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# THE BUSY PERIOD FOR THE $M^{\theta}$/G/1/m SYSTEM WITH SERVICE TIME DEPENDENT OF THE QUEUE LENGTH

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**Abstract.** We consider the $M^{\theta}$/G/1/m system wherein the service time depends on the queue length and it is determined at the beginning of customer service. Using an approach based on the potential method proposed by V. Korolyuk, the Laplace transforms for the distribution of the number of customers in the system on the busy period and for the distribution function of the busy period are found.

***Keywords:*** *queueing system, group arrival of customers, service time dependent on the queue length*

## Introduction

Models of queuing systems in which customers arrive in groups, and the intensity of service purposefully varies with the queue length, often used for the study of the telecommunication processes, in particular the processes of data transmission in ATM networks using multiplexing techniques (see references in [1]). Continuing the research started in [1, 2], in this paper we study the characteristics of the busy period of the system in which the service time of each customer is determined according to the rule: if at the moment of a customer service start there are $n$ customers in the system, the distribution function of the customer service time is $F_n(x)$. Queueing systems with switching of operation regimes were studied by conventional methods, in particular, in [3, 4, 5, p. 56-61, 6]. In this paper we use the idea of the potential method proposed by V. Korolyuk [7].

## 1. Description of the model and basic notations

We consider an $M^{\theta}$/G/1/m queueing system that is formally described as follows. Let sequences of random variables $\{\alpha_i\}$, $\{\theta_i\}$, $\{\beta_{in}\}$, $i, n \geq 1$, be specified, where $\alpha_i$ is the time between arrivals of the $(i-1)$-th and $i$-th groups, $\theta_i$ is the number of customers in the $i$-th group, and $\beta_{in}$ is the service time of the $i$-th customer on condition, that at the moment of its service start there are $n$ customers in the system. All these random variables are supposed to be totally independent and $\mathbf{P}\{\alpha_i < x\} = 1 - e^{-\lambda x}$, $\lambda > 0$, $\mathbf{P}\{\theta_i = n\} = a_n$, $\sum_{n=1}^{\infty} a_n = 1$. If $\mathbf{P}\{\theta_i = 1\} = a_1 = 1$, then customers arrive at the system one by one.

Denote the number of customers in the system at time $t$ as $\xi(t)$. If $t$ is the instant of the $i$-th customer service start and $\xi(t) = n$, then $\mathbf{P}\{\beta_{in} < x\} = F_n(x)$ $(x \geq 0)$, $F_n(0) = 0$ $(n \in \{1, 2, \ldots, m+1\})$.

Customers are served one by one, a served customer leaves the system, and the server immediately starts serving a customer from the queue, if one exists, or waits for the arrival of the next customers group. The first-in first-out (FIFO) service discipline is used. A queue inside one customer group can be arbitrarily organized, since the characteristics under study are independent of the way in which the queue is organized. Let $m$ be the maximum number of customers that can simultaneously be in the queue. Denote the queueing system described above as $\mathrm{M}^{\theta}/\mathrm{G}_1, \ldots, \mathrm{G}_{\mathrm{m}+1}/1/\mathrm{m}$.

Denote by $\mathbf{P}_n$ the conditional probability, provided that at the initial time the number of customers of queueing system is $n \geq 0$, and by $\mathbf{P}$ the conditional probability if the system starts to work at the time of arrival of the first group of customers.

Suppose that there are finite mathematical expectations

$$\int_0^{\infty} x dF_n(x) < \infty, \quad 1 \leq n \leq m+1; \qquad \sum_{k=1}^{\infty} k a_k < \infty.$$

We introduce the following notations: $\eta(x)$ is the number of customers arriving in the system during the time interval $[0;x)$; $a_n^{k*}$ is the $k$-fold convolution of the sequence $a_n$;

$$f_n(s) = \int_0^{\infty} e^{-sx} dF_n(x), \qquad \overline{F}_n(x) = 1 - F_n(x), \qquad \overline{a}_n = \sum_{k=n}^{\infty} a_k;$$

$$p_{ni}(s) = \frac{1}{f_n(s)} \int_0^{\infty} e^{-sx} \mathbf{P}_n\{\eta(x) = i+1\} dF_n(x) =$$

$$= \frac{1}{f_n(s)} \sum_{k=0}^{i+1} a_{i+1}^{k*} \int_0^{\infty} e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} dF_n(x), \qquad n \geq 1,\ i \geq -1;$$

$$q_{ni}(s) = \int_0^{\infty} e^{-sx} \mathbf{P}_n\{\eta(x) = i\} \overline{F}_n(x) dx =$$

$$=\sum_{k=0}^{i} a_i^{k*}\int_0^\infty e^{-(\lambda+s)x}\frac{(\lambda x)^k}{k!}\overline{F}_n(x)dx, \qquad n\ge 1,\ i\ge 0;$$

$$R_{n1}(s)=\frac{1}{f_n(s)p_{n,-1}(s)}, \tag{1}$$

$$R_{n,k+1}(s)=\frac{R_{nk}(s)-f_n(s)\sum_{i=0}^{k-1}p_{ni}(s)R_{n,k-i}(s)}{f_n(s)p_{n,-1}(s)}, \quad n\ge 1,\ k\ge 1.$$

All functions on $s$ we consider for values of the argument satisfying the condition $\operatorname{Re} s\ge 0$.

