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NUMERICAL SOLUTION OF A FRACTIONAL COUPLED SYSTEM WITH THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract. Within this work, we discuss the existence of solutions for a coupled system of linear fractional differential equations involving Caputo-Fabrizio fractional orders. We prove the existence and uniqueness of the solution by using the Picard-Lindelöf method and fixed point theory. Also, to compute an approximate solution of problem, we utilize the Adomian decomposition method (ADM), as this method provides the solution in the form of a series such that the infinite series converge to the exact solution. Numerical examples are presented to illustrate the validity and effectiveness of the proposed method.

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1. Introduction

Fractional order differential equations are generalizations of ordinary differential equations of non-integer order. They have attracted considerable interest due to the non-localization properties of the fractional derivatives contrary to the integer-order derivatives [1–3]. Indeed, fractional differential equations play an important role in the various fields of engineering and sciences including applications in physics, economics, chemistry, biology, and more, see [4–6]. The theory of existence of linear and nonlinear problems in the field of fractional differential equations has been studied and cited by many researchers. For an in-depth study of various linear and nonlinear problems in fractional differential equations and applications, see [7–9]. Another interesting field of recent research is the coupled system of fractional differential equations. It has been proven to be more accurate and realistic that have many applications in real-world problems such as: [10–13]. However, due to the complexities of the nonlinear term, many fractional differential equations do not have exact analytic solutions. Because of this difficulty, many researchers developed numerical schemes to find an approximate solution. There are numerous methods for calculating the numerical solution of fractional differential equations and integral equations, such as the iteration techniques [14], numerical bifurcation [15], difference methods [16], etc.

The Adomian decomposition method (ADM) was first introduced by George Adomian in the beginning of 1980's and developed in [17]. The method is a kind of algorithm, and it is an advantageous method for solving a linear and nonlinear differential equations of fractional order, which gives the approximate solution and even exact solution. This method has many advantages, such as: it's quite straight forward to write computer codes, it is also avoids the cumbersome integration of the Picard method, and it can solve some nonlinear problems which cannot be solved by other numerical methods.

In 2015, Caputo and Fabrizio [18] suggested a new fractional derivative with non-singular kernel. On the other hand, the Caputo-Fabrizio fractional derivative has many significant properties, such as its ability to describe matter hetrogeneities and configuration with different scales, many works studied the existence and uniqueness solution of boundary value problems involving such operator [19–21]. In the articles [22–24] several methods and issues for solving and modeling solutions to problems in applied mathematics and proofs for important theories were presented. Moreover, the authors investigated the existence and uniqueness results for some coupled systems involving C-F derivatives, for instance see [25–29] and the references cited therein.

In this paper, we will provide a state of the art that can be easily used as a basis to familiarize oneself with couple system of linear fractional order of Caputo-Fabrizio type with boundary conditions, as follows:

$$\begin{cases} \mathscr{D}^{(\alpha)} r(t) = c_1 r(t) + c_2 w(t) + f(t), & t \in \Omega := [0,1] \\ \mathscr{D}^{(\alpha)} w(t) = c_3 r(t) + c_4 w(t) + g(t), & t \in \Omega := [0,1] \\ r(0) = w(0) = 0, \end{cases} \tag{1}$$

where $0 < \alpha < 1$ is a real number, $\mathscr{D}^{(\alpha)}$ is the new fractional derivative of Caputo Fabrizio, $f, g : \Omega \to \mathbb{R}$ are given continuous functions, and c_i real constants and $i = 1,2,3,4$.

The paper is organized as follows: In the second Section, we present some useful definitions and lemmas of fractional calculus. In the third Section, the Picard-Lindelöf technique and the Banach fixed point theorem is applied to obtain uniqueness of solutions for system (1). In the articles [17, 29, 30], several methods and issues for solving and modeling solutions to problems in applied mathematics and proofs for important theories were presented, and we choose in the fourth Section, the Adomin Decomposition Method (ADM) to construct approximate solutions of the problem (1). A numerical example is given in last Section to illustrate our main results.

2. Preliminaries

We begin this section by reviewing the definitions of the Caputo-Fabrizio fractional derivative and integral and investigate their main characteristics as well.

