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NUMERICAL SCHEME METHODS FOR SOLVING NONLINEAR PSEUDO-HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. The numerical solutions to the nonlinear pseudo-hyperbolic partial differential equation with nonlocal conditions are presented in this study. This equation is solved using the homotopy analysis technique (HAM) and the variational iteration method (VIM). Both strategies are compared and contrasted in terms of approximate and accurate solutions. The results show that the HAM technique is more appropriate, effective, and close to the exact solution than the VIM method. Finally, the graphical representations of the obtained results are given.

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1. Introduction

Most of the problems in physics and engineering can be described by means of nonlinear differential equations. The nonlinear partial differential equations are able to describe various phenomena such as sound, diffusion, electrodynamics, fluid dynamics, elasticity, gravity and quantum mechanics. These physical phenomena can be shaped to some extent in partial differential equations. But arriving at accurate solutions to nonlinear partial differential equations is not easy. Consequently, analytical methods were used to find approximate solutions.

In recent years, many authors have been interested in testing partial differential equations using different numerical and analytical methods, among these are the Adomian decomposition method [1], the homotopy perturbation method [2], the multiple Exp-function method [3], the Homotopy analysis Rangaig transform method [4] the finite difference method [5, 6], the variational iteration method [7], the residual power series method [8], the He's variational iteration method [9] and the homotopy analysis method [10] have been applied to solve linear and nonlinear partial differential equations.

The homotopy analysis method has been suggested by Liao [11] and is an effective method for solving nonlinear problems. This analytical technique is based on the homotopy of the topology. From the beginning, the auxiliary parameter h is not given, but simply followed the conventional concept of Homotopy to build the parameter h from the equations. The variational iteration method was first proposed by He [12] and provides a quick, successive approximation of an accurate solution if such a solution exists. The suggested method has been successfully developed for creating analytical solutions for both linear and nonlinear partial differential equations.

The hyperbolic partial differential equations represent a model for the Longitudinal vibrations of structures, such as buildings, beams, and machines and are the basis for equations of atomic physics [13, 14], and hyperbolic partial differential equations were studied to model the real engineering problems [15–17].

The aim of this paper is to find the approximate solutions of the nonlinear pseudo--hyperbolic partial differential equation depending on nonlocal conditions by the homotopy perturbation method and the variational iteration method. Finally, compare the result obtained from both methods with the exact solutions and present them graphically.

2. The variational iteration method for solving nonlinear pseudo-hyperbolic partial differential equations

In this section, we develop a VIM for a nonlinear pseudo-hyperbolic equation, to illustrate the basic idea of the VIM method [7].

Firstly, we consider the generalized nonlinear pseudo-hyperbolic partial differential equations depending on nonlocal conditions in [18] as follows:

$$u_{tt}(x,t) - u(x,t)u_{txx}(x,t) - u_{xx}(x,t) - f(t,x) = 0, \quad x \in (0,X), \quad t \in (0,T) \tag{1}$$

with the initial and boundary conditions

$$u(x,0) = V_0(x),\ u_t(x,0) = V_1(x),\ x \in [0,X], \tag{2}$$

$$u(0,t) = \alpha(t) + \int_0^X u(x,t)dx,\ t \in [0,T], \tag{3}$$

$$u(X,t) = \beta(t) + \int_0^X u(x,t)dx,\ t \in [0,T]. \tag{4}$$

The basic structure of VIM is a functional correction construct of (1), as

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t)+\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial\xi^2}-u_n(x,\xi)\frac{\partial}{\partial\xi}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)\right. \\ &\quad \left.-\frac{\partial^2 u_n(x,\xi)}{\partial x^2}-f(x,\xi)\right]\partial\xi. \end{aligned} \tag{5}$$

Where λ is the Lagrange multiplier, which can be specified optimally through the variational theories and $\tilde{u}_n$ is a finite difference requiring $\delta\tilde{u}_n=0$.

By solving equation (5), the Lagrange λ multiplier must be defined which will be determined optimally through integration by the parts formula. The consecutive estimates $u_n(x,t), n>0$ for solution $u(x,t)$ will be obtained quickly by utilizing the multiple of the obtained Lagrange and any selective function $u_0(x,t)$. The solution is then given by

$$u(x,t)=\lim_{n\to\infty} u_n(x,t). \tag{6}$$

According to the variational iteration method of Equation (1), this formula will be completed as follows

$$\begin{cases} u(x,0)=V_0(x)+tV_1(x) \quad \text{is initial guess} \\ u_{n+1}(x,t)=u_n(x,t)+\displaystyle\int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial\xi^2}-u_n(x,\xi)\frac{\partial}{\partial\xi}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)\right. \\ \left.-\dfrac{\partial^2 u_n(x,\xi)}{\partial x^2}-f(x,\xi)\right]\partial\xi. \end{cases} \tag{7}$$

The Lagrange multiplier λ must be specified in the next section.

