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# ANALYTICAL AND NUMERICAL STUDY FOR A FRACTIONAL BOUNDARY VALUE PROBLEM WITH A CONFORMABLE FRACTIONAL DERIVATIVE OF CAPUTO AND ITS FRACTIONAL INTEGRAL

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**Abstract.** We study the existence and uniqueness of the solution of a fractional boundary value problem with conformable fractional derivation of the Caputo type, which increases the interest of this study. In order to study this problem we have introduced a new definition of fractional integral as an inverse of the conformable fractional derivative of Caputo, therefore, the proofs are based upon the reduction of the problem to a equivalent linear Volterra-Fredholm integral equations of the second kind, and we have built the minimum conditions to obtain the existence and uniqueness of this solution. The analytical study is followed by a complete numerical study.

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## 1. Introduction

The usual integral and derivative are (to say the least) a staple for the new technology, essential as a means of understanding and working with natural and artificial systems. Recently, many authors have participated in the development of the fractional calculus (differentiation and integration of arbitrary order) [1, 2]. The applications of fractional calculus often appear in the fields such as generalized voltage dividers [3], electric conductance of biological systems [4], capacitor theory [5, 6], engineering [7, 8], electrode-electrolyte interface models [9], feedback amplifiers [10], medical [11], fractional order models of neurons [12], analysis of special functions [13], and fitting experimental data [14].

Recently, papers have been published that deal with the existence and multiplicity of the solution of nonlinear initial fractional differential equation by the use of techniques of nonlinear analysis, see [15–17]. However, most of the papers offer the problem using the standard Riemann-Liouville differentiation, see [18, 19]. However, Our aim is to study the existence and the uniqueness of the solution for a class of fractional boundary value problems. To the best of our knowledge, this is the first work that solves problem with the conformable fractional derivative by Caputo and Fabrizio in paper [20], which has many properties mentioned in the article [9]. The interest for in this new approach is due to the necessity of using a model to describe the behavior of classical viscoelastic materials, electromagnetic systems, thermal media, etc. In fact, the original definition of Caputo's fractional derivative appears to be particularly convenient for those mechanical phenomena, related to damage and with electromagnetic hysteresis, fatigue and plasticity. When these effects are not present it seems more appropriate to use the new fractional derivative [20].

In this paper, We study the existence and uniqueness of the solution of the fractional differential equation boundary value problem, as follows:

$$\begin{cases} \mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $1 < \gamma < 2$ is a real number, $q$ is the potential function, and $f : [0,1] \to \mathbb{R}$ is continuous. and $\mathscr{D}^{(\gamma)}$ is the new fractional derivative, and we introduce a new definition of its fractional integral with some properties, using this fractional integral upon problem (1) to obtain an equivalent linear Volterra-Fredholm integral equations of second kind [21, 22]. Finally, by the means of some theorems, the existence and uniqueness of solutions are obtained, and we introduce an algorithm for finding a numerical solution of this problem class.

## 2. Preliminaries

For the convenience of the reader, we present here the necessary definitions and lemmas from fractional calculus theory. These definitions can be found in the recent literature.

**Definition 1** [20] For $\alpha \in [0,1]$, the fractional time derivative $\mathscr{D}^{(\alpha)}u(x)$ of order $(\alpha)$ is defined as follows:

$$\mathscr{D}^{(\alpha)} u(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x u'(s) \exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right] \mathrm{d}s \tag{2}$$

where $\alpha \in [0,1]$ and $a \in ]-\infty, x)$, $u \in H^1(a,b)$, $b > a$, and $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. □

**Definition 2** [20] Let $n \geq 1$, and $\alpha \in [0,1]$ the fractional derivative $\mathscr{D}^{(\alpha+n)} f$ of order $(n+\alpha)$ is defined by

$$\mathscr{D}^{(\alpha+n)} f(x) := \mathscr{D}_t^{(\alpha)} \left( \mathscr{D}^{(n)} f(x) \right) = \frac{M(\alpha)}{1-\alpha} \int_a^x f^{(n+1)}(s) \exp\left[ -\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s$$

Such that

$$\mathscr{D}^{(\alpha+n)} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x f^{(n+1)}(s) \exp\left[ -\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s \tag{3}$$

## 3. A new definition of fractional integral

In this section, We introduce a new definition of a fractional integral as a theorem:

