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FOX H-FUNCTIONS AS EXACT SOLUTIONS FOR CAPUTO TYPE MASS SPRING DAMPER SYSTEM UNDER SUMUDU TRANSFORM

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Abstract. Closed form solutions for mathematical systems are not easy to find in many cases. In particular, linear systems such as the population growth/decay model, RLC circuit, mixing problems in chemistry, first-order kinetic reactions, and mass spring damper system in mechanical and mechatronic engineering can be handled with tools available in theoretical study of linear systems. One such linear system has been investigated in the present research study. The second order linear ordinary differential equation called the mass spring damper system is explored under the Caputo type differential operator while using the Sumudu integral transform. The closed form solution has been found in terms of the Fox H-function wherein different aspects of the solution can be obtained with variation in $\alpha \in (1,2]$ and $\beta \in (0,1]$. The classical mass spring damper model is retrieved for $\alpha = \beta = 1$.

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1. Introduction

Mathematical systems based upon ordinary differential equations of first and higher order play an important role in comprehending different types of physical and natural phenomena. Among the physical phenomena, one can observe that the flow of current in an RLC circuit can be modeled with the help of a second order ordinary differential equation shown by $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$; temperature of an object can be determined by a first order ordinary differential equation with the help of Newton's warming law as shown by $\frac{dT}{dt} = k(T - T_m)$; mixing of two salt solutions in chemistry give rise to a first order ordinary differential equation as given by $\frac{dA}{dt} =$ $=$ (input rate)-(output rate); velocity of a falling object under air resistance in Physics is once again a first order ordinary differential equation given by $m\frac{dv}{dt} + kv = mg$, where $k > 0$. Among natural systems, one of the vibrant fields of mathematical modeling is epidemiology wherein infectious diseases are modeled in terms of compartmental models of a deterministic type with the help of a system of nonlinear ordinary differential equations such as the following epidemiological model which is designed for understanding transmission dynamics of rubella disease.

$$\begin{cases} S'(t) &= A(1-\rho)-\beta(t)I(t)S(t)-\mu S(t),\ S(0)=S_0>0, \\ E'(t) &= \beta(t)I(t)S(t)-(\sigma+\mu)E(t),\ E(0)=E_0\geq 0, \\ I'(t) &= \sigma E(t)-(\gamma+\mu)I(t),\ I(0)=I_0\geq 0, \\ R'(t) &= A\rho+\gamma I(t)-\mu R(t),\ R(0)=R_0>0, \end{cases} \tag{1}$$

where each compartment shows a first order ordinary differential equation. The description of state variables and related biological parameters in (1) can be found in [1] and [2]. Various other mathematical systems can be obtained in research areas including computational biology, medicine, microbiology, biophysics, physiology, environmental science, and many more.

Classical systems modeled with ordinary derivatives have been investigated under operators from fractional calculus [3–21]. These fractional operators called Riemann--Liouville, Caputo, Caputo-Fabrizio, Atangana-Baleanu, Atangana-Gomez and a few others have characteristics of retaining memory unlike classical operators. The nonlocal nature of fractional operators have earned them recognition for being one of the most suitable mathematical tools to be used for analyzing various mathematical systems such as those discussed above.

In this research investigation, the Caputo differential operator is used to study a very important model from mechanical engineering. The Mass-Spring-Damper model having properties of being linear is fractionalyzed while using Caputo operator. The exact solution of the model under investigation is obtained in terms of the Fox H-function with the help of the Sumudu integral transform. The motivation behind the present investigation comes from published research work [22] wherein authors have used the Sumudu integral transform to find an exact solution for the RLC linear circuit model under the Caputo differential operator, however, they have expressed their computed solution in terms of the Mittag-Leffler function.

2. Mathematical preliminaries

Some important concepts are presented in this section which will subsequently be used in sections to come. These concepts are related with the basic theory of fractional calculus and the Sumudu integral transform.