3. Applications of the VIM

In this section, we give the applications of the variational iteration method for the nonlinear pseudo-hyperbolic partial differential equation.

Example 3.1. We consider the nonlinear pseudo-hyperbolic partial differential equation as

$$u_{tt}(x,t)-u(x,t)u_{txx}(x,t)-u_{xx}(x,t)-e^{-t}sin(x)(e^{-t}sin(x)-2)=0, \quad x\in(0,\pi), \quad t\in(0,1) \tag{8}$$

with the initial and boundary conditions

$$u(x,0)=\sin(x) \quad u_t(x,0)=-\sin(x), \quad x\in[0,\pi], \tag{9}$$

$$u(0,t)=\int_0^\pi u(x,t)dx-2e^{-t}=u(\pi,t), \quad 0\le t\le 1. \tag{10}$$

The corrective function of a certain equation is roughly expressed as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\xi)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - u_n(x,\xi)\frac{\partial}{\partial \xi}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)\right.$$
$$\left. - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - f(x,\xi)\right]\partial \xi.$$

The fixed conditions give

$$\lambda(\xi) = \xi - t.$$

By substituting the value of the Lagrange multiplier with the correction function, the iteration formula can be given as follows

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - u_n(x,\xi)\frac{\partial}{\partial \xi}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)\right.$$
$$\left. - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} - f(x,\xi)\right]\partial \xi.$$

According to the VIM method, $u_0(x,t)$ can be determined from the initial condition (9). With this determination, the successive estimates should be obtained as

$$u_0(x,t) = \sin x - t\sin x + \frac{t^2}{2}\sin x,$$

$$u_1(x,t) = u_0(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - u_n(x,\xi)\frac{\partial}{\partial \xi}\left(\frac{\partial^2 u_n(x,\xi)}{\partial x^2}\right)\right.$$
$$\left. - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + (e^{-\xi}\sin(x)(e^{-\xi}\sin(x) - 2))\right]\partial \xi,$$

by simplifying the above equation, it gives the following form

$$\begin{aligned} u_1(x,t) \quad = \quad & -\frac{1}{120}\left(\left(\left(\left(\left(\left(45t + \frac{45}{2}\right)\sin^2(x) + (t^6 - 10t^4 + 40t^3 - 60t^2\right.\right.\right.\right.\right. \quad (11) \\ & \left.\left. +240t + 480)\sin(x) + 60t^2\right)e^{2t} - 120\sin(x)\left(\left(-\frac{t^2}{2} - \frac{3}{16}\right)\right.\right. \\ & \left.\left.\left.\left.\sin(x) + e^t\left(t^2 - 2t + 4\right)\right)\right)e^{-2t} - 60t^2 + 120t - 120\right)\sin(x)\right). \end{aligned}$$

4. The homotopy analysis method for solving nonlinear pseudo-hyperbolic partial differential equations with nonlocal conditions

This section includes a discussion of the basic scheme of homotopy analysis method [10] for equations (1) and (2) as follows:

to solve equation (1) by means of the homotopy analysis method, according to the initial conditions (2), we have

$$L[\varphi(x,t,q)] = \frac{\partial^2 \varphi(x,t,q)}{\partial t^2}, L[c_1 + c_2 t] = 0, \tag{12}$$

where L is linear operator, $c_1$and c_2 are constants coefficients and φ is the real function. Now define the nonlinear operator as:

$$L[\varphi(x,t,q)] = \frac{\partial^2 \varphi(x,t,q)}{\partial t^2} - \varphi(x,t,q)\frac{\partial^3 \varphi(x,t,q)}{\partial t \partial x^2} - \frac{\partial^2 \varphi(x,t,q)}{\partial x^2} - f(x,t), \tag{13}$$

by using (13) with the assumption, we construct the zeroth order deformation equation

$$(1-q)L[\varphi(r,t,q) - u_0(r,t)] = qhH(r,t)N[\varphi(r,t,q)]. \tag{14}$$

Assuming $H(r,t) = 1$. It is important to have great freedom in choosing the auxiliary parameter in the HAM. Clearly, when $p = 0$ and $p = 1$, it holds

$$\begin{cases} \varphi_1(x,t,0) = u_0(x,t) \\ \varphi_2(x,t,1) = u(x,t). \end{cases} \tag{15}$$