**Theorem 1** *Let $n \geq 1$, $\alpha \in [0,1]$, and $f \in \mathscr{C}^1[a,b]$. The formula:*

$$I_a^{n+\alpha} f(x) = \frac{1}{M(\alpha) \cdot n!} \int_a^x (x-s)^{n-1} [\alpha(x-s) + n(1-\alpha)] f(s) \, \mathrm{d}s$$

*where $f \in \mathscr{C}^1[a,b]$, and $M(\alpha)$, is a normalization function such that $M(0) = M(1) = = 1$ is a new fractional integral of order $(n+\alpha)$, and it's as an inverse of the conformable fractional derivative of Caputo of order $(n+\alpha)$.* □

PROOF From Definitions 1 and 2, we obtain

$$\mathscr{D}^{(\alpha+n)} f(x) = \frac{M(\alpha)}{1-\alpha} \int_a^x f^{(n+1)}(s) \exp\left[ -\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s$$

and the Leibniz integral rule gives the formula

$$\frac{d}{dt} \left( \mathscr{D}^{(\alpha+n)} f(x) \right) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(x) - \frac{\alpha}{1-\alpha} \frac{M(\alpha)}{1-\alpha} \int_a^x f^{(n+1)}(s) \exp\left[ -\frac{\alpha(x-s)}{1-\alpha} \right] \mathrm{d}s$$

$$\Rightarrow \frac{d}{dt} \left( \mathscr{D}^{(\alpha+n)} f(x) \right) = \frac{M(\alpha)}{1-\alpha} f^{(n+1)}(x) - \frac{\alpha}{1-\alpha} \mathscr{D}^{(\alpha+n)} f(x).$$

$$\Rightarrow f^{(n+1)}(x) = \frac{1}{M(\alpha)} \left[ (1-\alpha) \frac{d}{dt} \left( \mathscr{D}^{(\alpha+n)} f(x) \right) + \alpha \mathscr{D}^{(\alpha+n)} f(x) \right].$$

we now use the Cauchy formula for evaluating the $(n+1)^{th}$ integration of the function $f^{(n+1)}(x)$

$$f(x) = \frac{1}{n! M(\alpha)} \int_a^x (x-s)^n \left[ (1-\alpha) \frac{d}{ds} \left( \mathscr{D}^{(\alpha+n)} f(s) \right) + \alpha \mathscr{D}^{(\alpha+n)} f(s) \right] \mathrm{d}s.$$

$$
\begin{aligned}
\Rightarrow I_a^{n+\alpha} f(x) &= \frac{1}{M(\alpha)\cdot n!}\int_a^x (x-s)^n[\alpha f(s)+(1-\alpha)f'(s)]\,\mathrm{d}s\\
&= \frac{1}{M(\alpha)\cdot n!}\int_a^x (x-s)^{n-1}[\alpha(x-s)+n(1-\alpha)]f(s)\,\mathrm{d}s.
\end{aligned}
$$

**Lemma 1** *Let $\gamma\in(n,n+1), n=[\gamma]\geqslant 0$. Assume that $u\in\mathscr{C}^n[a,b]$, then those statements hold:*

*1. if $u(a)=0$, then $\mathscr{D}^{(\gamma)}\ (I_a^\gamma u(x))=u(x)$.*

*2. $I_a^\gamma\left(\mathscr{D}^{(\gamma)}u(x)\right)=u(x)+\sum_{i=0}^n a_i x^i,\ a_i\in\mathbb{R}\ i=0,1,\dots,n.$* □

PROOF Let $\gamma\in]n,n+1[$, it can be written in the form: $\gamma=n+\alpha$ where $\alpha\in]0,1[$, and $n=[\gamma]$, we have

$$
\begin{aligned}
1.\mathscr{D}^{(\gamma)}\ (I_a^\gamma u(x)) &= \frac{\alpha}{(1-\alpha)}\int_a^x \frac{d^{(n+1)}}{ds^{(n+1)}}\left[\frac{1}{n!}\int_a^s (s-x)^n u(x)\mathrm{d}x\right]\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\\
&+\int_a^x \frac{d^{(n+1)}}{ds^{(n+1)}}\left[\frac{1}{n!}\int_a^s (s-x)^n u'(x)\mathrm{d}x\right]\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\\
&= \frac{\alpha}{(1-\alpha)}\int_a^x u(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s+\int_a^x u'(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\\
&= \frac{\alpha}{(1-\alpha)}\int_a^x u(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\\
&+u(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\Bigg|_a^x-\frac{\alpha}{(1-\alpha)}\int_a^x u(s)\exp\left[-\frac{\alpha(x-s)}{1-\alpha}\right]\mathrm{d}s\\
&= u(x)-u(a)\exp\left[-\frac{\alpha(x-a)}{1-\alpha}\right]=u(x).
\end{aligned}
$$

