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NUMERICAL STUDY FOR A SECOND ORDER FREDHOLM INTEGRO-DIFFERENTIAL EQUATION BY APPLYING GALERKIN-CHEBYSHEV-WAVELETS METHOD

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Abstract. In the present paper, we apply the Galerkin method using Chebyshev wavelets to approximate the exact solution for a second order Fredholm integro-differential equation with initial conditions. This numerical method gives us a nonlinear algebraic system that would be solved using the Picard successive approximations technique. Furthermore, we show the validity and the ability of the proposed method through some illustrative examples.

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1. Introduction

Recently, there has been a growing interest in the Integro-Differential Equations (IDEs), for these kinds of equations can be found in modeling real phenomena in many fields of sciences, physics, chemistry, biology and engineering problems, such as epidemic models [1, 2], the Boltzmann kinetic equation [3], and the Vlasov and Landau equations [4]. However, the exact solution to such equations is usually difficult to obtain, so the researchers use different numerical methods to approach the exact solution such as the Pade approximation, the Legendre-Galerkin method [5], the Hermite wavelet [6], the Haar wavelet [7], the Chebyshev wavelet collocation method [8, 9], the wavelet-Galerkin method [10], the Laguerre wavelets collocation method [11], the Laplace decomposition method [12], Bernoulli polynomials [13], and the B-spline method [14, 15].

In this article, we consider the following Fredholm integro-differential equation with initial conditions:

$$\begin{cases} \phi(x) = f(x) + \displaystyle\int_0^1 K(x,y,\phi(y),\phi'(y),\phi''(y))dy, \\ \phi(0) = \rho_1, \quad \phi'(0) = \rho_2, \end{cases} \tag{1}$$

where $\phi(x)$, $f(x) \in H^2([0,1])$, $K, \partial_x K, \partial_x^2 K \in C([0,1]^2 \times \mathbb{R}^3)$, and ρ_1, $\rho_2 \in \mathbb{R}$.

We mention that this type of equation has a specific form, whereas the unknown function and its derivatives appear inside of the nonlinear kernel, so the main result of this paper is to introduce a numerical method to approach the exact solution of the equation (1) by applying the Galerkin method with Chebyshev-Wavelets.

This paper is structured as follows: In section 2, we introduce some Chebyshev wavelet properties and function approximation. In section 3, we build the integration operational matrices. In section 4, we explain our numerical method, and in section 5, we prove the convergence analysis of the proposed method. Finally, in section 6, we demonstrate the accuracy and efficiency of our method with some illustrative examples.

2. Properties of Chebyshev wavelets

2.1. Chebyshev polynomials

The Chebyshev polynomials are obtained by expanding the formula:

$$T_n(x) = \cos(n \arccos(x)),$$

and satisfy the following recurrence relation:

$$\begin{cases} T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \\ T_1(x) = x, \quad T_0(x) = 1, \quad \text{for} \quad n \geqslant 1. \end{cases}$$

These polynomials are orthogonal with respect to the weight function $w(x) = \dfrac{1}{\sqrt{1-x^2}}$ on $[-1,1]$, and

$$\int_{-1}^{1} w(x)T_n(x)T_m(x)dx = \begin{cases} \pi, & n = m = 0, \\ \pi/2, & n = m \neq 0, \\ 0, & n \neq m. \end{cases}$$

The first few Chebyshev polynomials are:

$$\begin{aligned}
T_0(x) &= 1,\\
T_1(x) &= x,\\
T_2(x) &= 2x^2 - 1,\\
T_3(x) &= 4x^3 - 3x,\\
T_4(x) &= 8x^4 - 8x^2 + 1,\\
T_5(x) &= 16x^5 - 20x^3 + 5x.
\end{aligned}$$

2.2. Chebyshev wavelets

The wavelets are stretched versions of the original wavelet T with the same basic shape but at a different scale and frequency. We construct them from the translation parameter b and the scaling parameter a, then we have the following family of wavelets:

$$\theta_{a,b}(x) = \frac{1}{\sqrt{|a|}} T\left(\frac{x-b}{a}\right), \qquad a, b \in \mathbb{R}, \quad a \neq 0.$$

