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ANALYSIS OF THE QUEUEING NETWORK WITH A RANDOM WAITING TIME OF NEGATIVE CUSTOMERS AT A NON-STATIONARY REGIME

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Abstract. In the article a queueing network (QN) with positive customers and a random waiting time of negative customers has been investigated. Negative customers destroy positive customers on the expiration of a random time. Queueing systems (QS) operate under a heavy-traffic regime. The system of difference-differential equations (DDE) for state probabilities of such a network was obtained. The technique of solving this system and finding mean characteristics of the network, which is based on the use of multivariate generating functions was proposed.

Keywords: *G-network, positive customers, negative customers, random waiting time, heavy-traffic regime, state probabilities, mean characteristics, non-stationary regime*

1. Network description

Consider an open G-network [1] with n single-queues QS. An independent Poisson flow of positive customers with rate λ_{0i}^{+} and a Poisson flow of negative customers with rate λ_{0i}^{-} arrive to QS S_i from outside (system S_0), $i=\overline{1,n}$. All arriving to QS customer flows are assumed to be independent. The probability that the positive customer serviced in S_i during time $[t,t+\Delta t)$, if at the current moment t in the system there are k_i customers, are equal to $\mu_i^{+}(k_i)\Delta t+o(\Delta t)$. The positive customer gets serviced in S_i with probability p_{ij}^{+} move to QS S_j as a positive customer and with probability p_{ij}^{-} - as a negative customer and with probability $p_{i0}=1-\sum_{j=1}^{n}\left(p_{ij}^{+}+p_{ij}^{-}\right)$ come out of the network to the external environment, $i,j=\overline{1,n}$.

A negative customer is arriving to QS increases the length of the queue of negative customers for one, and requires no service. Each negative customer, located in i-th QS, stays in the queue for a random time according to a Poisson process of rate $\mu_i^-(l_i)$, $i=\overline{1,n}$. By the end this time, the negative customer destroys one positive customer in the QS S_i and leaves the network. If after this random time in the system there are no positive customers, then a given negative customer leaves the network, without exerting any influence on the operation of the network as a whole. Wherein the probability that in QS S_i, negative customer leaves the queue during $[t,t+\Delta t)$, on the condition that, in this QS at time t there are l_i negative customers, equals $\mu_i^-(l_i)\Delta t+o(\Delta t)$.

The network state at time t described by the vector $k(t)=(k,l,t)=$ $=((k_1,l_1,t),(k_2,l_2,t),...,(k_n,l_n,t))$, which forms a homogeneous Markov process with a countable number of states, where the state (k_i,l_i,t) means that at time t in QS S_i, there are k_i positive customers and l_i negative customers, $i=\overline{1,n}$. We introduce the vectors $(k,t)=(k_1,k_2,...,k_n,t)$ and $(l,t)=(l_1,l_2,...,l_n,t)$, I_i - vector, which is i-th component equal to 1, all the others are 0, $i=\overline{1,n}$.

Negative customers may describe the behavior of computer viruses, whose impact on the information (positive customers) occurs through a random time.

It should be noted that analisys at a stationary regime of QN with positive and negative customers excluding random queueing time, and also with signals has been carried out in [2, 3] and at non-stationary regime in [4-5].

2. State probabilities of the network operating under a heavy-traffic regime

Lemma. *Let* $P(k,l,t)$ *- state probability* (k,l) *at time* t. *State probabilities of considered network are satisfy system of DDE:*

$$\frac{dP(k,l,t)}{dt}=-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i^{+}(k_i)\left(1-p_{ii}^{+}\right)+\mu_i^{-}(l_i)\right]P(k,l,t)+$$
$$+\sum_{i=1}^{n}\lambda_{0i}^{+}u(k_i(t))\,P(k-I_i,l,t)+\sum_{i=1}^{n}\lambda_{0i}^{-}u(l_i(t))\,P(k,l-I_i,t)+$$
$$+\sum_{i=1}^{n}\mu_i^{+}(k_i+1)p_{i0}P(k+I_i,l,t)+\sum_{i=1}^{n}\mu_i^{-}(l_i+1)P(k+I_i,l+I_i,t)+ \quad (1)$$
$$+\sum_{i=1}^{n}\mu_i^{-}(l_i+1)(1-u(k_i(t)))\,P(k,l+I_i,t)+$$