$$
\begin{aligned}
2.\ I_a^\gamma\left(\mathscr{D}^{(\gamma)}u(x)\right) &= \frac{1}{M(\alpha)\cdot n!}\int_a^x (x-s)^{n-1}[\alpha(x-s)+n(1-\alpha)]\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s\\
&= \frac{\alpha}{M(\alpha)\cdot n!}\int_a^x (x-s)^n\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s+\frac{(1-\alpha)}{M(\alpha)\cdot n!}\int_a^x (x-s)^n\frac{d}{ds}\left(\mathscr{D}^{(\gamma)}u(s)\right)\mathrm{d}s\\
&= \frac{\alpha}{M(\alpha)\cdot n!}\int_a^x (x-s)^n\mathscr{D}^{(\gamma)}u(s)\,\mathrm{d}s\\
&+\frac{(1-\alpha)}{M(\alpha)\cdot n!}\int_a^x (x-s)^n\left(\frac{M(\alpha)}{1-\alpha}u^{(n+1)}(s)-\frac{\alpha}{1-\alpha}\mathscr{D}^{(\gamma)}u(s)\right)\mathrm{d}s\\
&= \frac{1}{n!}\int_a^x (x-s)^n u^{(n+1)}(s)\,\mathrm{d}s=u(x)+\sum_{i=0}^n a_i x^i,\ a_i\in\mathbb{R}\ i=0,1,\dots,n.
\end{aligned}
$$

## 4. Analytic study

In the following, we suppose the function $M(\alpha) = 1$.

**Lemma 2** *Given* $q \in \mathscr{C}[0,1]$*, and* $1 < \gamma < 2$*, the solution of*

$$\begin{aligned} &\mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{4}$$

*satisfies the following linear Volterra-Fredholm integral equations of the second kind*

$$u(x) + \int_0^x G(x,s)u(s)\,\mathrm{d}s + \int_0^1 K(x,s)u(s)\,\mathrm{d}s = g(x) \tag{5}$$

*where* $g(x) = \int_0^x (x-1)(\alpha s - 1 + \alpha) f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1) f(s)\,\mathrm{d}s,$

$$G(x,s) = (x-1)(\alpha s - 1 + \alpha)q(s) \text{ and } K(x,s) = x(\alpha s - 1)q(s).$$

PROOF We may apply Lemma 1 to reduce Eq. (4) to an equivalent integral equation

$$\begin{aligned} &I_0^\gamma \left( \mathscr{D}^{(\gamma)} u(x) \right) = I_0^\gamma \left( f(x) - q(x)u(x) \right) \\ &\Rightarrow u(x) + cx + d = \int_0^x \left[ \alpha(x-s) + (1-\alpha) \right] \left( f(s) - q(s)u(s) \right) \mathrm{d}s \end{aligned}$$

Using boundary conditions $u(0) = u(1) = 0$, we have $d = 0$, and

$$c = \int_0^1 (1 - \alpha s) \left( f(s) - q(s)u(s) \right) \mathrm{d}s.$$

Therefore, the unique solution of problem (4) is

$$\begin{aligned} u(x) &= \int_0^x \left[ \alpha(x-s) + (1-\alpha) \right] \left( f(s) - q(s)u(s) \right) \mathrm{d}s \\ &\quad + \int_0^1 x(\alpha s - 1) \left( f(s) - q(s)u(s) \right) \mathrm{d}s \\ &= \int_0^x \left[ \alpha(x-s) + (1-\alpha) \right] f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s - 1) f(s)\,\mathrm{d}s \\ &\quad - \int_0^x \left[ \alpha(x-s) + (1-\alpha) \right] q(s)u(s)\,\mathrm{d}s - \int_0^1 x(\alpha s - 1) q(s)u(s)\,\mathrm{d}s \\ &= g(x) - \int_0^x G(x,s)u(s)\,\mathrm{d}s - \int_0^1 K(x,s)u(s)\,\mathrm{d}s. \end{aligned}$$