So, the Chebyshev wavelets are defined as:

$$\theta_{i,j}(x) = \begin{cases} 2^{\frac{k}{2}} \widetilde{T}_j(2^k x - 2i + 1), & \dfrac{i-1}{2^{k-1}} \leqslant x \leqslant \dfrac{i}{2^{k-1}},\\ 0, & \text{otherwise}, \end{cases}$$

and

$$\widetilde{T}_j(t) = \begin{cases} \dfrac{1}{\sqrt{\pi}}, & j = 0,\\ \sqrt{\dfrac{2}{\pi}} T_j(t), & j > 0, \end{cases}$$

where k is a positive integer number, $j = 0, 1, 2, \ldots, n-1$, $i = 1, 2, \ldots, 2^{k-1}$ and T_j is the Chebyshev polynomial of degree j. However, the family of Chebyshev wavelets $\{\theta_{i,j}\}$ defines an orthonormal basis for $L^2_{w_k}([0,1])$ such that:

$$w_k(x) = \begin{cases} w_{1,k}(x), & 0 \leq x < \dfrac{1}{2^{k-1}},\\ w_{2,k}(x), & \dfrac{1}{2^{k-1}} \leq x < \dfrac{2}{2^{k-1}},\\ \vdots & \vdots\\ w_{2^{k-1},k}(x), & \dfrac{2^{k-1}-1}{2^{k-1}} \leq x < 1, \end{cases}$$

where $w_{i,k}(x) = w\left(2^k x - 2i + 1\right)$. The Chebyshev wavelets charts are proven for $k = 1$ and $n = 1, 2, 3, 4$ in Figure 1.

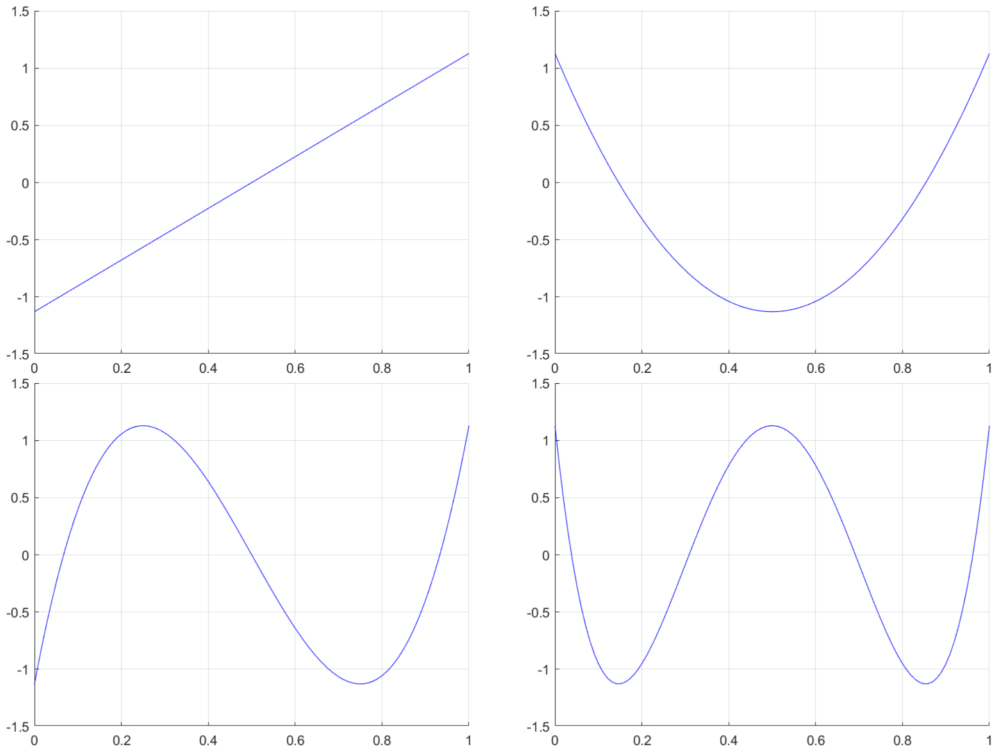

Fig. 1. Chebyshev wavelets for $k = 1$

For any function $\phi(x)$ in $L^2_{w_k}([0,1])$ can be expanded as:

