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HAMACHER OPERATIONS ON PYTHAGOREAN FUZZY MATRICES

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Abstract. In this paper, we introduce the Hamacher operations on Pythagorean fuzzy matrices and prove some desirable properties of these operations, such as commutativity, idempotancy and monotanicity. Further, we prove De Morgan's laws for these operations over complement.

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Keywords: *intuitionistic fuzzy matrix, Pythagorean fuzzy matrix, Hamacher sum and Hamacher product*

1. Introduction

The concept of an intuitionistic fuzzy matrix (IFM) was introduced by Khan et al. [1] and simultaneously Im et al. [2] to generalize the concept of Thomason's [3] fuzzy matrix. Each element in an IFM is expressed by an ordered pair $\langle a_{ij}, a'_{ij} \rangle$ with $a_{ij}, a'_{ij} \in [0,1]$ and $0 \leq a_{ij} + a'_{ij} \leq 1$. Khan and Pal [4] defined some basic operations and relations of IFMs including maxmin, minmax, complement, algebraic sum, algebraic product etc. and proved equality between IFMs. Mondal and Pal [5] studied the similarity relations, together with invertibility conditions and eigenvalues of IFMs. Zhang and Xu [6] studied the intuitionistic fuzzy value and IFMs, they defined intuitionistic fuzzy similarity relation and utilised it in clustering analysis. Emam and Fndh [7] defined some kinds of IFMs, also they construct an idempotent IFM from any given one through the minmax composition. Muthuraji et al. [8] obtained a decomposition of intuitionistic fuzzy matrices. In [9], we defined Hamacher operations on fuzzy matrices and investigated their algebraic properties. In [10], we extended Hamacher operations to IFMs.

Yager [11] introduced the concept of a Pythagorean fuzzy set (PFS) and developed some aggregation operations for PFS. The PFS characterized by a membership degree and a nonmembership degree satisfying the condition that the square sum of its membership degree and nonmembership degree is equal to or less than 1, has much stronger ability than Intuitionistic fuzzy set (IFS) to model such uncertain information in multi-criteria decision making problems. Zhang and Xu [12] studied various binary operations over PFS and also proposed a decision making algorithm based on PFS. Using the theory of PFS, [13] we defined the Pythagorean fuzzy matrix (PFM) and its algebraic operations. Also, we constructed nA and A^n of a Pythagorean fuzzy matrix A and using these operations. Further, we studied the properties of the algebraic operations on Pythagorean fuzzy matrices and proved that the set of all PFMs forms a commutative monoid [14].

This paper is organized as follows: In Section 2, Some basic definitions and operations are given. In Section 3, we have introduced the Hamacher operations on Pythagorean fuzzy matrices and investigated their algebraic properties. In Section 4, we proved De Morgan's laws for these operations over complement. Finally, the conclusion is given in Section 5.

1.1. Motivation

The Pythagorean fuzzy matrix (PFM) has emerged as an effective tool for depicting uncertainty of the multi-criteria decision making problems. The PFM is also characterized by the membership degree and the non-membership degree, whose sum of squares is less than or equal to 1. The PFM is more general than the IFM. In some cases, the PFM can solve the problems that the IFM cannot, for example, if a DM gives the membership degree and the non-membership degree as 0.7 and 0.6, respectively, then it is only valid for the PFM. In other words, all the intuitionistic fuzzy degrees are a part of the Pythagorean fuzzy degrees, which indicates that the PFM is more powerful in handling the uncertain problems.

The difference between IFM and PFM as shown in Figure 1.

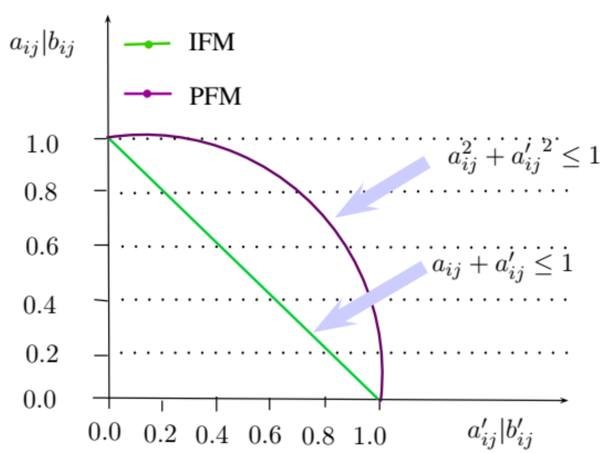

Fig. 1. Comparison of spaces of the PFMs and IFMs

Wu and Wei [15] investigated the multiple attribute decision making (MADM) problem based on the Hamacher aggregation operators with Pythagorean fuzzy information. Then, motivated by the ideal of Hamacher operations [16], we have developed Hamacher operations on Pythagorean fuzzy matrices and analyze some desirable properties of these operations.