The proof is complete. ∎

The classical approach to proving the existence and uniqueness of the solution of equation (5) is the Picard method. This consists of the simple iteration for $n = 1, 2, \dots$

$$u_n(x) = g(x) + \int_0^x G(x,s)u_{n-1}(s)\,\mathrm{d}s + \int_0^1 K(x,s)u_{n-1}(s)\,\mathrm{d}s \tag{6}$$

with $u_0(x) = g(x)$. For ease of manipulation, it is convenient to introduce

$$v_n(x) = v_n(x) - v_{n-1}(x), \quad n = 1, 2, \dots \tag{7}$$

with $v_0(x) = g(x)$. On subtracting from (6), the same equation with $n$ replaced by $n-1$, an we see that

$$v_n(x) = \int_0^x k(x,s)v_{n-1}(s)ds, \quad n = 1, 2, \dots$$

Also, from (7)

$$u_n(x) = \sum_{i=0}^{n} v_i(x) \tag{8}$$

The following theorem uses this iteration to prove the existence and uniqueness of the solution under quite restrictive conditions, namely that $G(x,s)$, $K(x,s)$ and $g(x)$ are continuous.

**Theorem 2** *If $g(x)$ is continuous in $0 \leqslant x \leqslant 1$, and the function $K(x,s)$, $G(x,s)$ are continuous in $0 \leqslant s \leqslant x \leqslant 1$, and $\max\limits_{0\leqslant s\leqslant x\leqslant 1} |K(x,s)| < 1$, then the integral equation (5) possesses a unique continuous solution for $0 \leqslant x \leqslant 1$.* □

PROOF Choose $M_1$, $M_2$ and $M_3$ such that

$$\begin{aligned} |g(x)| &\leqslant M_1, \quad 0 \leqslant x \leqslant 1 \\ |G(x,s)| &\leqslant M_2, \quad 0 \leqslant s \leqslant x \leqslant 1 \\ |K(x,s)| &\leqslant M_3, \quad 0 \leqslant s \leqslant x \leqslant 1 \text{ where } M_3 < 1. \end{aligned}$$

We first prove by induction that

$$|v_n(x)| \leqslant \frac{M_1(M_2x)^n}{n!} + M_1M_3^n, \quad 0 \leqslant x \leqslant 1, \quad n = 0, 1, \dots \tag{9}$$

this bound makes it obvious that the sequence $u_n(x)$ in (8) converges, and we can write

$$u(x) = \sum_{i=0}^{\infty} v_i(x) \tag{10}$$

We now show that this $u(x)$ satisfies equation (5). The series (10) is uniformly convergent since the terms $v_i(x)$ are dominated by $M_1(M_2x)^i/i! + M_1M_3^i$. Consequently, we can interchange the order of integration and summation in the following expression to obtain

$$\begin{aligned}\int_0^x G(x,s)\sum_{i=0}^{\infty} v_i(s)\,\mathrm{d}s+\int_0^1 K(x,s)\sum_{i=0}^{\infty} v_i(s)\,\mathrm{d}s &= \sum_{i=0}^{\infty}\int_0^x G(x,s)v_i(s)\,\mathrm{d}s\\ &\quad+\sum_{i=0}^{\infty}\int_0^1 K(x,s)v_i(s)\,\mathrm{d}s\\ &= \sum_{i=0}^{\infty} v_{i+1}(s)\\ &= \sum_{i=0}^{\infty} v_i(s)-g(x).\end{aligned}$$

Each of the $v_i(x)$ is clearly continuous. Therefore $u(x)$ is continuous, since it is the limit of a uniformly convergent sequence of continuous functions.