Thus, we obtain the m^{th} order deformation equation

$$L[u_m - x_m u_{m-1}] + hH(r,t)R_m(\vec{u}_{m-1},t), \tag{16}$$

where

$$x_m = \begin{cases} 0, & \text{if } m \leq 1 \\ 1, & \text{if } m > 1, \end{cases} \tag{17}$$

and

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} - \sum_{k=1}^{m-1} u_{m-1}(x,t)\frac{\partial^3 u_{m-1-k}(x,t)}{\partial t \partial x^2} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - f(x,t). \tag{18}$$

Applying L^{-1} on both sides of (18) and using the (HAM) to Equations (1) and (2), we have

$$u_m(x,t) = x_m u_{m-1}(x,t) + hH(r,t)L^{-1}[R_m(\vec{u}_{m-1},x,t)]. \tag{19}$$

Now, apply HAM of equation (1), (2), and since $m > 1$, $x_m = 1$, $H(r,t) = 1$, in equation (19), we get

$$u_m(x,t) = u_{m-1}(x,t) + hL^{-1}[R_m(u_{m-1},x,t)]. \tag{20}$$

And

$$L^{-1} = \int_0^t \int_0^t (.)dtdt.$$

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t). \tag{21}$$

5. Applications of the HAM

In this section, we give applications of the homotopy analysis method for (Example 3.1).

To solve equation (8) according to initial condition (9) by the HAM method, we have

$$L[\varphi(x,t,q)] = \frac{\partial^2 \varphi(x,t,q)}{\partial t^2}, \quad L[c_1 + c_2 t] = 0, \tag{22}$$

where L is linear operator, $c_1$and c_2 are constant coefficients and φ is the real function. Now define the nonlinear operator as:

$$\begin{aligned} L[\varphi(x,t,q)] \quad = \quad & -\frac{\partial^2 \varphi(x,t,q)}{\partial t^2} - \varphi(x,t,q)\frac{\partial^3 \varphi(x,t,q)}{\partial t \partial x^2} - \frac{\partial^2 \varphi(x,t,q)}{\partial x^2} \\ & - e^{-t}\sin(x)(e^{-t}\sin(x) - 2). \end{aligned} \tag{23}$$

by using the above equation, it constructs the zeroth order deformation equations

$$(1-q)L[\varphi(r,t,q) - u_0(r,t)] = qhH(r,t)N[\varphi(r,t,q)]. \tag{24}$$

Assuming $H(r,t) = 1$. We can choose the auxiliary parameter h in the HAM method. Clearly, when $p = 0$ and $p = 1$, then

$$\begin{cases} \varphi_1(x,t,0) = u_0(x,t) \\ \varphi_2(x,t,1) = u(x,t). \end{cases} \tag{25}$$

Thus, we obtain the m^{th} order distortion equation

$$L[u_m - x_m u_{m-1}] + hH(r,t)R_m(\vec{u}_{m-1},t), \tag{26}$$

where

$$x_m = \begin{cases} 0, & \text{if } m \le 1, \\ 1, & \text{if } m > 1, \end{cases} \tag{27}$$

and

$$R_m(u_{m-1},x,t) = \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} - \sum_{k=1}^{m-1} u_{m-1}(x,t)\frac{\partial^3 u_{m-1-k}(x,t)}{\partial t \partial x^2} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - e^{-t}\sin(x)(e^{-t}\sin(x)-2). \tag{28}$$

Applying L^{-1} to both sides of (26) and also applying the HAM method to equations (8) and (9), we get

$$u_m(x,t) = x_m u_{m-1}(x,t) + hH(r,t)L^{-1}[R_m(\vec{u}_{m-1},x,t)]. \tag{29}$$