Definition 1 [18] The CF (Caputo-Fabrizio) fractional derivative of α order for a function $s \in H^1(a,b)$, $b > a$ and $\alpha \in]0,1[$ is given as

$$\mathscr{D}^{(\alpha)} s(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t s'(\eta) \exp\left[-\frac{\alpha(t-\eta)}{1-\alpha}\right] \mathrm{d}\eta, \tag{2}$$

the respective CF fractional integral is defined by:

$$I_a^\alpha s(t) = \frac{1}{M(\alpha)} \left[(1-\alpha)(s(t)-s(a)) + \alpha \int_a^t s(\eta)\,\mathrm{d}\eta\right], \tag{3}$$

where $M(\alpha)$ is a normalization function with the main properties of $M(0) = M(1) = 1$. □

Lemma 1 [20] *Let $\gamma \in (n, n+1), n = \lfloor \gamma \rfloor \geqslant 0$. Assume that $s \in \mathscr{C}^n[a,b]$, then those statements hold:*

1. *if $s(a) = 0$, then $\mathscr{D}^{(\gamma)}\left(I_a^\gamma s(t)\right) = s(t)$.*
2. $I_a^\gamma\left(\mathscr{D}^{(\gamma)} s(t)\right) = s(t) + \sum_{i=0}^{n} a_i t^i,\ a_i \in \mathbb{R}\ i = 0,1,\ldots,n.$ □

3. Study of the associated linear system

In the following, we suppose the function $M(\alpha) = 1$.

Lemma 2 *Let $0 < \alpha < 1$, $r, w \in \mathscr{C}^1(\Omega)$, $f, g : \Omega \to \mathbb{R}$ be given continuous functions, and c_i real constants and $i = 1,2,3,4$. Then the solution of linear coupled system (1) is given by*

$$\begin{aligned} r(t) &= K(t) + (1-\alpha)(c_1 r(t) + c_2 w(t)) + \alpha \int_0^t (c_1 r(\eta) + c_2 w(\eta))\,\mathrm{d}\eta, \\ w(t) &= G(t) + (1-\alpha)(c_3 r(t) + c_4 w(t)) + \alpha \int_0^t (c_3 r(\eta) + c_4 w(\eta))\,\mathrm{d}\eta, \end{aligned} \tag{4}$$

where $K(t) = (1-\alpha)(f(t) - f(0)) + \alpha \int_0^t f(\eta)\,\mathrm{d}\eta$, *and* $G(t) = (1-\alpha)(g(t) - g(0)) + \alpha \int_0^t g(\eta)\,\mathrm{d}\eta$. □

Proof From Lemma 1, we can reduce Eq. (1) to the following equivalent integral equations:

$$\begin{aligned} r(t)+a_1 &= (1-\alpha)(c_1 r(t)+c_2 w(t)+f(t)-f(0)) \\ &\quad +\alpha\int_0^t (c_1 r(\eta)+c_2 w(\eta)+f(\eta))\,\mathrm{d}\eta, \\ w(t)+a_2 &= (1-\alpha)(c_3 r(t)+c_4 w(t)+g(t)-g(0)) \\ &\quad +\alpha\int_0^t (c_3 r(\eta)+c_4 w(\eta)+g(\eta))\,\mathrm{d}\eta, \end{aligned} \tag{5}$$

where $a_1, a_2 \in \mathbb{R}$. Using initial conditions $r(0) = w(0) = 0$, we obtain $a_1 = a_2 = 0$. Thus (5) reduces to

$$\begin{aligned} r(t) &= (1-\alpha)(c_1 r(t)+c_2 w(t)+f(t)-f(0))+\alpha\int_0^t (c_1 r(\eta)+c_2 w(\eta)+f(\eta))\,\mathrm{d}\eta, \\ w(t) &= (1-\alpha)(c_3 r(t)+c_4 w(t)+g(t)-g(0))+\alpha\int_0^t (c_3 r(\eta)+c_4 w(\eta)+g(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{6}$$