To show that $u(x)$ is the only continuous solution, suppose there exists another continuous solution $\tilde{u}(x)$ of (5). Then

$$u(x)-\tilde{u}(x)=\int_0^x G(x,s)(u(s)-\tilde{u}(s))\,\mathrm{d}s+\int_0^1 K(x,s)(u(s)-\tilde{u}(s))\,\mathrm{d}s \tag{11}$$

since $f(x)$ and $\tilde{f}(x)$ are both continuous, there exists a constant $C$ such that

$$|u(x)-\tilde{u}(x)|\leqslant C,\quad 0\leqslant x\leqslant 1$$

Substituting this into (11)

$$|u(x)-\tilde{u}(x)|\leqslant C(M_2x+M_3),\quad 0\leqslant x\leqslant 1$$

and repeating the step shows that

$$|u(x)-\tilde{u}(x)|\leqslant C\left(\frac{(M_2x)^n}{n!}+M_3^n\right),\quad 0\leqslant x\leqslant 1,\ \text{ for any } n.$$

For a large enough $n$, the right-hand side is arbitrarily small, therefore, we must have

$$u(x)-\tilde{u}(x),\quad 0\leqslant x\leqslant 1$$

**Theorem 3** *If $f(x)$, $q(x)$ are continuous in $[0,1]$, and $\max\limits_{0\leqslant x\leqslant 1}|q(x)|<1$, then the fractional boundary value problem (1) possesses a unique continuous solution for $0\leqslant x\leqslant 1$.* □

PROOF If $f(x),\ q(x)$ are continuous in $[0,1]$, then it is clear that the following functions

$$\begin{aligned} g(x) &= \int_0^x (x-1)(\alpha s-1+\alpha) f(s)\,\mathrm{d}s + \int_0^1 x(\alpha s-1) f(s)\,\mathrm{d}s,\\ G(x,s) &= (x-1)(\alpha s-1+\alpha)q(s),\\ K(x,s) &= x(\alpha s-1)q(s), \end{aligned}$$

are continuous, and $|K(x,s)| = |x(\alpha s-1)q(s)| \leqslant |q(s)| < 1, \quad \forall x,s \in [0,1]$, which means that integral equation (5) possesses a unique continuous solution for $0 \leqslant x \leqslant 1$. Therefore, there is a unique continuous solution of the fractional boundary value problem (1) for $0 \leqslant x \leqslant 1$. ∎

## 5. Numerical study

In this section, we introduce an algorithm for finding a numerical solution of linear Volterra-Fredholm integral equations of the second kind, the methods based upon trapezoidal rule. For all $N \in \mathbb{N}$, Here the interval $[0,1]$ in to $N$ equal sub-intervals, where $h = (b-a)/N$, and $x_i = a + i \cdot h$ for all $i \in \{0 \cdots N\}$.

The formula of the numerical integration is:

$$\int_a^b f(s)\,\mathrm{d}s \approx \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{N} f(x_j) + f(b)\right]$$

we apply this formula in eq. (5), and we obtain:

$$\begin{aligned} g(x_i) = u(x_i) &+ \frac{h}{2}\left[G(x_i,x_0)u(x_0) + 2\sum_{j=1}^{i-1} G(x_i,x_j)u(x_j) + G(x_i,x_i)u(x_i)\right]\\ &+ \frac{h}{2}\left[K(x_i,x_0)u(x_0) + 2\sum_{j=1}^{N-1} K(x_i,x_j)u(x_j) + K(x_i,x_N)u(x_N)\right] \end{aligned}$$

$$\begin{aligned} \Rightarrow \forall i = 0,\ldots,N,\ g_i = u_i &+ \frac{h}{2}\left[G_{i0}u_0 + 2\sum_{j=1}^{i-1} G_{ij}u_j + G_{ii}u_i\right]\\ &+ \frac{h}{2}\left[K_{i0}u_0 + 2\sum_{j=1}^{N-1} K_{ij}u_j + K_{iN}u_N\right] \end{aligned}$$

This leads to

$$\frac{h}{2}(G_{i0}+K_{i0})u_0+h\sum_{j=1}^{i-1}(G_{ij}+K_{ij})u_j+\frac{h}{2}\left(\frac{2}{h}+G_{ii}+2K_{ii}\right)u_i$$
$$+h\sum_{j=i+1}^{N-1}K_{ij}u_j+\frac{h}{2}K_{iN}u_N=g_i$$