$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i^{+}(k_i+1)u(k_j(t))p_{ij}^{+}\,P(k+I_i-I_j,l,t)+$$

$$+\sum_{i,j=1}^{n}\mu_i^{+}(k_i+1)u(l_j(t))p_{ij}^{-}P(k+I_i,l-I_j,t)$$

where $\mu_i^{+}(0)=0,\ \mu_i^{-}(0)=0.$

Proof. The possible transitions of our Markov process in the state $(k,l,t+\Delta t)$ during time Δt:

1) from the state Δt, in this case into QS S_i for the time Δt a positive customer will arrive with probability $\lambda_{0i}^{+}u(k_i(t))\Delta t+o(\Delta t),\ i=\overline{1,n}$;
2) from the state $(k,l-I_i,t)$, while to the QS S_i for the time Δt a negative customer will arrive with probability $\lambda_{0i}^{-}u(l_i(t))\Delta t+o(\Delta t),\ i=\overline{1,n}$;
3) from the state $(k+I_i,l,t)$, in this case the positive customer comes out of the network to the external environment with probability $\mu_i^{+}(k_i+1)p_{i0}\Delta t+o\,(\Delta t)$, $i=\overline{1,n}$;
4) from the state $(k+I_i,l+I_i,t)$, in the given case into QS S_i the negative customer, destroys in the QS S_i the positive customer, leaves the network; the probability of such an event is equal to $\mu_i^{-}(l_i+1)\Delta t+o(\Delta t),\ i=\overline{1,n}$;
5) from the state $(k,l+I_i,t)$, while in the QS S_i, the residence time in the queue of the negative customer finished, if in time t there were l_i+1 negative customers and there were no positive customers; the probability of such an event is equal to $\mu_i^{-}(l_i+1)(1-u(k_i(t)))\Delta t+o\,(\Delta t)$, $i=\overline{1,n}$;
6) from the state $(k+I_i-I_j,l,t)$, in given case after finishing the service of the positive customer in the QS S_i it moves to the QS S_j again as a positive customer with probability $\mu_i^{+}(k_i+1)u(k_j(t))p_{ij}^{+}\Delta t+o(\Delta t),\ i=\overline{1,n}$;
7) from the state $(k+I_i,l-I_j,t)$, in this case the positive customer, which is serviced in QS S_i, moves to QS S_j as a negative customer; the probability of such an event is equal to $\mu_i^{+}(k_i+1)u(l_j(t))p_{ij}^{-}\Delta t+o\,(\Delta t),\ i=\overline{1,n}$;
8) from the state (k,l,t), while in each QS S_i, $i=\overline{1,n}$, do not arrive any positive nor any negative customers, and in which for the time Δt any customer didn't service, no negative customer will come out of the queue; the probability of such event is equal to

$$1-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i^{+}(k_i)\left(1-p_{ii}^{+}\right)+\mu_i^{-}(l_i)\right]\Delta t+o(\Delta t),\ i=\overline{1,n};$$

9) from other states with probability $o(\Delta t)$.

Then, using the formula of total probability, we can write

$$P(k,l,t+\Delta t)=\sum_{i=1}^{n}\lambda_{0i}^{+}u(k_i(t))P(k-I_i,l,t)\Delta t+\sum_{i=1}^{n}\lambda_{0i}^{-}u(l_i(t))P(k,l-I_i,t)\Delta t+$$
$$+\sum_{i=1}^{n}\mu_i^{+}(k_i+1)p_{i0}P(k+I_i,l,t)\Delta t+\sum_{i=1}^{n}\mu_i^{-}(l_i+1)P(k+I_i,l+I_i,t)\Delta t+$$
$$+\sum_{i=1}^{n}\mu_i^{-}(l_i+1)(1-u(k_i(t)))P(k,l+I_i,t)\Delta t+$$
$$+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i^{+}(k_i+1)u(k_j(t))p_{ij}^{+}P(k+I_i-I_j,l,t)\Delta t+$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i^{+}(k_i+1)u(l_j(t))p_{ij}^{-}\;P(k+I_i,l-I_j,t)\Delta t+$$
$$+\left(1-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{-}+\mu_i^{+}(k_i)(1-p_{ii}^{+})+\mu_i^{-}(l_i)\right]\Delta t\right)P(k,l,t)+o(\Delta t)$$

Taking the limit $\Delta t\to 0$, we obtain a system of equations for state probabilities of the network. (1). The lemma is proved.