Then the equation (6) can be written as follows

$$\begin{aligned} r(t) &= (1-\alpha)(f(t)-f(0))+\alpha\int_0^t f(\eta)\,\mathrm{d}\eta+(1-\alpha)(c_1 r(t)+c_2 w(t)) \\ &\quad +\alpha\int_0^t (c_1 r(\eta)+c_2 w(\eta))\,\mathrm{d}\eta, \\ w(t) &= (1-\alpha)(g(t)-g(0))+\alpha\int_0^t g(\eta)\,\mathrm{d}\eta+(1-\alpha)(c_3 r(t)+c_4 w(t)) \\ &\quad +\alpha\int_0^t (c_3 r(\eta)+c_4 w(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{7}$$

Hence, the unique solution of problem (1) is

$$\begin{aligned} r(t) &= K(t)+(1-\alpha)(c_1 r(t)+c_2 w(t))+\alpha\int_0^t (c_1 r(\eta)+c_2 w(\eta))\,\mathrm{d}\eta, \\ w(t) &= G(t)+(1-\alpha)(c_3 r(t)+c_4 w(t))+\alpha\int_0^t (c_3 r(\eta)+c_4 w(\eta))\,\mathrm{d}\eta. \end{aligned}$$

The proof is complete. ■

3.1. Existence and uniqueness of the solution

Here we analyze the existence of a unique solution using the Picard-Lindelöf technique and the fixed point theory.

Let $(r_0(t), w_0(t)) = (K(t), G(t))$; then the Picard iteration is defined by

$$\begin{aligned} r_{i+1}(t) &= (1-\alpha)(c_1 r_i(t) + c_2 w_i(t)) + \alpha \int_0^t (c_1 r_i(\eta) + c_2 w_i(\eta))\,\mathrm{d}\eta, \\ w_{i+1}(t) &= (1-\alpha)(c_3 r_i(t) + c_4 w_i(t)) + \alpha \int_0^t (c_3 r_i(\eta) + c_4 w_i(\eta))\,\mathrm{d}\eta. \end{aligned} \tag{8}$$

In order to show the existence of a unique solution, let us define

$$\begin{aligned} L_1(t,r,w) &= c_1 r + c_2 w, \\ L_2(t,r,w) &= c_3 r + c_4 w, \end{aligned} \tag{9}$$

and $\lambda(t) = (r(t), w(t))$.

Lemma 3 *Let $L_1, L_2 : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then $L_1(t,r,w)$ and $L_2(t,r,w)$ are contraction with respect to r and w if*

$$\mu_1 < 1 \text{ and } \mu_2 < 1, \tag{10}$$

where $\mu_1 = \max(|c_1|,|c_2|),\ \mu_2 = \max(|c_3|,|c_4|)$. □

Proof Let $r_i,\ w_i \in \mathbb{R},\ i = 1,2$ and for all $t \in \Omega$, we have

$$\begin{aligned} |L_1(t,r_1,w_1) - L_1(t,r_2,w_2)| &\leqslant |c_1| \parallel r_1(t) - r_2(t) \parallel + |c_2| \parallel w_1(t) - w_2(t) \parallel, \\ &\leqslant \max(|c_1|,|c_2|)\,(\parallel r_1(t) - r_2(t) \parallel + \parallel w_1(t) - w_2(t) \parallel) \\ &\leqslant \mu_1 \parallel \lambda_1(t) - \lambda_2(t) \parallel . \end{aligned}$$

In a similar manner:

$$|L_2(t,r_1,w_1) - L_2(t,r_2,w_2)| \leqslant \mu_2 \parallel \lambda_1(t) - \lambda_2(t) \parallel .$$

Hence,

$$\begin{aligned} \parallel L_1(t,r_1,w_1) - L_1(t,r_2,w_2) \parallel &\leqslant \mu_1 \parallel \lambda_1(t) - \lambda_2(t) \parallel, \\ \parallel L_2(t,r_1,w_1) - L_2(t,r_2,w_2) \parallel &\leqslant \mu_2 \parallel \lambda_1(t) - \lambda_2(t) \parallel . \end{aligned} \tag{11}$$

which, in view of (10), implies that $L_1(t,r,w,)$ and $L_2(t,r,w)$ are contraction with respect to r and w. ■