Finally, we get a system of $N+1$ equations, which is:

$$AU=B \tag{12}$$

when $B=(g_0,g_1,\dots,g_N)$, $U=(u_0,u_1,\dots,u_N)$, and $A=(a_{ij})_{i,j=0,\dots,N}$;

$$a_{ij}=\begin{cases} h\cdot K_{00}/2+1 & \text{if } i=j=0,\\ h\cdot K_{0j} & \text{if } j=1,\dots,N-1,\\ h\cdot K_{0j}/2 & \text{if } j=N,\\ h\cdot(G_{i0}+K_{i0})/2 & \text{if } i=1,\dots,N,\\ h\cdot(G_{ii}+2K_{ii})/2+1 & \text{if } i=j=1,\dots,N-1,\\ h\cdot(G_{ii}+K_{ii})/2+1 & \text{if } i=j=N,\\ h\cdot K_{iN}/2 & \text{if } i=1,\dots,N-1,\\ h\cdot(G_{ij}+K_{ij}) & \text{if } i=2,\dots,N,\ j=1,\dots,i-1,\\ h\cdot K_{ij} & \text{if } i=1,\dots,N-1,\ j=i+1,\dots,N-1,\end{cases}$$

We have chosen to write our system in its general matrix form without taking into account the fact that $u_0=u_N=0$. However, we can see that

$$g_0=K_{0j}=0,\ \forall j\in\{0\dots N\}\Rightarrow u_0=0.$$

In same way, we get $u_N=0$.

## 6. Numerical result

In this section, we give three numerical examples (Fig. 1) to illustrate the above methods for solve the linear Volterra-Fredholm integral equations of the second kind. The exact solution is known and used to show that the numerical solution obtained with our methods is correct. We used MATLAB to solve these examples.

**Example 1** Consider the following fractional boundary value problem:

$$\begin{array}{l}\mathscr{D}^{(\gamma)}u(x)+q(x)u(x)=f(x), \quad 0\leqslant x\leqslant 1\\ u(0)=u(1)=0\end{array} \tag{13}$$

where $\gamma = 1.5$, $q(x) = 1$, $f(x) = \sinh(x) + (e^x - 1)(x-1) + xe^x$, with the exact solution $u(x) = (e^x - 1)(x - 1)$. □

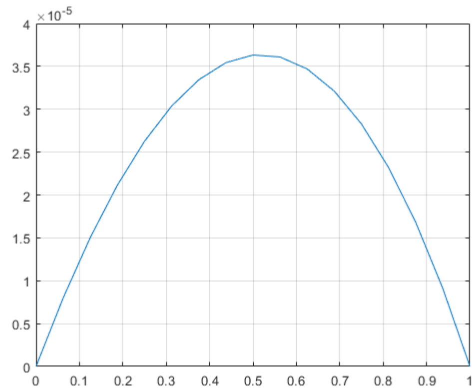


Fig. 1. The absolute error of test Example 1 with $N = 16$

**Example 2** Consider the following fractional boundary value problem (Fig. 2):

$$\begin{aligned} &\mathscr{D}^{(\gamma)}u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = u(1) = 0 \end{aligned} \tag{14}$$

where $\gamma = 1.9$, $q(x) = t - 1$, and $f(x) = 10x$. In this case, we don't know the exact solution. □

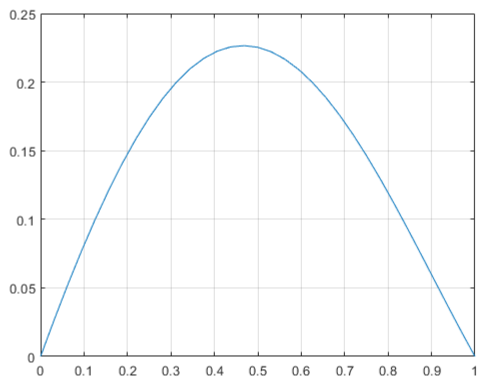


Fig. 2. The approximate solution of test Example 2 with $N = 32$

## 7. Conclusion

In this article, we proved the existence and uniqueness of the fractional boundary value problem with use of the minimum of hypotheses that ensure this, and using numerical methods and programming by Matlab to solve the problem.

As perspectives, we will try to study the following generalized version

$$\begin{aligned} &\mathscr{D}^{(\gamma)} u(x) + q(x)u(x) = f(x), \quad 0 \leqslant x \leqslant 1 \\ &u(0) = a,\ u(1) = b \end{aligned} \tag{15}$$

where, $a, b \in \mathbb{R}$. This generalized version represents a challenge from the analytical point of view, i.e. the existence and uniqueness of the solution. However, the numerical side remains the same.

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