We will assume, that all queuing network systems are single-queue, and customer service duration in the QS has an exponential distribution with the rate μ_i^{+}. Consequently, in this case $\mu_i^{+}(k_i)=\mu_i^{+}u(k_i)$, $i=\overline{1,n}$.

Denote by $\Psi_{2n}(z,t)$, where $z=(z_1,z_2,...,z_n,z_{n+1},...,z_{2n})$, the generating function of the dimension of $2n$:

$$\Psi_{2n}(z,t)=\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{l_1=0}^{\infty}...\sum_{l_n=0}^{\infty}P(z_1,z_2,...,z_n,z_{n+1},...,z_{2n})z_1^{k_1}...\,z_n^{k_n}\,z_{n+1}^{l_1}...z_{2n}^{l_n}=$$
$$=\sum_{k_1=0}^{\infty}...\sum_{k_n=0}^{\infty}\sum_{l_1=0}^{\infty}...\sum_{l_n=0}^{\infty}P\,(k,l,t)\prod_{i=1}^{n}z_i^{k_i}z_{n+i}^{l_i},\ |z|<1, \tag{2}$$

the summation is taking for each k_i, l_i from 0 to ∞, $i=\overline{1,n}$.

We will assume that $k_i(t)>0$, $l_i(t)>0$, $\forall t>0$, $i=\overline{1,n}$.

Multiplying each of the equations (1) to $\prod_{m=1}^{n}z_m^{k_m}z_m^{l_m}$ and summing up all possible values k_m and l_m from 1 to $+\infty$, $m=\overline{1,n}$. Here the summation for all k_m and l_m is taken from 1 to $+\infty$, i.e. all summands in (2), for which in the network state $k(t)$ there are components $k_m=0$ and $l_m=0$, due to the assumptions put forward above. Because, for example

$$P\left(k_1,...,k_{m-1},0,k_{m+1},...,k_n,l_1,...,l_{m-1},0,l_{m+1},...,l_n,t\right)=0,\ m=\overline{2,n}.$$

Then we obtain

$$\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}\frac{dP(k,l,t)}{dt}\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}=$$

$$\begin{aligned}
&=-\sum_{i=1}^{n}\left(\lambda_{0i}^{+}+\lambda_{0i}^{+}+\mu_i^{+}\left(1-p_{ii}^{+}\right)+\mu_i^{-}\right)\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\lambda_{0i}^{+}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k-I_i,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\lambda_{0i}^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l-I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\mu_i^{+}p_{i0}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\mu_i^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l+I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\mu_i^{-}\left(1-u(k_i(t))\right)\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l+I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i^{+}p_{ij}^{+}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i-I_j,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}+\\
&+\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i^{+}p_{ij}^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l-I_j,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}
\end{aligned}\tag{3}$$

Let's consider the sums, contained on the right side of the relation (3). Let

$$\sum\nolimits_1(z,t)=-\sum_{i=1}^{n}\left(\lambda_{0i}^{+}+\lambda_{0i}^{+}+\mu_i^{+}\left(1-p_{ii}^{+}\right)+\mu_i^{-}\right)\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}.$$

Then

$$\sum\nolimits_1(z,t)=-\sum_{i=1}^{n}\left(\lambda_{0i}^{+}+\lambda_{0i}^{+}+\mu_i^{+}\left(1-p_{ii}^{+}\right)+\mu_i^{-}\right)\Psi_{2n}(z,t).$$

Similarly for the sum $\sum_2(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{+}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k-I_i,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}$ we have:

$$\sum\nolimits_2(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{+}z_i\Psi_{2n}(z,t).$$

For the sum $\sum_3(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l-I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}$ we obtain:

$$\sum\nolimits_3(z,t)=\sum_{i=1}^{n}\lambda_{0i}^{-}z_{n+i}\Psi_{2n}(z,t).$$

The sum $\sum_4(z,t)=\sum_{i=1}^{n}\mu_i^{+}p_{i0}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}$ has the form:

$$\sum\nolimits_4(z,t)=\sum_{i=1}^{n}\mu_i^{+}\frac{p_{i0}}{z_i}\Psi_{2n}(z,t)$$

For the sum $\sum_5(z,t)=\sum_{i=1}^{n}\mu_i^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l+I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}$ we obtain:

$$\sum\nolimits_5(z,t)=\sum_{i=1}^{n}\mu_i^{-}\frac{1}{z_iz_{n+i}}\Psi_{2n}(z,t).$$