Theorem 1 *Let $L_1, L_2 : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Then, the system (1) has a unique solution in Ω, provided by*

$$\mu = \mu_1 + \mu_2 < 1,$$

where $\mu_1 = \max(|c_1|,|c_2|),\ \mu_2 = \max(|c_3|,|c_4|)$. □

Proof In order to show the existence of a unique solution, we have $L_1(t,r,w)$ and $L_2(t,r,w)$ are contraction with respect to r and w. Then the Picard operator can therefore be defined as follows:

$$\theta(\lambda(t)) = \lambda_0 + (1-\alpha)\varphi(t,\lambda(t)) + \alpha\int_0^t \varphi(\eta,\lambda(\eta))\,\mathrm{d}\eta, \tag{12}$$

where $\varphi(t\lambda(t)) = (L_1(t,r(t),w(t)),L_2(t,r(t),w(t)))$ and $\lambda_0 = (K(t),G(t))$. It is worth noting that the solution of the fractional problem is bounded. In addition, since L_1 and L_2 are contraction, we get

$$\| \varphi(t,\lambda_1(t)) - \varphi(t,\lambda_2(t)) \| \leqslant \mu \| \lambda_1(t) - \lambda_2(t) \|, \tag{13}$$

where $\mu = \mu_1 + \mu_2 < 1$. Also, by using Eq. (4), we obtain

$$\begin{aligned}
\| \lambda(t) - \lambda_0 \| &= \| (1-\alpha)\varphi(t,\lambda(t)) + \alpha\int_0^t \varphi(\eta,\lambda(\eta))\,\mathrm{d}\eta \| \\
&\leqslant (1-\alpha) \| \varphi(t,\lambda(t)) \| + \alpha\int_0^t \| \varphi(\eta,\lambda(\eta)) \| \,\mathrm{d}\eta \\
&\leqslant (1-\alpha+\alpha t)\mu \\
&\leqslant \mu,
\end{aligned} \tag{14}$$

where $\mu < 1$.

Now, by using the definition of the Picard operator (12), we prove the contraction property of θ. We have

$$\begin{aligned}
\| \theta(\lambda_1(t)) - \theta(\lambda_2(t)) \| &= \| (1-\alpha)(\varphi(t,\lambda_1(t)) - \varphi(t,\lambda_2(t))) \\
&\quad + \alpha\int_0^t (\varphi(\eta,\lambda_1(\eta)) - \varphi(\eta,\lambda_2(\eta)))\,\mathrm{d}\eta \| \\
&\leqslant (1-\alpha) \| \varphi(t,\lambda_1(t)) - \varphi(t,\lambda_2(t)) \| \\
&\quad + \alpha\int_0^t \| \varphi(\eta,\lambda_1(\eta)) - \varphi(\eta,\lambda_2(\eta)) \| \,\mathrm{d}\eta \\
&\leqslant (1-\alpha)\mu \| \lambda_1(t) - \lambda_2(t) \| \\
&\quad + \alpha\mu\int_0^t \| \lambda_1(\eta) - \lambda_2(\eta) \| \,\mathrm{d}\eta \\
&\leqslant (1-\alpha+\alpha t)\mu \| \lambda_1(t) - \lambda_2(t) \| \\
&\leqslant \mu \| \lambda_1(t) - \lambda_2(t) \|,
\end{aligned} \tag{15}$$

since $\mu < 1$ by Eq. (14). Therefore, the defined operator θ is a contraction. Thus, by Banach's fixed point theorem [30], the operator θ has a unique fixed point, which is the unique solution of (1). This completes the proof. ■

4. Numerical method

In this section, we apply the Adomian Decomposition Method (ADM) [17] in order to implement the fractional system (1) in an appropriate manner.

In the decomposition method, we usually express the solution $r(t)$ and $w(t)$ of the integral equation (4) in a series form defined by

$$r(t)=\sum_{i=0}^{\infty} r_i(t) \quad \text{and} \quad w(t)=\sum_{i=0}^{\infty} w_i(t). \tag{16}$$