Now, applying the HAM on equations (8) and (9), we chose $h = 1/13$, and since $m > 1$, $x_m = 1$, $H(r,t) = 1$, using iteration formula of the HAM method, then it gives

$$u_0(x,t) = \sin x - t\sin x + \frac{t^2}{2}\sin x, \tag{30}$$

$$\begin{aligned} u_1(x,t) = {} & 2h\sin(x) - \frac{h}{4}\sin^2(x) + \frac{ht^3}{3}\sin^2(x) + -2ht sin^2(x) + \\ & \frac{ht}{2}\sin^2(x) + ht^2\sin(x) - \frac{ht^3}{6}\sin^2(x) + \frac{ht^4}{24}\sin^2(x) - 2he^{-t}\sin(x) \\ & -\frac{ht^2}{2}\sin^2(x) - \frac{ht^4}{8}\sin^2(x) + \frac{ht^5}{40}\sin^2(x) + \frac{h}{4}e^{-2t}\sin^2(x) \\ & \vdots \end{aligned} \tag{31}$$

6. Numerical implementation

This section summarizes and discusses the main findings. The proposed methods for obtaining approximate analytical solutions by numerical simulation are explained in the following points. Table 1 shows a comparison between the exact and the approximate solutions at different times with the absolute error by the HAM method. The obtained result showed that the current method is accurate and provides effective results. Table 2 illustrates, a comparison between the exact and approximate solutions at various points with the absolute errors by the VIM method. Figure 1 shows the exact solution of the $u(x,t)$ at $x \in [0,\pi]$, and $t \in [0,1]$. While Figures 2 and 3 give the approximate solution of the $u(x,t)$ by the VIM method and the HAM method respectively.

Table 1. Error Analysis using HAM

x	t	Exact solutions	HAM solution	ε
$19/20$	$19\pi/20$	0.06049962510	0.05943495638	0.00106466872
$10/20$	$10\pi/20$	0.6065306597	0.6013998736	0.0051307861
$5/20$	$5\pi/20$	0.5506953150	0.5463858245	0.0043094905
$1/20$	$\pi/20$	0.1488050662	0.1487250445	0.0000800217
$1/40$	$\pi/40$	0.07652193376	0.07651115176	0.00001078200
$1/80$	$\pi/80$	0.03877212251	0.03877075664	1.36587×10^{-6}
$1/100$	$\pi/100$	0.03109821680	0.03109748955	7.2725×10^{-7}
$1/160$	$\pi/160$	0.01951136456	0.01951121756	1.4700×10^{-7}
$1/200$	$\pi/200$	0.01562897674	0.01562886910	1.0764×10^{-7}
$1/250$	$\pi/250$	0.01251587612	0.01251581998	5.614×10^{-8}
$1/400$	$\pi/400$	0.007834290661	0.007834263665	2.6996×10^{-8}

Table 1 gives a comparison between the exact solutions and approximate solutions by the HAM method to the nonlinear pseudo-hyperbolic partial differential equation at various points by absolute errors, where $h = 1/13$ and $m > 1$. Here, $\varepsilon =$Absolute errors$|u_{exact} - u_{HAM}|$.

Table 2. Error Analysis using VIM

x	t	Exact solutions	VIM solution	ε
$19/20$	$19\pi/20$	0.06049962510	0.02555367739	0.03494594771
$10/20$	$10\pi/20$	0.6065306597	0.6922237961	0.0856931364
$5/20$	$5\pi/20$	0.5506953150	0.5614145770	0.0107192620
$1/20$	$\pi/20$	0.1488050662	0.1486969600	0.0001081062
$1/40$	$\pi/40$	0.07652193376	0.07650314703	0.00001878673
$1/80$	$\pi/80$	0.03877212251	0.03876942167	2.70084×10^{-6}
$1/100$	$\pi/100$	0.03109821680	0.03109679697	1.41983×10^{-6}
$1/160$	$\pi/160$	0.01951136456	0.01951100424	3.6032×10^{-7}
$1/200$	$\pi/200$	0.01562897674	0.01562878990	1.8684×10^{-7}
$1/250$	$\pi/250$	0.01251587612	0.01251577949	9.663×10^{-8}
$1/400$	$\pi/400$	0.007834290661	0.007834266713	2.3948×10^{-8}

Table 2 illustrates the comparison between the exact solutions and approximate solutions by the VIM to the nonlinear pseudo-hyperbolic partial differential equation with non-local conditions at various points by absolute errors.

Here, $\varepsilon =$Absolute errors$|u_{exact} - u_{VIM}|$.