## 2. The distribution of the number of customers in the system on the busy period

Let $\tau(m)=\inf\{t\ge 0:\xi(t)=0\}$ denote the first busy period for the system M$^\theta$/G$_1$,…,G$_{m+1}$/1/m;

$$\varphi_n(t,k)=\mathbf{P}_n\{\xi(t)=k,\tau(m)>t\},$$

$$\Phi_n(s,k)=\int_0^\infty e^{-st}\varphi_n(t,k)dt, \qquad 1\le n,k\le m+1.$$

It is obvious that $\varphi_0(t,k)=0$. Using the formula of total probability, we obtain the equalities

$$\begin{aligned}\varphi_n(t,k)&=\sum_{j=0}^{m-n}\int_0^t \mathbf{P}_n\{\eta(x)=j\}\varphi_{n+j-1}(t-x,k)dF_n(x)+\\&+\int_0^t \mathbf{P}_n\{\eta(x)\ge m+1-n\}\varphi_m(t-x,k)dF_n(x)+\big(\mathbf{P}_n\{\eta(t)=k-n\}+\\&+I\{k=m+1\}\mathbf{P}_n\{\eta(t)\ge m+2-n\}\big)\overline{F}_n(t), \quad 1\le n\le m.\end{aligned} \tag{2}$$

Here, $I\{A\}$ is 1 or 0, depending on whether the event $A$ occurs or not.

Introducing the notation

$$f_{(n)}(s,k,m)=q_{n,k-n}(s)+I\{k=m+1\}\overline{q}_{n,m+2-n}(s),$$

and using (1), from (2) we obtain the system of equations for the functions $\Phi_n(s,k)$

$$\Phi_n(s,k) = f_n(s)\sum_{j=0}^{m-n} p_{n,j-1}(s)\Phi_{n+j-1}(s,k) + \\ + f_n(s)\overline{p}_{n,m-n}(s)\Phi_m(s,k) + f_{(n)}(s,k,m), \qquad 1 \le n \le m, \tag{3}$$

with the boundary condition

$$\Phi_0(s,k) = 0. \tag{4}$$

Let us solve the system of equations (3), (4) and find the functions $\Phi_n(s,k)$.

We will use the functions $\mathcal{R}_{ni}(s)$, defined by the recurrence relations:

$$\mathcal{R}_{n,1}(s) = R_{n+1,1}(s);$$
$$\mathcal{R}_{n,j+1}(s) = R_{n+1,1}(s)\left(\mathcal{R}_{n+1,j}(s) - f_{n+1}(s)\sum_{i=0}^{j-1} p_{n+1,i}(s)\mathcal{R}_{n+1+i,j-i}(s)\right), \tag{5}$$
$$1 \le j \le m-n-1,\ 0 \le n \le m-1.$$

**Theorem 1.** For all $1 \le k \le m+1$ and $\operatorname{Re} s > 0$ functions $\Phi_n(s,k)$ are defined as

$$\Phi_n(s,k) = \left(\mathcal{R}_{n,m-n}(s) - \sum_{i=1}^{m-n}\mathcal{R}_{ni}(s) f_{n+i}(s)\overline{p}_{n+i,m-n-i}(s)\right)\Phi_m(s,k) - \\ - \sum_{i=1}^{m-n}\mathcal{R}_{ni}(s) f_{(n+i)}(s,k,m), \qquad 1 \le n \le m-1, \tag{6}$$

where

$$\Phi_m(s,k) = \frac{\sum_{i=1}^{m}\mathcal{R}_{0i}(s) f_{(i)}(s,k,m)}{\mathcal{R}_{0m}(s) - \sum_{i=1}^{m}\mathcal{R}_{0i}(s) f_i(s)\overline{p}_{i,m-i}(s)}. \tag{7}$$

*Proof.* Let us prove the equalities (6) using the method of mathematical induction. Equation (3) for $n = m$ takes the form

$$\Phi_m(s,k) = f_m(s)p_{m,-1}(s)\Phi_{m-1}(s,k) + f_m(s)\overline{p}_{m,0}(s)\Phi_m(s,k) + f_{(m)}(s,k,m).$$

Taking into account that $f_m(s)p_{m,-1}(s) = 1/R_{m,1}(s) = 1/\mathcal{R}_{m-1,1}(s)$, we find

$$\Phi_{m-1}(s,k) = \mathcal{R}_{m-1,1}(s)\Big((1 - f_m(s)\overline{p}_{m,0}(s))\Phi_m(s,k) - f_{(m)}(s,k,m)\Big).$$

This equality is identical to (6) for $n = m-1$.