The sum $\sum_6(z,t)=\sum_{i=1}^{n}\mu_i^{-}(1-u(k_i(t)))\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k,l+I_i,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}=0.$

For the sum $\sum_7(z,t)=\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mu_i^{+}p_{ij}^{+}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i-I_j,l,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}$ we shall obtain:

$$\sum\nolimits_7(z,t)=\sum_{i,j=1}^{n}\mu_i^{+}p_{ij}^{+}\frac{z_j}{z_i}\Psi_{2n}(z,t).$$

And, finally, for the last sum we shall have:

$$\sum\nolimits_8(z,t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i^{+}p_{ij}^{-}\sum_{k_1=1}^{\infty}...\sum_{k_n=1}^{\infty}\sum_{l_1=1}^{\infty}...\sum_{l_n=1}^{\infty}P(k+I_i,l-I_j,t)\prod_{m=1}^{n}z_m^{k_m}z_{n+m}^{l_m}=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\mu_i^{+}p_{ij}^{-}\frac{z_{n+j}}{z_i}\Psi_{2n}(z,t)$$

Using these sums, we obtain a homogeneous linear differential equation:

$$\frac{d\,\Psi_{2n}(z,t)}{dt} = -\sum_{i=1}^{n}\Bigg[\lambda_{0i}^{+} + \lambda_{0i}^{+} + \mu_i^{+}\left(1 - p_{ii}^{+}\right) + \mu_i^{-} - \lambda_{0i}^{+} z_i - \lambda_{0i}^{-} z_{n+i} - \frac{\mu_i^{-}}{z_i z_{n+i}} - \\ - \mu_i^{+}\sum_{j=1}^{n}\left(p_{ij}^{+}\frac{z_j}{z_i} + p_{ij}^{-}\frac{z_{n+j}}{z_i}\right)\Bigg]\Psi_{2n}(z,t).$$

Its solution has the form

$$\Psi_n(z,t) = C_n \exp\Bigg\{-\sum_{i=1}^{n}\Bigg[\lambda_{0i}^{+} + \lambda_{0i}^{+} + \mu_i^{+}\left(1 - p_{ii}^{+}\right) + \mu_i^{-} - \lambda_{0i}^{+} z_i - \lambda_{0i}^{-} z_{n+i} - \frac{\mu_i^{-}}{z_i z_{n+i}} - \\ - \mu_i^{+}\sum_{j=1}^{n}\left(p_{ij}^{+}\frac{z_j}{z_i} + p_{ij}^{-}\frac{z_{n+j}}{z_i}\right)\Bigg]t\Bigg\}.$$

Let's consider, that at the initial moment of time, the network is in a state $(\alpha_1, \alpha_2, ..., \alpha_{2n}, 0)$, $\alpha_i > 0$, $\alpha_{n+i} > 0$,

$$P(\alpha_1, \alpha_2, ..., \alpha_{2n}, 0) = 1,\ P(k_1, k_2, ..., k_n, l_1, l_2, ..., l_n\ 0) = 0,\ \forall \alpha_i \neq k_i, l_i,\ i = \overline{1,n}.$$

Then the initial condition for the last equation will be

$$\Psi_{2n}(z,0) = P(\alpha_1, \alpha_2, ..., \alpha_{2n}, 0)\prod_{m=1}^{n} z_m^{\alpha_m} z_{n+m}^{\alpha_{n+m}} = \prod_{m=1}^{n} z_m^{\alpha_m} z_{n+m}^{\alpha_{n+m}},$$

from which we obtain $C_n = 1$.

Theorem. If at the initial moment of time the QN is in a state $(\alpha_1, \alpha_2, ..., \alpha_{2n}, 0)$, $\alpha_i > 0$, $\alpha_{n+i} > 0$, $i = \overline{1,n}$, then the expression for the generating function $\Psi_{2n}(z,t)$, taking into account the expansions appearing in it exponent Maclaurin, has the form