Definition 1 The integral operator called the Riemman-Liouville fractional of order $\alpha > 0$ is defined by the following integral equation [23]

$$\mathbb{J}^{\alpha}_{0,t}s(t)=\frac{1}{\Gamma(\alpha)}\int_0^t s(\delta)(t-\delta)^{\alpha-1}d\delta, \quad t>0, \tag{2}$$

where $\Gamma(\cdot)$ is a special function which is also known as the Euler Gamma function.□

Definition 2 The differential operator commonly used in fractional calculus is the Caputo operator defined by the following integral equation [23]

$$ {}^{\mathrm{C}}\mathbb{D}_{0,t}^{\alpha} s(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{s^{(m)}(\delta)}{(t-\delta)^{\alpha+1-m}} d\delta, \qquad m-1 < \alpha \leq m \in \mathbb{N}. \tag{3} $$

Theorem 1 *The Sumudu transform for the Caputo derivative of a function $s(t)$ with order $\alpha > 0$ is defined by the following relation [24]*

$$ \mathbb{S}[{}^{C}\mathbb{D}_{0,t}^{\alpha} s(t); u] = u^{-\alpha} S(u) - \sum_{k=0}^{m-1} u^{k-\alpha} s^{(k)}(0), \quad m-1 < \alpha \leq m, \tag{4} $$

where $S(u)$ is the Sumudu integral transform for $s(t)$. □

3. Research methodology

In this section, we investigate the fractional order version of the mass spring damper system [25] under the Caputo differential operator obtained by summing all the forces in vertical direction. The force acting on the object is from Newton's second law of motion $F = mx''(t)$, the force on the spring is from Hooke's law $F = -kx(t)$, and the damping force acting on the object is $F = -bx'(t)$. Thus, one obtains the following governing equation having no external influences:

$$ mx''(t) = -bx'(t) - kx(t), \tag{5} $$

where b shows a positive damping constant and k is the positive spring constant. Now, consider the external force $f(t)$ acting on the vibrating object on a spring. This inclusion of $f(t)$ will produce a second order linear ordinary differential equation of driven or forced motion as given below

$$ mx''(t) + bx'(t) + kx(t) = f(t), x(0) = p, \ \ x'(0) = q, \tag{6} $$

where $x(0) = p$ and $x'(0) = q$ stand for initial displacement and initial velocity of the object, respectively.

Using the Caputo differential operator for the first and second term of the above equation, one obtains the following fractional order mass spring damper model

$$ \mathbb{D}_{0,t}^{\alpha} x(t) + \frac{b}{m}\mathbb{D}_{0,t}^{\beta} x(t) + \frac{k}{m} x(t) = \frac{1}{m} f(t), \ \alpha \in (1,2], \ \beta \in (0,1]. \tag{7} $$

Upon the use of the Sumudu transform, one obtains the following

$$ \left[\frac{X(u)}{u^{\alpha}} - \frac{p}{u^{\alpha}} - \frac{q}{u^{\alpha-1}}\right] + \frac{b}{m}\left[\frac{X(u)}{u^{\beta}} - \frac{p}{u^{\beta}}\right] + \frac{k}{m} X(u) = \frac{1}{m} F(u) \tag{8} $$

Simplification of the above equation produces the following

$$\begin{aligned}X(u) = \frac{1}{m}\frac{F(u)}{u^{-\alpha}+\frac{b}{m}u^{-\beta}+\frac{k}{m}} + p\frac{u^{-\alpha}}{u^{-\alpha}+\frac{b}{m}u^{-\beta}+\frac{k}{m}}+ \\ \frac{bp}{m}\frac{u^{-\beta}}{u^{-\alpha}+\frac{b}{m}u^{-\beta}+\frac{k}{m}} + q\frac{u^{-\alpha+1}}{u^{-\alpha}+\frac{b}{m}u^{-\beta}+\frac{k}{m}}\end{aligned} \tag{9}$$