$$\phi(x) = \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} c_{i,j} \theta_{i,j}(x), \tag{2}$$

where $c_{i,j} = \langle \phi, \theta_{i,j} \rangle$, such that $\langle ., . \rangle$ is the inner product in $L^2_{w_k}([0,1])$. So we approximate the function $\phi(x)$ by truncating the infinite series (2):

$$\phi_n(x) = \sum_{i=1}^{2^{k-1}} \sum_{j=0}^{n-1} c_{i,j} \theta_{i,j}(x) = C^T P(x), \tag{3}$$

where, C^T and $P(x)$ are $2^{k-1} n \times 1$ matrices:

$$C^T = [c_{1,0}, c_{1,1}, ..., c_{1,n-1}, c_{1,0}, c_{2,1}, ..., c_{2,n-1}, ..., c_{2^{k-1},0}, ..., c_{2^{k-1},n-1}],$$

and

$$P(x) = [\theta_{1,0}, \theta_{1,1}, ..., \theta_{1,n-1}, \theta_{2,0}, \theta_{2,1}, ..., \theta_{2,n-1}, ..., \theta_{2^{k-1},0}, \theta_{2^{k-1},1} ..., \theta_{2^{k-1},n-1}]^T.$$

3. Chebyshev wavelet operational matrix of integration

If we take $k=2$, then the matrices C^T and $P(x)$ would be:

$$C^T = [c_{1,0}, c_{1,1}, ..., c_{1,n-1}, c_{2,0}, c_{2,1}, ..., c_{2,n-1}].$$

$$P(x) = [\theta_{1,0}(x), \theta_{1,1}(x), ..., \theta_{1,n-1}(x), \theta_{2,0}(x), \theta_{2,1}(x), ..., \theta_{2,n-1}(x)].$$

Let W_n be a matrix that contains the coefficients of Chebyshev wavelets:

$$W_n = \begin{pmatrix} F_n & O_n \\ O_n & \tilde{F}_n \end{pmatrix},$$

$$F_n = \frac{2}{\sqrt{\pi}} \begin{pmatrix} 1 & -\sqrt{2} & \cdots & (-1)^{n-1}\sqrt{2} \\ 0 & 4\sqrt{2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 4^{n-1}\sqrt{2} \end{pmatrix}, \quad \tilde{F}_n = \frac{2}{\sqrt{\pi}} \begin{pmatrix} 1 & -3\sqrt{2} & \cdots & T_{n-1}(-3)\sqrt{2} \\ 0 & 4\sqrt{2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 4^{n-1}\sqrt{2} \end{pmatrix},$$

and

$$X_n(x) = (1, x, x^2, ..., x^{n-1}, 1, x, x^2, ..., x^{n-1}),$$

$$P_n(x) = [\theta_{1,0}, \theta_{1,1}, ..., \theta_{1,n-1}, \theta_{2,0}, \theta_{2,1}, ..., \theta_{2,n-1}].$$

So, we can write:

$$P_n(x) = X_n(x) W_n.$$

Let N_n be an integral matrix in classical basis for polynomial space:

$$N_n = \begin{pmatrix} G_n & O_n \\ O_n & \tilde{G}_n \end{pmatrix},$$

$$G_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{pmatrix}, \quad \tilde{G}_n = \begin{pmatrix} -\frac{1}{2} & 1 & 0 & 0 & \cdots & 0 \\ -\frac{1}{2^2} & 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{2^n} & 0 & 0 & 0 & \cdots & \frac{1}{n} \end{pmatrix},$$

then, we have the first integration matrix:

$$\int_0^x C^T P_n(y)dy = C^T M_n P_{n+1}(x) = C^T Q_1(x),$$

and the double integration matrix:

$$\int_0^x \int_0^z C^T P_n(y)dydz = C^T M_n M_{n+1} P_{n+2}(x) = C^T Q_2(x),$$

where,

$$M_n = W_n^{-1} N_n W_{n+1}.$$

4. Description of method

Let the following Fredholm integro-differential equation:

$$\begin{cases} \phi(x) = f(x) + \int_0^1 K(x,y,\phi(y),\phi'(y),\phi''(y))dy, \\ \phi(0) = \rho_1, \quad \phi'(0) = \rho_2. \end{cases} \tag{4}$$