$$\Psi_{2n}(z,t) = a_0(t) \sum_{\substack{\underline{b_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} \sum_{\substack{\underline{c_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} \sum_{\substack{\underline{d_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} \sum_{\substack{\underline{g_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} \sum_{\substack{\underline{h_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} \sum_{\substack{\underline{r_j = 0} \\ j=\overline{1,n}, j\neq i}}^{\infty} t^{\sum_{i=1}^{n}(b_i + c_i + d_i + g_i + h_i + r_i)} \times \\ \times \prod_{i=1}^{n}\left[\frac{\left(\prod_{j=1}^{n} p_{ij}^{+}\right)^{h_i}\left(\prod_{j=1}^{n} p_{ij}^{-}\right)^{r_i}}{b_i!c_i!d_i!g_i!h_i!r_i!}\left(\lambda_{0i}^{+}\right)^{b_i}\left(\lambda_{0i}^{-}\right)^{c_i}\left(\mu_i^{+}\right)^{h_i + r_i}\left(\mu_i^{-}\right)^{d_i + g_i} z_i^{\alpha_i + b_i - d_i - g_i + H - h_i - r_i} z_{n+i}^{\alpha_{n+i} + c_i - d_i + R}\right], \tag{4}$$

where

$$H=\sum_{i=1}^{n}h_i,\ R=\sum_{i=1}^{n}r_i,\ a_0(t)=\exp\left\{-\sum_{i=1}^{n}\left[\lambda_{0i}^{+}+\lambda_{0i}^{+}+\mu_i^{+}\left(1-p_{ii}^{+}\right)+\mu_i^{-}\right]t\right\}.$$

Proof. We have:

$$\Psi_n(z,t)=a_0(t)\prod_{i=1}^{5}a_i(z,t)\prod_{m=1}^{n}z_m^{\alpha_m}z_{n+m}^{\alpha_{n+m}},$$

where

$$a_1(z,t)=\exp\left\{t\sum_{i=1}^{n}\lambda_{0i}^{+}z_i\right\}=\prod_{i=1}^{n}\sum_{b_i=0}^{\infty}\frac{\left[\lambda_{0i}^{+}tz_i\right]^{b_i}}{b_i!}=\sum_{b_1=0}^{\infty}...\sum_{b_n=0}^{\infty}\prod_{i=1}^{n}\frac{\left[\lambda_{0i}^{+}tz_i\right]^{b_i}}{b_i!}=$$

$$=\sum_{b_1=0}^{\infty}...\sum_{b_n=0}^{\infty}\frac{t^{b_1+b_2+...+b_n}}{b_1!b_2!\cdot...\cdot b_n!}\left(\lambda_{01}^{+}\right)^{b_1}\cdot...\cdot\left(\lambda_{0n}^{+}\right)^{b_n}z_1^{b_1}\cdot...\cdot z_n^{b_n}$$

$$a_2(z,t)=\exp\left\{t\sum_{i=1}^{n}\lambda_{0i}^{-}z_{n+i}\right\}=\sum_{c_1=0}^{\infty}...\sum_{c_n=0}^{\infty}\frac{t^{c_1+c_2+...+c_n}}{c_1!c_2!\cdot...\cdot c_n!}\left(\lambda_{01}^{-}\right)^{c_1}\cdot...\cdot\left(\lambda_{0n}^{-}\right)^{c_n}z_{n+1}^{c_1}\cdot...\cdot z_{2n}^{c_n}$$

$$a_3(z,t)=\exp\left\{t\sum_{i=1}^{n}\mu_i^{-}\frac{1}{z_iz_{n+i}}\right\}=\prod_{i=1}^{n}\sum_{d_i=0}^{\infty}\frac{\left[\mu_i^{-}tz_i^{-1}z_{n+i}^{-1}\right]^{d_i}}{d_i!}=$$

$$=\sum_{d_1=0}^{\infty}...\sum_{d_n=0}^{\infty}\frac{t^{d_1+d_2+...+d_n}}{d_1!d_2!\cdot...\cdot d_n!}\left(\mu_1^{-}\right)^{d_1}\cdot...\cdot\left(\mu_n^{-}\right)^{d_n}z_1^{-d_1}\cdot...\cdot z_n^{-d_n}z_{n+1}^{-d_1}\cdot...\cdot z_{2n}^{-d_n}$$

$$a_4(z,t)=\exp\left\{\sum_{i,j=1}^{n}t\mu_i^{+}p_{ij}^{+}\frac{z_j}{z_i}\right\}=\prod_{i=1}^{n}\prod_{j=1}^{n}\sum_{h_i=0}^{\infty}\frac{\left[t\mu_i^{+}p_{ij}^{+}z_jz_i^{-1}\right]^{h_i}}{h_i!}=$$