Substituting the decomposition (16) into both sides of (4) yields

$$\begin{aligned}\sum_{i=0}^{\infty} r_i(t) &= K(t)+(1-\alpha)\left(c_1\sum_{i=0}^{\infty} r_i(t)+c_2\sum_{i=0}^{\infty} w_i(t)\right)\\ &\quad+\alpha\int_0^t\left(c_1\sum_{i=0}^{\infty} r_i(\eta)+c_2\sum_{i=0}^{\infty} w_i(\eta)\right)\mathrm{d}\eta,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=0}^{\infty} w_i(t) &= G(t)+(1-\alpha)\left(c_3\sum_{i=0}^{\infty} r_i(t)+c_4\sum_{i=0}^{\infty} w_i(t)\right)\\ &\quad+\alpha\int_0^t\left(c_3\sum_{i=0}^{\infty} r_i(\eta)+c_4\sum_{i=0}^{\infty} w_i(\eta)\right)\mathrm{d}\eta.\end{aligned}$$

So the above discussed scheme for the determination of the components $r_0(t), r_1(t)$, $r_2(t), \ldots$ and $w_0(t), w_1(t), w_2(t), \ldots$ of the solution $r(t)$ and $w(t)$ of Eq. (4) respectively can be written in a recursive manner by

$$\begin{aligned}r_0(t) &= K(t),\\ r_{i+1}(t) &= (1-\alpha)(c_1 r_i(t)+c_2 w_i(t))+\alpha\int_0^t (c_1 r_i(\eta)+c_2 w_i(\eta))\,\mathrm{d}\eta, \quad i\geqslant 0,\end{aligned}$$

and

$$\begin{aligned}w_0(t) &= G(t),\\ w_{i+1}(t) &= (1-\alpha)(c_3 r_i(t)+c_4 w_i(t))\alpha\int_0^t (c_3 r_i(\eta)+c_4 w_i(\eta))\,\mathrm{d}\eta, \quad i\geqslant 0.\end{aligned}$$

5. Numerical illustrations

In this section, we apply the technique discussed in the previous section to find the numerical solution of the linear integral equations and compare our results with exact solutions.We used MATLAB to solve these examples.

Example 1 Consider the linear fractional differential equation, described as

$$\begin{cases} \mathscr{D}^{(\alpha)} r(t) = c_1 r(t) + c_2 w(t) + f(t), & t \in \Omega := [0,1] \\ \mathscr{D}^{(\alpha)} w(t) = c_3 r(t) + c_4 w(t) + g(t), & t \in \Omega := [0,1] \\ r(0) = 0, \ w(0) = 0, \end{cases}$$

where $\alpha = 0.75, \ c_1 = -\frac{1}{5}, \ c_2 = -\frac{4}{5}, \ c_3 = \frac{1}{8}, \ c_4 = -\frac{1}{7},$
$f(t) = 6(t - \frac{8}{5})\sin(t) - 2(t + \frac{7}{5})\cos(t) - \frac{58}{15}e^{-3t} + \frac{20}{3},$
$g(t) = 2(t + \frac{12}{5})\sin(t-1) + 6(t - \frac{3}{5})\cos(t-1) + \frac{6}{5}(4\sin(1) + 3\cos(1))e^{-3t}.$

The exact solution of problem is given by $r(t) = 5(x-1)(1-\cos(x))$ and $w(t) = 5x\sin(x)$. The obtained errors are presented in Figure 1.