First, we derive the equation (4) twice to obtain the following equation:

$$\phi''(x) = f''(x) + \int_0^1 \partial_x^2 K(x,y,\phi(y),\phi'(y),\phi''(y))dy, \tag{5}$$

we approach the function $\phi''(x)$ using Chebyshev wavelets as:

$$\phi_n''(x) = C^T P_n(x), \tag{6}$$

integrate the equation (6) from 0 to x, we obtain:

$$\phi_n'(x) = \rho_2 + C^T Q_1(x), \tag{7}$$

integrate again (7) from 0 to x, then we get:

$$\phi_n(x) = \rho_1 + \rho_2 x + C^T Q_2(x), \tag{8}$$

by substituting (6), (7) and (8) in (5), we obtain:

$$C^T P_n(x) = f''(x) + \int_0^1 \partial_x^2 K(x,y,\rho_1+\rho_2 y + C^T Q_2(y), \rho_2 + C^T Q_1(y), C^T P_n(y))dy. \tag{9}$$

Multiplying equation (9) by $\theta_{i,j}(x)w_{i,2}(x)$ for $i = 1,2$ and $j = 0,1,\cdots,n-1$, after that we integrate with respect to x from 0 to 1, then we get the following nonlinear algebraic system:

$$c_{i,j} = y_{i,j} + \int_0^1 \int_0^1 \theta_{i,j}(x) w_{i,2}(x) \partial_x^2 K\left(x, y, \rho_1 + \rho_2 y + C^T Q_2(y), \rho_2 + C^T Q_1(y), C^T P_n(y)\right) dy dx, \tag{10}$$

where $y_{i,j} = \langle f'', \theta_{i,j} \rangle$. However, we can find the vector C^T solution of the above system (10) by using the Picard successive approximations method, then we substitute them in (8) to get the numerical solution of the proposed equation (1).

5. The convergence analysis

First, to prove the convergence analysis of the proposed numerical process described above, consider the Sobolev space $\mathscr{H} = H^2([0,1], \mathbb{R})$ equipped with the following norm:

$$\forall \phi \in \mathscr{H},\ \|\phi\|_{\mathscr{H}} = \|\phi\|_{L^2[0,1]} + \|\phi'\|_{L^2[0,1]} + \|\phi''\|_{L^2[0,1]}$$

Furthermore, let's suppose the following assumptions:

$$(\mathscr{A}) \begin{cases} \exists A_m, B_m, C_m > 0, \text{where } m = 0,1,2. \forall x, y \in [0,1], \forall u, \bar{u}, v, \bar{v}, w, \bar{w} \in \mathbb{R}, \\ |K(x,y,u,v,w) - K(x,y,\bar{u},\bar{v},\bar{w})| \leqslant A_0|u-\bar{u}| + B_0|v-\bar{v}| + C_0|w-\bar{w}|, \\ |\partial_x K(x,y,u,v,w) - \partial_x K(x,y,\bar{u},\bar{v},\bar{w})| \leqslant A_1|u-\bar{u}| + B_1|v-\bar{v}| + C_1|w-\bar{w}|, \\ |\partial_x^2 K(x,y,u,v,w) - \partial_x^2 K(x,y,\bar{u},\bar{v},\bar{w})| \leqslant A_2|u-\bar{u}| + B_2|v-\bar{v}| + C_2|w-\bar{w}|, \\ 0 < \gamma = \max\left\{ \sum_{m=0}^{2} A_m, \sum_{m=0}^{2} B_m, \sum_{m=0}^{2} C_m \right\} < 1. \end{cases}$$

Theorem 1 *According to assumptions $(\mathscr{A})$, the approximate solution ϕ_n converges to the exact solution ϕ in the space $\mathscr{H}$.* □

