_1+1}-f(b))\\
&= \varphi_{\overline{\sigma}_m(r_1)}\left(\left|\overline{y}_{r_1+1}-f(a)\right|\right)+\varphi_{\overline{\sigma}_m(r_1+1)}(f(\left|\overline{y}_{r_1+1}-f(b)\right|),
\end{aligned}
$$

by (16). Since σ_m is a permutation, by (9) and (14), we have either $\overline{\sigma}_m(r_1)=1$ or $\overline{\sigma}_m(r_1+1)=1$. Observe that $\overline{\sigma}_m(r_1)=1$ implies $\overline{\sigma}_m(r_1+1)\geq 2$ and so, from (2), we find $\varphi_{\overline{\sigma}_m(r_1+1)}\leq \varphi_2$. Similarly, $\overline{\sigma}_m(r_1+1)=1$ implies $\overline{\sigma}_m(r_1)\geq 2$ and, consequently, also by (2), $\varphi_{\overline{\sigma}_m(r_1)}\leq \varphi_2$. Finally, we obtain one of the following possibilities:

$$\sum_{i=1}^{m}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)\leq\varphi_{\sigma_2(1)}\left(\left|\bar{y}_{r_1}-f(a)\right|\right)+\varphi_{\sigma_2(2)}\left(\left|\bar{y}_{r_1}-f(b)\right|\right),$$

or

$$\sum_{i=1}^{m}\varphi_{\sigma_m(i)}(|f(t_i)-f(t_{i-1})|)\leq\varphi_{\sigma_2(1)}\left(\left|\bar{y}_{r_1+1}-f(a)\right|\right)+\varphi_{\sigma_2(2)}(f(\left|\bar{y}_{r_1+1}-f(b)\right|),$$

for any permutation σ_2.

If $r_2<m$, then $f(t_{r_2})>f(t_{r_2+1})$, and we have to repeat the above procedure several times until we get $r_s=m$, $s\in\{3,...,n\}$; by Definition 1, the proof is completed. □

Corollary 1. The family $PL([a,b])$ of all continuous piecewise linear functions on $[a,b]$ and the family $P_n([a,b])$ of polynomials of degree not exceeding n, $n\in\mathbb{N}$, are contained in $\Phi BV(I)$ for any Schramm sequence Φ.

3. Composition operators in the spaces of continuous functions of bounded Schramm variation

In the following, let $C\Phi BV(I)$ denote $\Phi BV(I)\cap C(I)$. Since the convergence with respect to the norm $\|\cdot\|_\Phi$ implies the uniform convergence on I [[5], Proposition 2.44], the pair $(C\Phi BV(I),\|\cdot\|_\Phi)$ is a closed subspace of $(\Phi BV(I),\|\cdot\|_\Phi)$, so it forms a Banach space.

From now on, given a closed interval $I=[a,b]\subset\mathbb{R}$, we denote by $X(I)$ and $Y(I)$ two spaces of functions $\varphi:I\to\mathbb{R}$.

Definition 2. An operator $H:X(I)\to Y(I)$ given by

$$H(f)(x):=h(x,f(x)),\quad f\in X(I),\ (x\in I),$$

for some function $h:I\times\mathbb{R}\to\mathbb{R}$ is said to be a *composition (Nemytskij or superposition) operator*. The function h is referred to as the *generator of the operator H*.

Now, we are in a position to give the following

Theorem 1. *Let $\Phi=(\varphi_i)_{i=1}^{\infty}$ and $\Psi=(\psi_i)_{i=1}^{\infty}$ be two Schramm sequences and $I=[a,b]$ $(a,b\in\mathbb{R},\ a<b)$ be a closed interval. If the composition operator H of the generator $h:I\times\mathbb{R}\to\mathbb{R}$ maps a Banach space $C\Phi BV(I)$ into $C\Psi BV(I)$, then h is continuous.*

Proof. Applying Lemmas 2 and 3, it is enough to argue along the lines of the proof of Theorem 1 of [9].

In 1982, Matkowski [10] proved that if the Nemytskij composition operator H maps the space $Lip(I)$ of Lipschitz continuous functions with Lip-norm into itself and is globally Lipschitzian, i.e., if there is a constant $\mu\geq 0$ such that

$$\|H(f_1)-H(f_2)\|_{Lip} \leq \mu \|f_1-f_2\|_{Lip}, \quad f_1,f_2 \in Lip(I),$$

then the generating function h is of the form

$$h(x,y)=\alpha(x)y+\beta(x), \quad x\in[0,1],\ y\in\mathbb{R}, \tag{17}$$

for some functions $\alpha,\beta \in Lip(I)$.