2. Preliminaries

In this section, some basic concepts related to the intuitionistic fuzzy matrix (IFM), Pythagorean fuzzy set(PFS) and Pythagorean fuzzy matrix (PFM) have been given.

Definition 1 [1] An intuitionistic fuzzy matrix (IFM) is a matrix of pairs $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right)$ of a non negative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satifying $0 \leq a_{ij} + a'_{ij} \leq 1$ for all i, j. □

Yager [11] proposed a novel concept of Pythagorean fuzzy set as follows.

Definition 2 Let a set X be a universe of discourse. A Pythagorean fuzzy set (PFS) P is an object having the form

$$P = (\langle x, P(\mu_p(x), \nu_p(x)) | (x \in X) \rangle)$$

where the function $\mu_p : X \to [0,1]$ defines the degree of membership and $\nu_p : X \to [0,1]$ defines the degree of non-membership of the element $x \in X$ to P, respectively, and for every $x \in X$, it holds that

$$0 \leq (\mu_p(x))^2 + (\nu_p(x))^2 \leq 1$$ □

Definition 3 [13] A Pythagorean fuzzy matrix (PFM) is a pair $A = \left(\left\langle a_{ij}, a'_{ij}\right\rangle\right)$ of non negative real numbers $a_{ij}, a'_{ij} \in [0,1]$ satisfying the condition $0 \leq {a_{ij}}^2 + {a'_{ij}}^2 \leq 1$, for all i, j □

Definition 4 [13] Let A, B and C be any three Pythagorean fuzzy matrices of the same size, then
$(i) A \vee B = \left(\left\langle \max\left\{a_{ij}, b_{ij}\right\}, \min\left\{a'_{ij}, b'_{ij}\right\}\right\rangle\right)$
$(ii) A \wedge B = \left(\left\langle \min\left\{a_{ij}, b_{ij}\right\}, \max\left\{a'_{ij}, b'_{ij}\right\}\right\rangle\right)$
$(iii) A^C = \left(\left\langle a'_{ij}, a_{ij}\right\rangle\right)$ (the complement of A)
$(iv) A \leq B$ if and only if $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.
$(v) A \oplus_P B = \left(\left\langle \sqrt{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}, a'_{ij} b'_{ij}\right\rangle\right)$
$(vi) A \odot_P B = \left(\left\langle a_{ij} b_{ij}, \sqrt{{a'_{ij}}^2 + {b'_{ij}}^2 - {a'_{ij}}^2 {b'_{ij}}^2}\right\rangle\right)$, where $+, -$ and $.$ are ordinary addition, subtraction and multiplication respectively. □

Definition 5 [14] The $m \times n$ zero PFM O is a PFM all of whose entries are $\langle 0,1\rangle$. The $m \times n$ universal PFM J is a PFM all of whose entries are $\langle 1,0\rangle$. □

2.1. Hamacher operations

T-norm and T-conorm are an important notion in the intuitionistic fuzzy set theory, which is used to define a generalized union and intersection of intuitionistic fuzzy sets [17]. Roychowdhury and Wang [18] gave the definition and conditions of $t-norm$ and $t-conorm$. Deschrijver and Kerre [19] introduced and analyzed the properties of a generalized union and a generalized intersection of intuitionistic fuzzy sets using the general triangular norm ($t-norm$) and triangular conorm ($t-conorm$). Hamacher [16] proposed a more generalized $t-norm$ and $t-conorm$. Hamacher operation includes the Hamacher product and Hamacher sum, which are examples of $t-norms$ and $t-conorms$, respectively. They are defined as follows:
Hamacher product $\odot$ is a $t-norm$ and Hamacher sum $\oplus$ is a $t-conorm$, where:

$$T(a,b) = a \odot b = \frac{ab}{\gamma + (1-\gamma)(a+b-ab)}, \gamma = 0.$$

$$T^*(a,b) = a \oplus b = \frac{a+b-ab-(1-\gamma)ab}{1-(1-\gamma)ab}, \gamma = 0.$$