$$=\sum_{h_1=0}^{\infty}...\sum_{h_n=0}^{\infty}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{\left[t\mu_i^{+}p_{ij}^{+}z_jz_i^{-1}\right]^{h_i}}{h_i!}=\sum_{h_1=0}^{\infty}...\sum_{h_n=0}^{\infty}t^{h_1}\cdot...\cdot t^{h_n}\frac{\left(\prod_{j=1}^{n}\mu_1^{+}p_{1j}^{+}\right)^{h_1}\cdot...\cdot\left(\prod_{j=1}^{n}\mu_n^{+}p_{nj}^{+}\right)^{h_n}}{h_1!\cdot...\cdot h_n!}\times$$

$$\times z_1^{h_1+h_2+...+h_n}z_2^{h_1+h_2+...+h_n}\cdot...\cdot z_n^{h_1+h_2+...+h_n}z_1^{-h_1}z_2^{-h_2}...z_n^{-h_n}=$$

$$\sum_{h_1=0}^{\infty}...\sum_{h_n=0}^{\infty}t^{h_1}\cdot...\cdot t^{h_n}\frac{\left(\prod_{j=1}^{n}\mu_1^{+}p_{1j}^{+}\right)^{h_1}\cdot...\cdot\left(\prod_{j=1}^{n}\mu_n^{+}p_{nj}^{+}\right)^{h_n}}{h_1!\cdot...\cdot h_n!}z_1^{H-h_1}\cdot...\cdot z_n^{H-h_n},$$

$$a_5(z,t)=\exp\left\{\sum_{i,j=1}^{n} t\mu_i^+ p_{ij}^- \frac{z_{n+j}}{z_i}\right\}=\prod_{i=1}^{n}\prod_{j=1}^{n}\exp\left\{t\mu_i^+ p_{ij}^- \frac{z_{n+j}}{z_i}\right\}=\prod_{i=1}^{n}\prod_{j=1}^{n}\sum_{r_i=0}^{\infty}\frac{\left[t\mu_i^+ p_{ij}^- z_{n+j} z_i^{-1}\right]^{r_i}}{r_i!}=$$

$$=\sum_{r_1=0}^{\infty}\ldots\sum_{r_n=0}^{\infty}\prod_{i=1}^{n}\prod_{j=1}^{n}\frac{\left[t\mu_i^+ p_{ij}^- z_{n+j} z_i^{-1}\right]^{r_i}}{r_i!}=\sum_{r_1=0}^{\infty}\ldots\sum_{r_n=0}^{\infty} t^{r_1}\cdot\ldots\cdot t^{r_n}\frac{\left(\prod_{j=1}^{n}\mu_1^+ p_{1j}^-\right)^{r_1}\cdot\ldots\cdot\left(\prod_{j=1}^{n}\mu_n^+ p_{nj}^-\right)^{r_n}}{r_1!\cdot\ldots\cdot r_n!}\times$$

$$\times z_1^{-r_1} z_2^{-r_2}\ldots z_n^{-r_n} z_{n+1}^{r_1+r_2+\ldots+r_n} z_{n+2}^{r_1+r_2+\ldots+r_n}\cdot\ldots\cdot z_{2n}^{r_1+r_2+\ldots+r_n}$$

Multiplying $a_0(t)$, $a_i(z,t)$, and $\prod_{m=1}^{n} z_m^{\alpha_m} z_{n+m}^{\alpha_{n+m}}$ we will obtain an expression (4), $i=\overline{1,5}$.

State probability of $P(k_1,k_2,..,k_n,l_1,l_2,...,l_n,t)$ is the coefficient of $z_1^{k_1} z_2^{k_2},...,z_n^{k_n} z_{n+1}^{l_1}, z_{n+2}^{l_2},...,z_{2n}^{l_n}$ in the expansion of $\Psi_{2n}(z,t)$ in multiple series (4), with the proviso, that at the initial time the network is in a state $(\alpha_1,\alpha_2,...,\alpha_{2n},0)$.