Inverting the Sumudu transform, one obtains the following closed form solution for the equation (7):

$$\begin{aligned}x(t) = \frac{1}{m}\int_0^t f(t-\tau)\sum_{i=0}^{\infty}\Big(-\frac{b}{m}\Big)^i \tau^{(\alpha-\beta)i+\alpha+1}\mathbb{E}^{i+1}_{\alpha,(\alpha-\beta)i+\alpha}\Big(-\frac{k}{m}t^{\alpha}\Big)d\tau+ \\ p\sum_{i=0}^{\infty}\Big(-\frac{b}{m}\Big)^i t^{(\alpha-\beta)i}\mathbb{E}^{i+1}_{\alpha,(\alpha-\beta)i+1}\Big(-\frac{k}{m}t^{\alpha}\Big)+ \\ \frac{bp}{m}\sum_{i=0}^{\infty}\Big(-\frac{b}{m}\Big)^i t^{(\alpha-\beta)(i+1)}\mathbb{E}^{i+1}_{\alpha,(\alpha-\beta)(i+1)+1}\Big(-\frac{k}{m}t^{\alpha}\Big)+ \\ q\sum_{i=0}^{\infty}\Big(-\frac{b}{m}\Big)^i t^{(\alpha-\beta)i+1}\mathbb{E}^{i+1}_{\alpha,(\alpha-\beta)i+2}\Big(-\frac{k}{m}t^{\alpha}\Big)\end{aligned} \tag{10}$$

Using the relation

$$\mathbb{E}^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!\Gamma(\alpha n+\beta)}z^n,\ Re(\alpha)>0,\ Re(\beta)>0,\ \gamma>0, \tag{11}$$

where $(\gamma)_n = \gamma(\gamma+1)\cdot(\gamma+n+1)$ is known as the Pochhammer notation and $(\gamma)_0 = = 1, (\gamma)_n = \dfrac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$. We have the following simplified version for the exact solution (10):

$$\begin{aligned}x(t) = \frac{1}{m}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\frac{b}{m})^i(-\frac{k}{m})^n\Gamma(i+n+1)}{n!\Gamma(i+1)\Gamma(\alpha(i+n+1)-i\beta)}\int_0^t \tau^{\alpha(i+n+1)-i\beta+1}f(t-\tau)d\tau + \\ p\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\frac{b}{m})^i(-\frac{k}{m})^n\Gamma(i+n+1)}{n!\Gamma(i+1)\Gamma(\alpha n+(\alpha-\beta)i+1)}t^{\alpha n+(\alpha-\beta)i} + \\ \frac{bp}{m}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\frac{b}{m})^i(-\frac{k}{m})^n\Gamma(i+n+1)}{n!\Gamma(i+1)\Gamma(\alpha n+(\alpha-\beta)(i+1)+1)}t^{\alpha n+(\alpha-\beta)(i+1)} + \\ q\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\frac{b}{m})^i(-\frac{k}{m})^n\Gamma(i+n+1)}{n!\Gamma(i+1)\Gamma(\alpha n+(\alpha-\beta)i+2)}t^{\alpha n+(\alpha-\beta)i+1} +\end{aligned} \tag{12}$$

Alternatively, the exact solution of the Caputo type mass-spring-damper system (7) can be written in the following form

$$
\begin{aligned}
x(t) = & \frac{1}{m}\int_0^t f(t-\tau)\sum_{i=0}^{\infty}\frac{(-\frac{b}{m})^i\tau^{(\alpha-\beta)i+\alpha+1}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\frac{k}{m}\tau^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (1-(\alpha-\beta)i-\alpha,\alpha)\end{matrix}\right]d\tau + \\
& p\sum_{i=0}^{\infty}\frac{(-\frac{b}{m})^i t^{(\alpha-\beta)i}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\frac{k}{m}t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)i,\alpha)\end{matrix}\right] + \\
& \frac{bp}{m}\sum_{i=0}^{\infty}\frac{(-\frac{b}{m})^i t^{(\alpha-\beta)(i+1)}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\frac{k}{m}t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)(i+1),\alpha)\end{matrix}\right] + \\
& q\sum_{i=0}^{\infty}\frac{(-\frac{b}{m})^i t^{(\alpha-\beta)i+1}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\frac{k}{m}t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)i-1,\alpha)\end{matrix}\right],
\end{aligned}
\tag{13}
$$