Especially, when $\gamma = 1$, then Hamacher t-norm and t-conorm will reduce to

$$T(a,b) = a \odot b = ab$$
$$T^*(a,b) = a \oplus b = a+b-ab$$

which are the algebraic t-norm and t-conorm respectively; when $\gamma = 2$, then Hamacher t-norm and t-conorm will reduce to

$$T(a,b) = a \odot b = \frac{ab}{1+(1-a)(1-b)}$$
$$T^*(a,b) = a \oplus b = \frac{a+b}{1+ab}$$

which are called the Einstin t-norm and t-conorm respectively.

2.2. Hamacher operations of Pythagorean fuzzy set

In [15], let $a_1 = (\mu_1, \nu_1), a_2 = (\mu_2, \nu_2)$, and $a = (\mu, \nu)$ be three PFNs, $\gamma = 0$, and based on the traditional Hamacher operations [4], Hamacher product $\odot$ is a $t-norm$ and Hamacher sum $\oplus$ is a $t-conorm$ of PFNs are defined as follows:

$$a_1 \oplus a_2 = \left(\left\langle \sqrt{\frac{(\mu_1)^2+(\mu_2)^2-(\mu_1)^2(\mu_2)^2-(1-\gamma)(\mu_1)^2(\mu_2)^2}{1-(1-\gamma)(\mu_1)^2(\mu_2)^2}}, \sqrt{\frac{(\nu_1)^2(\nu_2)^2}{\gamma+(1-\gamma)((\nu_1)^2+(\nu_2)^2-(\nu_1)^2(\nu_2)^2)}}\right\rangle\right)$$

$$a_1 \odot a_2 = \left(\left\langle \sqrt{\frac{(\mu_1)^2(\mu_2)^2}{\gamma+(1-\gamma)((\mu_1)^2+(\mu_2)^2-(\mu_1)^2(\mu_2)^2)}}, \right.\right.$$
$$\left.\left. \sqrt{\frac{(\nu_1)^2+(\nu_2)^2-(\nu_1)^2(\nu_2)^2-(1-\gamma)(\nu_1)^2(\nu_2)^2}{1-(1-\gamma)(\nu_1)^2(\nu_2)^2}} \right\rangle\right)$$

Lemma 1 *[9] For any two real numbers $a,b \in [0,1]$, the following inequality holds,*

$$\sqrt{\frac{a^2b^2}{a^2+b^2-a^2b^2}} \leq \sqrt{\frac{a^2+b^2-2a^2b^2}{1-a^2b^2}}. \qquad \square$$

Lemma 2 *[9] For any three real numbers $a,b,c \in [0,1]$, the following inequalities hold, If $a \leq b$ then,*

(i) $\sqrt{\frac{a^2c^2}{a^2+c^2-a^2c^2}} \leq \sqrt{\frac{b^2c^2}{b^2+c^2-b^2c^2}}$

(ii) $\sqrt{\frac{a^2+c^2-2a^2c^2}{1-a^2c^2}} \leq \sqrt{\frac{b^2+c^2-2b^2c^2}{1-b^2c^2}}$. $\square$

3. Hamacher operations on Pythagorean fuzzy matrices

In this section, we shall develop the Hamacher operations on Pythagorean fuzzy matrices based on the operations of PFNs and analyze some desirable properties.

Definition 6 Let $A = (\langle a_{ij}, a'_{ij}\rangle)$ and $B = (\langle b_{ij}, b'_{ij}\rangle)$ be any two Pythagorean fuzzy matrices of same size, then

(i) The Hamacher sum of A and B is defined by
$A \oplus_H B = (c_{ij})$,

$$\text{where } c_{ij} = \begin{cases} \langle 1,0\rangle, & if\ \langle a_{ij}, a'_{ij}\rangle = \langle 1,0\rangle, \langle b_{ij}, b'_{ij}\rangle = \langle 1,0\rangle \\ \left\langle \sqrt{\frac{a_{ij}^2+b_{ij}^2-2a_{ij}^2b_{ij}^2}{1-a_{ij}^2b_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2+{b'_{ij}}^2-{a'_{ij}}^2{b'_{ij}}^2}} \right\rangle, & otherwise \end{cases}$$

for all i,j.