PROOF Consider the following operator T defined from $\mathscr{H}$ to itself by:

$$\forall x \in [0,1], \quad T : \mathscr{H} \longrightarrow \mathscr{H}$$

$$\phi \longmapsto T(\phi)(x) = f(x) + \int_0^1 K\left(x, y, \phi(y), \phi'(y), \phi''(y)\right) dy.$$

So, the exact solution ϕ of our equation (1) and its derivatives ϕ' and ϕ'' can be represented by the following system:

$$\begin{cases} \phi(x) = T(\phi)(x), \\ \phi'(x) = T'(\phi)(x), \\ \phi''(x) = T''(\phi)(x). \end{cases}$$

By applying the Galerkin projection method using Chebyshev wavelets given by (3), the previous system will be approached by:

$$\begin{cases} \phi_n(x) = T_n(\phi_n)(x), \\ \phi_n'(x) = T_n'(\phi_n)(x), \\ \phi_n''(x) = T_n''(\phi_n)(x). \end{cases}$$

It's clear that by applying the triangle inequality, we get:

$$\begin{aligned} |\phi_n(x) - \phi(x)| = |T_n(\phi_n) - T(\phi)| &= |T_n(\phi_n) - T(\phi_n) + T(\phi_n) - T(\phi)|, \\ &\leqslant |T_n(\phi_n) - T(\phi_n)| + |T(\phi_n) - T(\phi)|. \end{aligned}$$

According to assumptions $(\mathscr{A})$ and Cauchy Schwarz inequality, we achieve:

$$\begin{aligned} |T(\phi_n) - T(\phi)| &= \left| \int_0^1 (K(x,y,\phi_n(y),\phi_n'(y),\phi_n''(y)) - K(x,y,\phi(y),\phi'(y),\phi''(y)))dy \right|, \\ &\leqslant A_0 \int_0^1 |\phi_n(y) - \phi(y)|dy + B_0 \int_0^1 |\phi_n'(y) - \phi'(y)|dy + C_0 \int_0^1 |\phi_n''(y) - \phi''(y)|dy, \\ &\leqslant A_0 \|\phi_n - \phi\|_{L^2[0,1]} + B_0 \|\phi_n' - \phi'\|_{L^2[0,1]} + C_0 \|\phi_n'' - \phi''\|_{L^2[0,1]}. \end{aligned} \tag{11}$$

On the other hand, the study [16] assumes that sequence $S_n = T_n(\phi_n)$ is convergent, and gives us the following error convergence order:

$$|T_n(\phi_n) - T(\phi_n)| \leqslant \mathscr{O}(n^{\mu_0}) \to 0. \tag{12}$$

So, from inequalities (11) and (12), we obtain:

$$\|\phi_n - \phi\|_{L^2[0,1]} \leqslant A_0 \|\phi_n - \phi\|_{L^2[0,1]} + B_0 \|\phi_n' - \phi'\|_{L^2[0,1]} + C_0 \|\phi_n'' - \phi''\|_{L^2[0,1]} + \mathscr{O}(n^{\mu_0}).$$

Similarly, we find:

$$\begin{aligned} \|\phi_n' - \phi'\|_{L^2[0,1]} &\leqslant A_1 \|\phi_n - \phi\|_{L^2[0,1]} + B_1 \|\phi_n' - \phi'\|_{L^2[0,1]} + C_1 \|\phi_n'' - \phi''\|_{L^2[0,1]} + \mathscr{O}(n^{\mu_1}), \\ \|\phi_n'' - \phi''\|_{L^2[0,1]} &\leqslant A_2 \|\phi_n - \phi\|_{L^2[0,1]} + B_2 \|\phi_n' - \phi'\|_{L^2[0,1]} + C_2 \|\phi_n'' - \phi''\|_{L^2[0,1]} + \mathscr{O}(n^{\mu_2}). \end{aligned}$$

Therefore, if we put $\mathscr{O}(n^{\mu}) = \mathscr{O}(n^{\mu_0}) + \mathscr{O}(n^{\mu_1}) + \mathscr{O}(n^{\mu_2}) \to 0$, and according to $0 < \gamma < 1$, we can conclude that:

$$\|\phi_n - \phi\|_{\mathscr{H}} \leqslant \frac{\mathscr{O}(n^{\mu})}{1-\gamma} \to 0.$$

Which confirms the convergence of the approximate solution ϕ_n to ϕ in the space $\mathscr{H}$. ∎

6. Examples

In this section, we apply our numerical method on two illustrative examples, in order to verify its accuracy and validity. For this reason, we define the error function

E_n by:

$$E_n = \|\phi_n - \phi\|_{\mathscr{H}},$$

where ϕ is the exact solution of the example, and ϕ_n is the numerical solution obtained using Matlab computation software.