The graphical representation of the solutions is as follows:

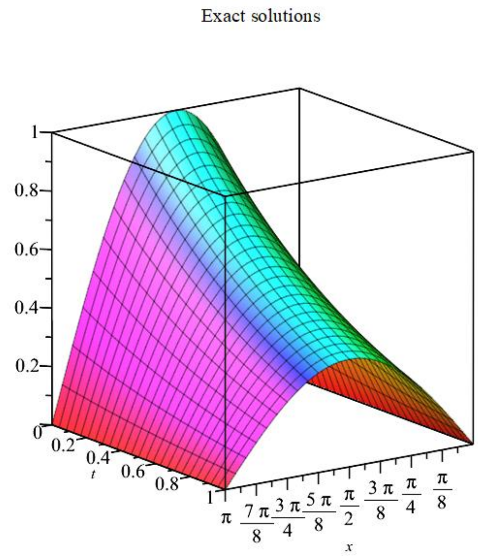

Fig. 1. The exact solution $u(x;t)$ for Example 3.1, at $0 \leq t \leq 1$ and $0 \leq x \leq \pi$

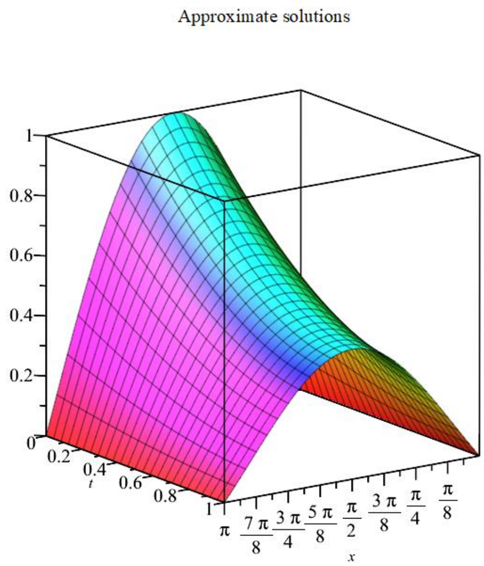

Fig. 2. The approximate solution of the $u(x;t)$ by the HAM method, at $0 \leq t \leq 1$ and $0 \leq x \leq \pi$

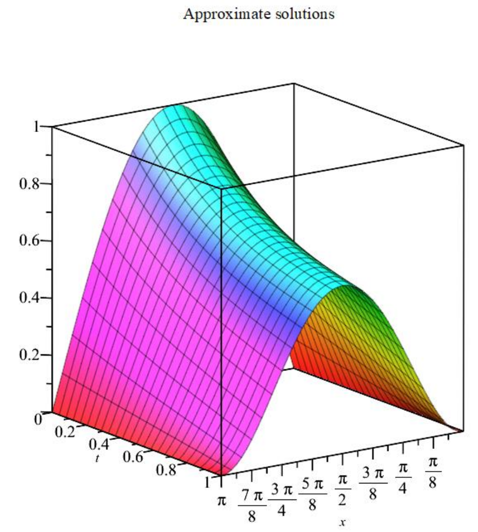

Fig. 3. The approximate solutions of the $u(x;t)$ by the VIM method, at $0 \leq t \leq 1$ and $0 \leq x \leq \pi$

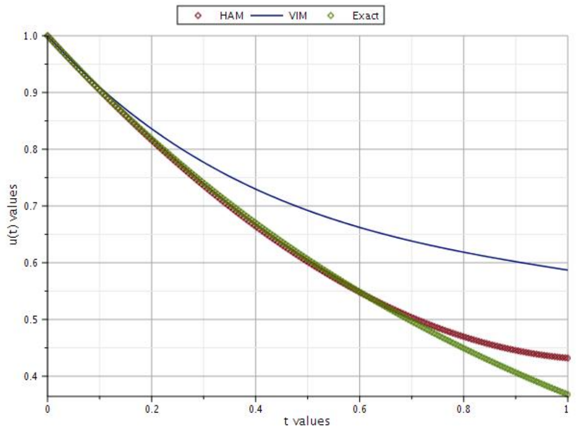

Fig. 4. The comparison between the exact solutions and the approximate solutions obtained using the HAM method and the VIM method together, for $0 \leq t \leq 1$

7. Conclusions

In this paper, numerical solutions of the nonlinear pseudo-hyperbolic partial differential equation depending on nonlocal conditions are presented. The homotopy analysis method HAM and the variational iteration method VIM are constructed for this equation. The approximation solution is obtained via both methods. The comparison between the approximate solutions of both methods with the exact solution is given to obtain the error estimations. The obtained result demonstrated that the homotopy analysis method HAM is more accurate and effective and is closer to the exact solution than the variational iteration method VIM by absolute errors.

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