(ii) The Hamacher product of A and B is defined by
$A \odot_H B = (c_{ij})$,

$$\text{where } c_{ij} = \begin{cases} \langle 0,1\rangle, & if\ \langle a_{ij}, a'_{ij}\rangle = \langle 0,1\rangle, \langle b_{ij}, b'_{ij}\rangle = \langle 0,1\rangle \\ \left\langle \sqrt{\frac{a_{ij}^2b_{ij}^2}{a_{ij}^2+b_{ij}^2-a_{ij}^2b_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2+{b'_{ij}}^2-2{a'_{ij}}^2{b'_{ij}}^2}{1-{a'_{ij}}^2{b'_{ij}}^2}} \right\rangle, & otherwise \end{cases}$$

for all i,j. $\square$

The relation between Hamacher sum and Hamacher product is established by the following theorem.

Theorem 1 *Let A,B be any two Pythagorean fuzzy matrices of the same size, then $A \odot_H B \leq A \oplus_H B$.*

PROOF By using Lemma 1,

$$\sqrt{\frac{a_{ij}^2 b_{ij}^2}{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}} \leq \sqrt{\frac{a_{ij}^2 + b_{ij}^2 - 2a_{ij}^2 b_{ij}^2}{1 - a_{ij}^2 b_{ij}^2}} \text{ and}$$
$$\sqrt{\frac{{a'_{ij}}^2 + {b'_{ij}}^2 - 2{a'_{ij}}^2 {b'_{ij}}^2}{1 - {a'_{ij}}^2 {b'_{ij}}^2}} \geq \sqrt{\frac{{a'_{ij}}^2 {b'_{ij}}^2}{{a'_{ij}}^2 + {b'_{ij}}^2 - {a'_{ij}}^2 {b'_{ij}}^2}} \text{ for all } i, j$$

By Definition 6, it follows that $A \odot_P B \leq A \oplus_P B$. ∎

Theorem 2 *For any Pythagorean fuzzy matrix A,*
(i) $A \oplus_H A \geq A$,
(ii) $A \odot_H A \leq A$.

PROOF (i)
$$\begin{aligned} A \oplus_H A &= \left(\left\langle \sqrt{\frac{2a_{ij}^2 - 2a_{ij}^4}{1 - a_{ij}^4}}, \sqrt{\frac{{a'_{ij}}^4}{2{a'_{ij}}^2 - {a'_{ij}}^4}} \right\rangle\right) \\ &= \left(\left\langle \sqrt{\frac{2a_{ij}^2}{1 + a_{ij}^2}}, \sqrt{\frac{{a'_{ij}}^2}{2 - {a'_{ij}}^2}} \right\rangle\right) \\ &\geq (< a_{ij}^2, {a'_{ij}}^2 >), \text{ Since } a_{ij}^2 \leq \frac{2a_{ij}^2}{1 + a_{ij}^2} \text{ and } {a'_{ij}}^2 \geq \frac{{a'_{ij}}^2}{2 - {a'_{ij}}^2} \\ &\geq (< a_{ij}, a'_{ij} >) \text{ for all } i, j \\ &\geq A. \end{aligned}$$

(ii) It can be proved similarly. ∎

The following theorem is obvious.
The operations Hamacher sum and Hamacher product of PFMs are commutative as well as associative, and the identities for $\oplus_H$ and $\odot_H$ exist.

Theorem 3 *Let A,B and C be any three Pythagorean fuzzy matrices of the same size, then*
(i) $A \oplus_H B = B \oplus_H A$,
(ii) $(A \oplus_H B) \oplus_H C = A \oplus_H (B \oplus_H C)$,
(iii) $A \odot_H B = B \odot_H A$,
(iv) $(A \odot_H B) \odot_H C = A \odot_H (B \odot_H C)$. □

The proof of the following theorem follows from the Definition 6.

Theorem 4 *For any Pythagorean fuzzy matrix A,*
(i) $A \oplus_H O = O \oplus_H A = A$,
(ii) $A \odot_H J = J \odot_H A = A$,
(iii) $A \oplus_H J = J$,
(iv) $A \odot_H O = O$. □

The set of all PFMs with respect to the Hamacher sum and Hamacher product forms a commutative monoid.
The Hamacher operations do not obey De Morgan's laws over transpose.

Theorem 5 *Let* A, B *be any two Pythagorean fuzzy matrices of the same size, then*
(i) $(A \oplus_H B)^T = A^T \oplus_H B^T$
(ii) $(A \odot_H B)^T = A^T \odot_H B^T$,
where A^T *is the transpose of* A.