6.1. Example 1

Consider the following equation:

$$\begin{cases} \phi(x) = f(x) + \int_0^1 K(x,y,\phi(y),\phi'(y),\phi''(y))dy, & \forall x \in [0,1], \\ \phi(0) = 0, \quad \phi'(0) = 0, \end{cases}$$

with

$$f(x) = x\sin(x) + \frac{1}{2}\sin(x)(2ln(2)-1),$$

$$K(x,y,u,v,w) = -\frac{1}{2}\sin(x)ln[1+\sin(y)u+\cos(y)v - \frac{1}{2}\sin(y)(w-\sin(y))],$$

and the exact solution is $\phi(x) = x\sin(x)$.

6.2. Example 2

Let the Fredholm integro-differential equation:

$$\begin{cases} \phi(x) = f(x) - \int_0^1 \frac{3}{4}\cos(x)\sin\left[\frac{1}{2}e^{-y}(2\phi''(y)-\phi'(y)-\phi(y))\right]ds, & \forall x \in [0,1], \\ \phi(0) = \frac{1}{4}, \quad \phi'(0) = -\frac{3}{4}, \end{cases}$$

with

$$f(x) = \begin{cases} \frac{1}{4}\left[(2x-1)^2e^x + \cos(x)[\cos(\frac{1}{2}) - 2\cos(2) + \cos(\frac{7}{2})]\right], & 0 \leqslant x \leqslant \frac{1}{2}, \\ \frac{1}{4}\left[-(2x-1)^2e^x + \cos(x)[\cos(\frac{1}{2}) - 2\cos(2) + \cos(\frac{7}{2})]\right], & \frac{1}{2} \leqslant x \leqslant 1, \end{cases}$$

and the exact solution is:

$$\phi(x) = \begin{cases} \frac{1}{4}(2x-1)^2e^x, & 0 \leqslant x \leqslant \frac{1}{2}, \\ -\frac{1}{4}(2x-1)^2e^x, & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