3. Finding the main characteristics

With the help of the generating function a different mean network characteristics can also be found at the transient regime. The expectation of a component with the number x of a multivariate random variable can be found, differentiating (4) by z_x and suppose $z_i=1$, $i=\overline{1,2n}$. Therefore for the mean number of positive customers in the network system S_x we will use the relation:

$$N_x^+(t)=\left.\frac{\partial\Psi_{2n}(z,t)}{\partial z_x}\right|_{z=(1,1,\ldots,1)}=$$

$$=a_0(t)\sum_{\substack{b_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{\substack{c_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{\substack{d_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{\substack{g_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{\substack{h_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty}\sum_{\substack{r_j=0\\ j=\overline{1,n},\,j\neq i}}^{\infty} t^{\sum_{i=1}^{n}(b_i+c_i+d_i+g_i+h_i+r_i)}\times$$
$$\times\left(\alpha_x+b_x-d_x-g_x+H-h_x-r_x\right)\times$$
$$\times\prod_{i=1}^{n}\left[\frac{\left(\lambda_{0i}^+\right)^{b_i}\left(\lambda_{0i}^-\right)^{c_i}\left(\mu_i^+\right)^{h_i+r_i}\left(\mu_i^-\right)^{d_i+g_i}}{b_i!c_i!d_i!g_i!h_i!r_i!}\left(\prod_{j=1}^{n}p_{ij}^+\right)^{h_i}\left(\prod_{j=1}^{n}p_{ij}^-\right)^{r_i}\right],\ x=\overline{1,n}. \quad (5)$$

The change of variables will be done in the expression (5) $k_x=\alpha_x+b_x-d_x-g_x+H-h_x-r_x$, then $b_x=k_x-\alpha_x+d_x+g_x-H+h_x+r_x$ and

$$N_x^+(t)=a_0(t)\sum_{\substack{c_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{d_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{g_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{h_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{r_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{k_j=\alpha_j-d_j-g_j+H-h_j-r_j\\ j=\overline{1,n},j\neq i}}^{\infty}k_x\times$$

$$\times t^{\sum_{i=1}^{n}(k_i-\alpha_i+2d_i+c_i+2g_i+2h_i+2r_i-H)}\times$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\lambda_{0i}^{+}\right)^{k_i-\alpha_i+d_i+g_i-H+h_i+r_i}\left(\lambda_{0i}^{-}\right)^{c_i}\left(\mu_i^{+}\right)^{h_i+r_i}\left(\mu_i^{-}\right)^{d_i+g_i}}{(k_i-\alpha_i+d_i+g_i-H+h_i+r_i)!c_i!d_i!g_i!h_i!r_i!}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{h_i}\left(\prod_{j=1}^{n}p_{ij}^{-}\right)^{r_i}\right],\ x=\overline{1,n}.$$

So like all network QS operating under heavy-traffic regime, we obtain, then $k_i=\alpha_i-d_i-g_i-h_i-r_i+H\geq 1$ and, consequently, $d_i\leq\alpha_i-g_i-h_i-r_i+H-1$, therefore

$$N_x^+(t)=a_0(t)\sum_{\substack{c_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{d_j=0\\ j=\overline{1,n},j\neq i}}^{\alpha_j-g_j-h_j-r_j+H-1}\sum_{\substack{g_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{h_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{r_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{k_j=1\\ j=\overline{1,n}}}^{\infty}k_x\times$$

$$\times t^{\sum_{i=1}^{n}(k_i-\alpha_i+2d_i+c_i+2g_i+2h_i+2r_i-H)}\times$$

$$\times\prod_{i=1}^{n}\left[\frac{\left(\lambda_{0i}^{+}\right)^{k_i-\alpha_i+d_i+g_i-H+h_i+r_i}\left(\lambda_{0i}^{-}\right)^{c_i}\left(\mu_i^{+}\right)^{h_i+r_i}\left(\mu_i^{-}\right)^{d_i+g_i}}{(k_i-\alpha_i+d_i+g_i-H+h_i+r_i)!c_i!d_i!g_i!h_i!r_i!}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{h_i}\left(\prod_{j=1}^{n}p_{ij}^{-}\right)^{r_i}\right],\ x=\overline{1,n}. \quad (6)$$