Theorem 6 *Let* A, B *be any two Pythagorean fuzzy matrices of the same size, if* $A \leq B$, *then* $A \odot_H C \leq B \odot_H C$.

PROOF Let $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.
By using Lemma 2(i),

$$\sqrt{\frac{a_{ij}^2 c_{ij}^2}{a_{ij}^2 + c_{ij}^2 - a_{ij}^2 c_{ij}^2}} \leq \sqrt{\frac{b_{ij}^2 c_{ij}^2}{b_{ij}^2 + c_{ij}^2 - b_{ij}^2 c_{ij}^2}}$$

$$\Rightarrow \sqrt{\frac{{b'_{ij}}^2 {c'_{ij}}^2}{{b'_{ij}}^2 + {c'_{ij}}^2 - {b'_{ij}}^2 {c'_{ij}}^2}} \geq \sqrt{\frac{{a'_{ij}}^2 {c'_{ij}}^2}{{a'_{ij}}^2 + {c'_{ij}}^2 - {a'_{ij}}^2 {c'_{ij}}^2}} \text{ for all } i, j.$$

Therefore, $A \odot_H C \leq B \odot_H C$.
Hence the result. ■

Theorem 7 *Let* A, B *be any two Pythagorean fuzzy matrices of the same size, if* $A \leq B$, *then* $A \oplus_H C \leq B \oplus_H C$.

PROOF Let $a_{ij} \leq b_{ij}$ and $a'_{ij} \geq b'_{ij}$ for all i, j.
By using Lemma 2(ii),

$$\sqrt{\frac{a_{ij}^2 + c_{ij}^2 - 2a_{ij}^2 c_{ij}^2}{1 - a_{ij}^2 c_{ij}^2}} \leq \sqrt{\frac{b_{ij}^2 + c_{ij}^2 - 2b_{ij}^2 c_{ij}^2}{1 - b_{ij}^2 c_{ij}^2}} \text{ and}$$

$$\Rightarrow \sqrt{\frac{{b'_{ij}}^2 + {c'_{ij}}^2 - 2{b'_{ij}}^2 {c'_{ij}}^2}{1 - {b'_{ij}}^2 {c'_{ij}}^2}} \geq \sqrt{\frac{{a'_{ij}}^2 + {c'_{ij}}^2 - 2{a'_{ij}}^2 {c'_{ij}}^2}{1 - {a'_{ij}}^2 {c'_{ij}}^2}}$$

Therefore, $A \oplus_H C \leq B \oplus_H C$.
Hence the result. ■

Theorem 8 *Let A,B be any two Pythagorean fuzzy matrices of the same size, then*
(i) $(A\wedge B)\oplus_H(A\vee B)=A\oplus_H B$,
(ii) $(A\wedge B)\odot_H(A\vee B)=A\odot_H B$.

PROOF (i) $(A\wedge B)\oplus_H(A\vee B)$

$$=\left(\left\langle \min\{a_{ij},b_{ij}\},\max\{a'_{ij},b'_{ij}\}\right\rangle \oplus_H \left\langle \max\{a_{ij},b_{ij}\},\min\{a'_{ij},b'_{ij}\}\right\rangle\right)$$

$$=\left(\left\langle\sqrt{\frac{\min\left\{a_{ij}^2,b_{ij}^2\right\}+\max\left\{a_{ij}^2,b_{ij}^2\right\}-2\min\left\{a_{ij}^2,b_{ij}^2\right\}\max\left\{a_{ij}^2,b_{ij}^2\right\}}{1-\min\left\{a_{ij}^2,b_{ij}^2\right\}\max\left\{a_{ij}^2,b_{ij}^2\right\}}},\right.\right.$$
$$\left.\left.\sqrt{\frac{\max\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}\min\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}}{\max\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}+\min\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}-\max\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}\min\left\{{a'_{ij}}^2,{b'_{ij}}^2\right\}}}\right\rangle\right)$$

$$=\left(\left\langle\sqrt{\frac{a_{ij}^2+b_{ij}^2-2a_{ij}^2b_{ij}^2}{1-a_{ij}^2b_{ij}^2}},\sqrt{\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2+{b'_{ij}}^2-{a'_{ij}}^2{b'_{ij}}^2}}\right\rangle\right)$$
$$=A\oplus_H B.$$

(ii) It can be proved similarly. ■

4. Results on the complement of Pythagorean Fuzzy Matrix

In this section, the complement of a Pythagorean fuzzy matrix is used to analyze the complementing nature of any system. Using the following results, we can study the complementing nature of a system with the help of the original Pythagorean fuzzy matrix. The operator complement obey De Morgan's law for the operations $\oplus_H$ and $\odot_H$. This is established in the following theorem.