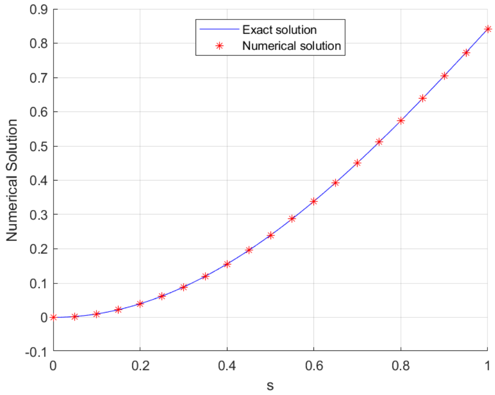

Fig. 2. Graphical comparison of ϕ vs ϕ_n for Example 1, with $n = 7$

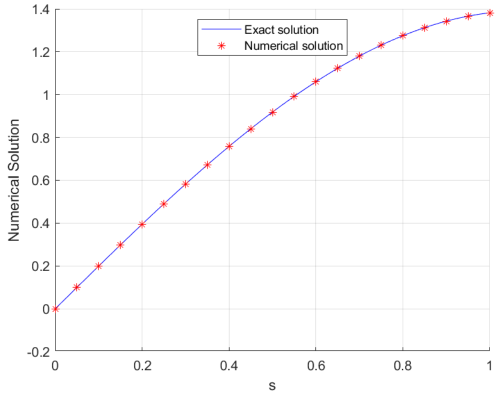

Fig. 3. Graphical comparison of ϕ' vs ϕ_n' for Example 1, with $n = 7$

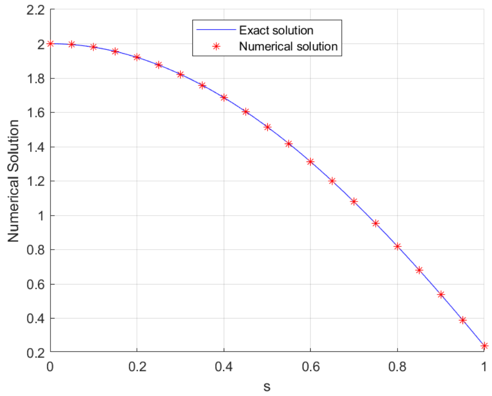

Fig. 4. Graphical comparison of ϕ'' vs ϕ_n'' for Example 1, with $n = 7$

Table 1. Numerical results of Example 1

n	E_n	CPU time
3	6.9888e-05	0.109
4	1.4336e-06	0.123
5	4.9340e-08	0.154
6	6.7237e-10	0.194
7	5.4441e-10	0.310

Table 2. Numerical results of Example 2

n	E_n	CPU time
3	7.9499e-04	0.081
4	2.7590e-05	0.108
5	7.6178e-07	0.129
6	1.7419e-08	0.169
7	8.5031e-09	0.372

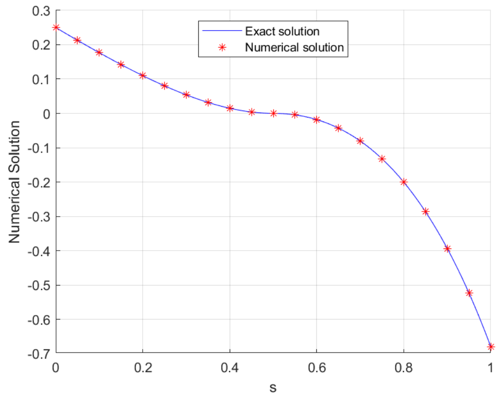

Fig. 5. Graphical comparison of ϕ vs ϕ_n for Example 2, with $n = 7$

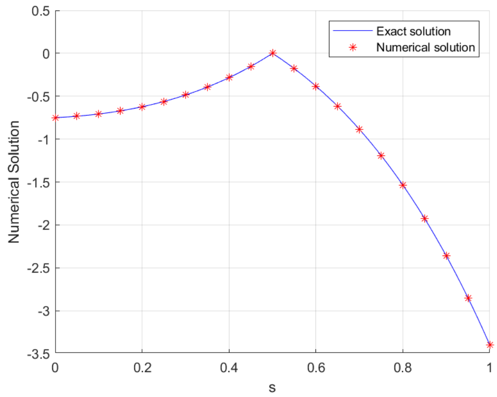

Fig. 6. Graphical comparison of ϕ' vs ϕ'_n for Example 2, with $n = 7$

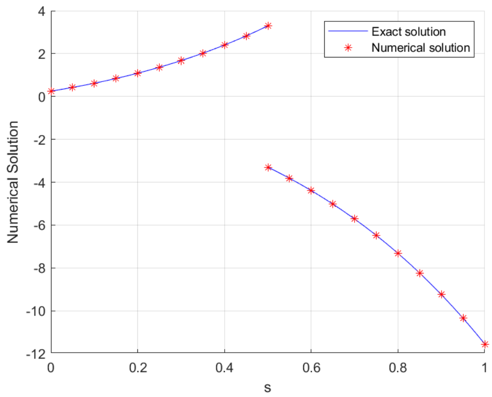

Fig. 7. Graphical comparison of ϕ'' vs ϕ''_n for Example 2, with $n = 7$

Discussion: We observe from the tables that the error function is almost null, especially for big values of the number n. Therefore, our proposed method is more effective whenever the degree of the polynomial (approximate solution) n is greater.

7. Conclusion

In this paper, the Galerkin method has been applied using the Chebyshev wavelets to approach the exact solution for a nonlinear Fredholm integro-differential equation of the second order with the initial conditions. The present method allowed us to reduce the equation into a nonlinear algebraic system, then we solve this system using the MATLAB tool. The numerical examples have proved the accuracy and validity of the proposed method.

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