Similarly, Matkowski and Miś [11] showed that if a globally Lipschitzian Nemytskij operator H maps the space $BV(I)$ into itself (where $BV(I)=\Phi BV(I)$ for $\varphi_i = id_I$, $i\in\mathbb{N}$), then there exist functions $\alpha,\beta : BV(I)\to BV(I)$ left-hand continuous on I such that the *left regularization of* h, i.e., the function $h^-: I^-\times\mathbb{R}\to\mathbb{R}$ defined by

$$h^-(x,y):=\lim_{s\uparrow x} h(s,y), \quad x\in I^-=(a,b]; \quad y\in\mathbb{R},$$

satisfies a condition

$$h^-(x,y)=\alpha(x)y+\beta(x), \quad x\in(a,b],\ y\in\mathbb{R}. \tag{18}$$

These results remain true if the Lipschitz norm-continuity of H is replaced by its uniform continuity [12].

Let us introduce the following

Definition 3 ([5], Definition 6.26). We say that a pair $(X(I),Y(I))$ of normed function spaces $(X(I),\|\cdot\|_X)$ and $(Y(I),\|\cdot\|_Y)$ has the *Matkowski property,* if whenever the Nemytskij superposition operator H maps the space $(X(I),\|\cdot\|_X)$ into the space $(Y(I),\|\cdot\|_Y)$ and there is a constant $\mu\geq 0$ such that

$$\|H(f_1)-H(f_2)\|_Y \leq \mu \|f_1-f_2\|_X, \quad f_1,f_2\in X(I),$$

the corresponding generating function h must have the form (17). In the case $X(I)=Y(I)$ we simply say that $X(I)$ has the Matkowski property.

Given two Schramm sequences $\Phi=(\varphi_i)_{i=1}^{\infty}$ and $\Psi=(\psi_i)_{i=1}^{\infty}$, by ([5], Proposition 2.45), we have $C\Phi BV(I)\subseteq C\Psi BV(I)$ if

$$\sum_{i=1}^{n}\varphi_i(t)\geq C\sum_{i=1}^{n}\psi_i(t), \quad t\in[0,T], \quad n\in\mathbb{N}, \tag{19}$$

for some $T>0$ and $C>0$.

Thus, under the additional assumption that the composition operator is uniformly continuous, we get a complete characterization of its generating function h. Namely, we get the following.

Theorem 2. *Let $\Phi=(\varphi_i)_{i=1}^{\infty}$ and $\Psi=(\psi_i)_{i=1}^{\infty}$ be two Schramm sequences and I be a compact interval. If a composition operator H mapping $C\Phi BV(I)$ into $C\Psi BV(I)$ is uniformly continuous, then there exist $\alpha(\cdot)$ and $\beta(\cdot)\in C\Psi BV(I)$ such that*

$$H(f)(x) = \alpha(x)f(x) + \beta(x), \quad f \in C\Phi BV(I), \quad (x \in I). \tag{20}$$

Conversely, if there exist $T > 0$ *and* $C > 0$ *such that (19) is fulfilled and an operator* $H : \mathbb{R}^I \to \mathbb{R}^I$ *is defined by (20) for some functions* $\alpha, \beta \in C\Psi BV(I)$, *then the operator* H *maps* $C\Phi BV(I)$ *into* $C\Psi BV(I)$, *and satisfies the global Lipschitz condition (so it is uniformly continuous).*

Proof. In view of Theorem 1, an operator H has the continuous generating function h. The existence of $\alpha, \beta : I \to \mathbb{R}$ satisfying (18) follows by the same methods as in the proof of Theorem 3.1 of [6], by Corollary 1. The continuity of h gives (20). Since $h(\cdot, y_0) = H(P_{y_0}(\cdot)) \in C\Psi BV(I)$ for all $y \in \mathbb{R}$ (where $P_{y_0} : I \to \mathbb{R}$ is defined by $P_{y_0}(t) := y_0$) and $\beta(x) = h(x,0)$; $\alpha(x) = h(x,1) - \beta(x)$, the functions $\alpha, \beta \in C\Psi BV(I)$, which gives a required claim.

In contrary, since the space $(C\Psi BV(I), \|\cdot\|_\Psi)$ is a Banach algebra, the proof is completed. □

Corollary 2. The pair $(C\Phi BV(I), C\Psi BV(I))$ of normed spaces $(C\Phi BV(I), \|\cdot\|_\Phi)$ and $(C\Psi BV(I), \|\cdot\|_\Psi)$ has the Matkowski property.

4. Conclusions

We prove that the family of all continuous piecewise linear functions on $[a,b]$ are contained in $\Phi BV(I)$ for any Schramm sequence Φ. Thus we get the continuity of the generating function h of the corresponding Nemytskij composition operator H acting between Banach spaces $C\Phi BV(I)$ of continuous functions of bounded variation in the sense of Schramm. Under the additional assumption that the composition operators are uniformly continuous, we observe that operators of such a type must be of the form $H(f) = \alpha \cdot f + \beta$, where α and β are the elements from the range. In particular, we note that a Banach space $C\Phi BV(I)$ has the Matkowski property. This extends the result of [6], where it was shown that the Banach spaces $\Phi BV(I)$ of functions of bounded Schramm variation have the weak Matkowski property.

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