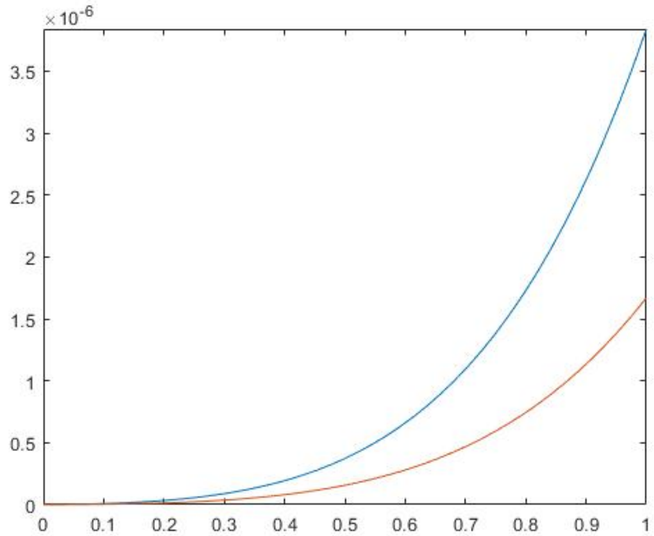

Fig. 1. The Absolute Error of test Example (1) with $N = 8$

Example 2 Consider the linear fractional differential equation, described as

$$\begin{cases} \mathscr{D}^{(\frac{1}{2})} r(t) = -\frac{1}{3} r(t) + \frac{2}{3} w(t) + ch(t) + \frac{1}{3}((2t+4)e^t - t - 7), & t \in \Omega := [0,1] \\ \mathscr{D}^{(\frac{1}{2})} w(t) = -\frac{1}{4} r(t) + \frac{1}{5} w(t) + sh(t) + \frac{(e^t - 1)(t+1)}{4} + \frac{4te^t}{5}, & t \in \Omega := [0,1] \\ r(0) = 0, \ w(0) = 0, \end{cases}$$

The exact solution of problem is given by $r(t) = (t+1)(e^t - 1)$ and $w(t) = te^t$.
The comparison of exact and numerical solution, and absolute errors are presented in Figures 2 and 3.

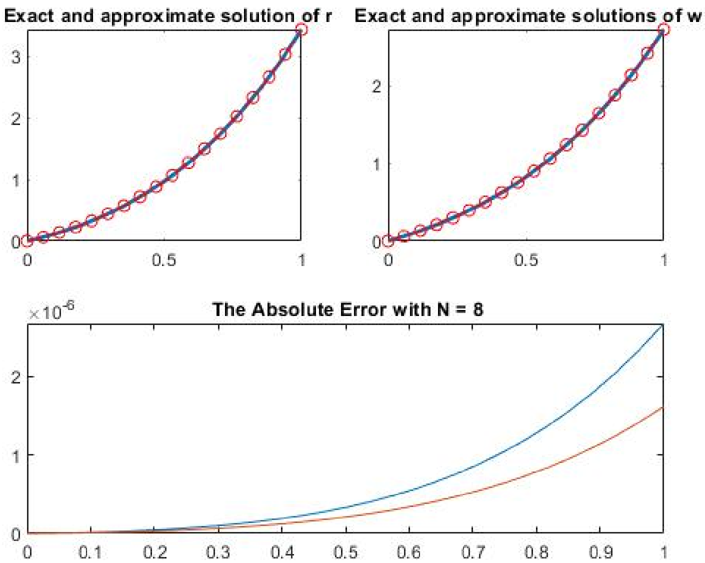

Fig. 2. Example (2) with $N = 8$

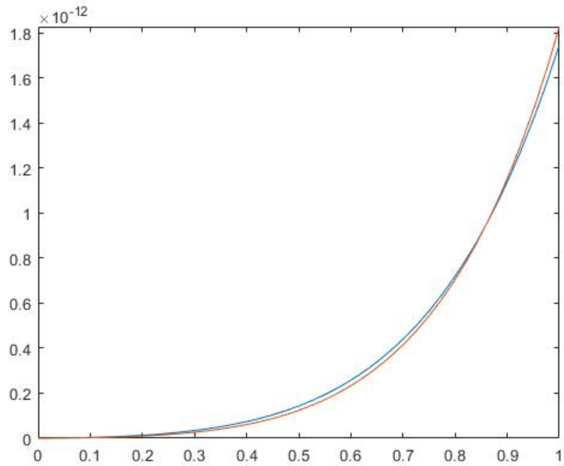

Fig. 3. The Absolute Error of test Example (2) with $N = 16$

6. Conclusion

In this study, we have investigated a new couple system of linear fractional differential equations involving Caputo-Fabrizio fractional derivatives with non-singular exponential kernels. The existence and uniqueness of solution was investigated by using the Picard-Lindelöf method and fixed point theory. Also, we used the Adomian decomposition method to calculate the approximate solution to the proposed system. The method provides a series of types of solution. In most cases, the series solution converges to the exact solution to the problem. From analysis and experimental results we can say that the proposed method provides a very high accurate estimate of the solution, and we observe that by using a high scale level, the iteration converges more rapidly, as shown by the Figure 3. Furthermore, the proposed method is easy to implement for the computation of solutions to various problems of fractional order differential equations.

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