where

$$
\mathbb{E}_{\alpha,\beta}^{\gamma}(z) = H_{1,2}^{1,1}\left[-z\middle|\begin{matrix}(1-\gamma,1)\\(0,1),\quad(1-\beta,\alpha)\end{matrix}\right]. \tag{14}
$$

The right hand side of the above equation is known as the Fox H-function and in this context, the three-parameter Mittag-Leffler (The Prabhakar) function can be taken as a special case of the Fox H-function. The relation (13) is the exact solution for the mass-spring-damper system (7) under the fractional order Caputo differential operator for $\alpha \in (1,2]$ and $\beta \in (0,1]$. Further, it is known that the natural frequency and the damping ratio of the mechanical system under investigation are respectively $\omega_n = \sqrt{\frac{k}{m}}$ and $\xi = \frac{b}{2\sqrt{km}}$, the exact solution shown in (13) can also be represented as follows:

$$
\begin{aligned}
x(t) = & \frac{1}{m}\int_0^t f(t-\tau)\sum_{i=0}^{\infty}\frac{(-2\xi\omega_n)^i\tau^{(\alpha-\beta)i+\alpha+1}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\omega_n^2\tau^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (1-(\alpha-\beta)i-\alpha,\alpha)\end{matrix}\right]d\tau + \\
& p\sum_{i=0}^{\infty}\frac{(-2\xi\omega_n)^i t^{(\alpha-\beta)i}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\omega_n^2 t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)i,\alpha)\end{matrix}\right] + \\
& \frac{bp}{m}\sum_{i=0}^{\infty}\frac{(-2\xi\omega_n)^i t^{(\alpha-\beta)(i+1)}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\omega_n^2 t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)(i+1),\alpha)\end{matrix}\right] + \\
& q\sum_{i=0}^{\infty}\frac{(-2\xi\omega_n)^i t^{(\alpha-\beta)i+1}}{\Gamma(i+1)}H_{1,2}^{1,1}\left[\omega_n^2 t^{\alpha}\middle|\begin{matrix}(-i,1)\\(0,1),\ (-(\alpha-\beta)i-1,\alpha)\end{matrix}\right].
\end{aligned}
\tag{15}
$$

4. Conclusion

This research study shows that the Caputo differential operator provides different kinds of behavior for the mass-spring-damper system based upon fractional parameters α and β. Upon taking different values of these parameters between $(1,2]$ and $(0,1]$ respectively, one can have various solutions for the system which were not possible to obtain under classical local operators having no memory effects in their very nature. The closed-form solution of the underlying Caputo system is achieved via the Sumudu integral transform in terms of double summation which was later reduced to the special Fox H-function. Natural frequency and the damping ratio of the system are also taken into consideration and different profiles from the exact solution can be obtained if mass m, damping constant b and spring constant k are assigned varying values.

5. Future directions

The present research investigation can be improved if fractional order operators called the Caputo-Fabrizio, Atangana-Baleanu, Atangana-Gomez, and Atangana--Koca are used to observe dynamical behavior of the mass-spring-damper system under different situations. Another recently introduced operator called the fractal--fractional operator is also a good candidate for the linear system under consideration. Moreover, the Caputo operator and other fractional operators can be used to analyze behavior of the underlying dynamical system when the mass is considered to be a variable. Under the effects of damping and stiffness of the spring, various profiles can be obtained for the mass-spring-damper system analyzed in this research study using the Caputo operator.

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