Theorem 9 *Let A,B be any two Pythagorean fuzzy matrices of the same size, then*
(i) $(A\oplus_H B)^C=A^C\odot_H B^C$,
(ii) $(A\odot_H B)^C=A^C\oplus_H B^C$,
(iii) $(A\oplus_H B)^C\le A^C\oplus_H B^C$,
(iv) $(A\odot_H B)^C\ge A^C\odot_H B^C$.

PROOF (i) $$A^C\odot_H B^C=\left(\left\langle\sqrt{\frac{{a'_{ij}}^2{b'_{ij}}^2}{{a'_{ij}}^2+{b'_{ij}}^2-{a'_{ij}}^2{b'_{ij}}^2}},\sqrt{\frac{a_{ij}^2+b_{ij}^2-2a_{ij}^2b_{ij}^2}{1-a_{ij}^2b_{ij}^2}}\right\rangle\right)$$
$$=(A\oplus_H B)^C.$$

(ii) $$A^C\oplus_H B^C=\left(\left\langle\sqrt{\frac{{a'_{ij}}^2+{b'_{ij}}^2-2{a'_{ij}}^2{b'_{ij}}^2}{1-{a'_{ij}}^2{b'_{ij}}^2}},\sqrt{\frac{a_{ij}^2b_{ij}^2}{a_{ij}^2+b_{ij}^2-a_{ij}^2b_{ij}^2}}\right\rangle\right)$$
$$=(A\odot_H B)^C.$$

(iii) $(A \oplus_H B)^C = \left(\left\langle \sqrt{\frac{a_{ij}'^2 b_{ij}'^2}{a_{ij}'^2 + b_{ij}'^2 - a_{ij}'^2 b_{ij}'^2}}, \sqrt{\frac{a_{ij}^2 + b_{ij}^2 - 2a_{ij}^2 b_{ij}^2}{1 - a_{ij}^2 b_{ij}^2}} \right\rangle\right)$

$A^C \oplus_H B^C = \left(\left\langle \sqrt{\frac{a_{ij}'^2 + b_{ij}'^2 - 2a_{ij}'^2 b_{ij}'^2}{1 - a_{ij}'^2 b_{ij}'^2}}, \sqrt{\frac{a_{ij}^2 b_{ij}^2}{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}} \right\rangle\right)$

By Lemma 1, $\sqrt{\frac{a_{ij}'^2 b_{ij}'^2}{a_{ij}'^2 + b_{ij}'^2 - a_{ij}'^2 b_{ij}'^2}} \leq \sqrt{\frac{a_{ij}'^2 + b_{ij}'^2 - 2a_{ij}'^2 b_{ij}'^2}{1 - a_{ij}'^2 b_{ij}'^2}}$ and

$\sqrt{\frac{a_{ij}^2 + b_{ij}^2 - 2a_{ij}^2 b_{ij}^2}{1 - a_{ij}^2 b_{ij}^2}} \geq \sqrt{\frac{a_{ij}^2 b_{ij}^2}{a_{ij}^2 + b_{ij}^2 - a_{ij}^2 b_{ij}^2}}$ for all i, j

Hence $(A \oplus_H B)^C \leq A^C \oplus_H B^C$.

(iv) It can be proved similarly. ■

The following theorem is obvious.

Theorem 10 *Let A, B be any two Pythagorean fuzzy matrices of the same size, then*
(i) $(A^C \odot_H B^C)^C = A \oplus_H B$,
(ii) $(A^C \oplus_H B^C)^C = A \odot_H B$.

5. Conclusions

In this paper, we have developed the Hamacher operations on Pythagorean fuzzy matrices based on the Hamacher operations of PFNs and investigated their algebraic properties. We also proved that the set of all PFMs with respect to Hamacher sum and Hamacher product forms a commutative monoid. A study of the algebraic structure of PFMs with respect to Hamacher operations gives us a deep insight into the applications. Finally, De Morgan's laws are verified.

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