Similarly, we can find the relation for the mean number of negative customers in the system S_x, that are awaiting:

$$N_x^-(t)=a_0(t)\sum_{\substack{b_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{d_j=0\\ j=\overline{1,n},j\neq i}}^{\alpha_{n+j}+R-1}\sum_{\substack{g_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{h_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{r_j=0\\ j=\overline{1,n},j\neq i}}^{\infty}\sum_{\substack{l_j=1\\ j=\overline{1,n}}}^{\infty}l_x\times$$

$$\times t^{\sum_{i=1}^{n}(l_i-\alpha_{n+i}+b_i+2d_i+g_i+h_i+r_i-R)}\times \quad (7)$$

$$\times\prod_{i=1}^{n}\frac{\left(\lambda_{0i}^{+}\right)^{b_i}\left(\lambda_{0i}^{-}\right)^{l_i-\alpha_{n+i}+d_i-R}\left(\mu_i^{+}\right)^{h_i+r_i}\left(\mu_i^{-}\right)^{d_i+g_i}}{b_i!(l_i-\alpha_{n+i}+d_i-R)!d_i!g_i!h_i!r_i!}\left(\prod_{j=1}^{n}p_{ij}^{+}\right)^{h_i}\left(\prod_{j=1}^{n}p_{ij}^{-}\right)^{r_i}.$$

Example. Let the number of QS in QN be $n=3$. Let external arrivals to the network of positive and negative customers respectively equal: $\lambda_{01}^{+}=1$, $\lambda_{02}^{+}=2$, $\lambda_{03}^{+}=0{,}5$, $\lambda_{01}^{-}=2$, $\lambda_{02}^{-}=1$, $\lambda_{03}^{-}=0{,}3$, and the service times of rates equal: $\mu_1^{+}=1$, $\mu_2^{+}=2$, $\mu_3^{+}=3$. Let negative customers stay in the queue for a random time, which has an exponential distribution with parameters equal: $\mu_1^- = 0{,}5$, $\mu_2^- = 0{,}2$, $\mu_3^- = 0{,}3$. We assume that the transition probability of positive customers p_{ij}^+ has the form: $p_{12}^+ = 0{,}1$, $p_{13}^+ = 0{,}25$, $p_{21}^+ = 0{,}3$, $p_{23}^+ = 0{,}2$, $p_{31}^+ = 0{,}1$, $p_{32}^+ = 0{,}4$; transition probabilities of negative customers equal: $p_{12}^- = \frac{1}{8}$, $p_{13}^- = \frac{1}{7}$, $p_{21}^- = \frac{3}{11}$, $p_{23}^- = \frac{1}{9}$, $p_{31}^- = \frac{2}{9}$, $p_{32}^- = \frac{2}{11}$; then the probabilities p_{i0} will be equal respectively: $p_{10} = 0{,}38$, $p_{20} = 0{,}12$, $p_{30} = 0{,}096$. In this case $a_0(t) = e^{-13{,}8t}$.

The mean number of customers in network systems (in the queue and in servicing), on the condition that $N_m(0) = 0$, $m = \overline{1,n}$, can be found by the formula (6), and the mean number of negative customers (waiting in the queue) may be found by the formula (7).

Figure 1 shows the chart of change of the mean number of positive customers in the QS S_1 (straight line) and the chart of change of the mean number of negative customers (dash line), which are awaiting in the queue of the QS S_1 respectively.

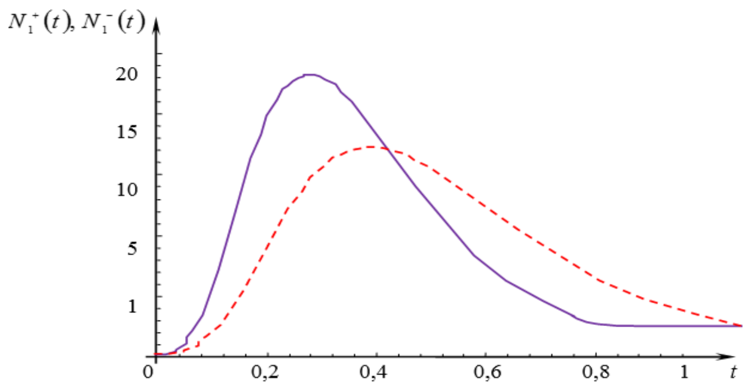

Fig. 1. The chats of changes of the mean number of positive customers and negative customers in the QS S_1

4. Conclusions

In the paper, the Markov network with positive customers with a random waiting time of negative customers at transient regime has been investigated. A technique of finding non-stationary state probabilities of the above network with single-queues of QS was proposed. It is based on the method of using the apparatus of multivariate generating functions. Relations for the mean characteristics depending on time of the considered G-network, on the condition that the network operates under heavy-traffic regime was obtained.

The practical significance of these results is that they can be used for modeling the functioning of various information networks and systems, a model of which is the aforementioned network taking into account the penetration of computer viruses into it.

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