Suppose that equalities (6) are true for all natural $n$ of the set $\{\tilde{n}, \tilde{n}+1, \ldots, m-2\}$, where $\tilde{n}$ is a fixed number ($1 < \tilde{n} < m-1$). Let us prove the equality of the form (6) for $n = \tilde{n}-1$. From (3) for $n = \tilde{n}$ we obtain:

$$\Phi_{\tilde{n}}(s,k) = f_{\tilde{n}}(s) p_{\tilde{n},-1}(s) \Phi_{\tilde{n}-1}(s,k) + f_{\tilde{n}}(s) p_{\tilde{n},0}(s) \Phi_{\tilde{n}}(s,k) +$$
$$+ f_{\tilde{n}}(s) \sum_{j=2}^{m-\tilde{n}} p_{\tilde{n},j-1}(s) \Phi_{\tilde{n}+j-1}(s,k) + f_{\tilde{n}}(s) \overline{p}_{\tilde{n},m-\tilde{n}}(s) \Phi_m(s,k) + f_{(\tilde{n})}(s,k,m).$$

Hence, we find

$$\Phi_{\tilde{n}-1}(s,k) = R_{\tilde{n},1}(s)\Big(\big(1 - f_{\tilde{n}}(s) p_{\tilde{n},0}(s)\big)\Phi_{\tilde{n}}(s,k) - f_{\tilde{n}}(s) \sum_{j=2}^{m-\tilde{n}} p_{\tilde{n},j-1}(s) \Phi_{\tilde{n}+j-1}(s,k) -$$
$$- f_{\tilde{n}}(s) \overline{p}_{\tilde{n},m-\tilde{n}}(s) \Phi_m(s,k) - f_{(\tilde{n})}(s,k,m)\Big) =$$
$$= R_{\tilde{n},1}(s)\Bigg(\big(1 - f_{\tilde{n}}(s) p_{\tilde{n},0}(s)\big)\Bigg(\Bigg(\mathcal{R}_{\tilde{n},m-\tilde{n}}(s) - \sum_{i=1}^{m-\tilde{n}} \mathcal{R}_{\tilde{n}i}(s) f_{\tilde{n}+i}(s) \overline{p}_{\tilde{n}+i,m-\tilde{n}-i}(s)\Bigg)\Phi_m(s,k) -$$
$$- \sum_{i=1}^{m-\tilde{n}} \mathcal{R}_{\tilde{n}i}(s) f_{(\tilde{n}+i)}(s,k,m)\Bigg) - f_{\tilde{n}}(s) \sum_{j=2}^{m-\tilde{n}} p_{\tilde{n},j-1}(s)\Big(\Big(\mathcal{R}_{\tilde{n}+j-1,m-\tilde{n}-j+1}(s) -$$
$$- \sum_{i=1}^{m-\tilde{n}-j+1} \mathcal{R}_{\tilde{n}+j-1,i}(s) f_{\tilde{n}+j-1+i}(s) \overline{p}_{\tilde{n}+j-1+i,m-\tilde{n}-j+1-i}(s)\Bigg)\Phi_m(s,k) -$$
$$- \sum_{i=1}^{m-\tilde{n}-j+1} \mathcal{R}_{\tilde{n}+j-1,i}(s) f_{(\tilde{n}+j-1+i)}(s,k,m)\Bigg) - f_{\tilde{n}}(s) \overline{p}_{\tilde{n},m-\tilde{n}}(s) \Phi_m(s,k) - f_{(\tilde{n})}(s,k,m)\Bigg).$$

If we gather all the terms in this expression in the form of coefficients at $\Phi_m(s,k)$, and separately write all other terms, after using (5) we find

$$\Phi_{\tilde{n}-1}(s,k) = R_{\tilde{n},1}(s)\Big(\big(1 - f_m(s) \overline{p}_{m,0}(s)\big)\Big((1 - f_{\tilde{n}}(s) p_{\tilde{n},0}(s)\big)\mathcal{R}_{\tilde{n},m-\tilde{n}}(s) -$$
$$- f_{\tilde{n}}(s) \sum_{j=2}^{m-\tilde{n}} p_{\tilde{n},j-1}(s) \mathcal{R}_{\tilde{n}+j-1,m-\tilde{n}-j+1}(s)\Bigg) - \sum_{i=2}^{m-\tilde{n}} f_{\tilde{n}-1+i}(s) \overline{p}_{\tilde{n}-1+i,m-\tilde{n}+1-i}(s)\Big(\mathcal{R}_{\tilde{n},i-1}(s) -$$
$$- f_{\tilde{n}}(s) \sum_{k=0}^{i-2} p_{\tilde{n},k}(s) \mathcal{R}_{\tilde{n}+k,i-1-k}(s)\Bigg) - f_{\tilde{n}}(s) \overline{p}_{\tilde{n},m-\tilde{n}}(s))\Phi_m(s,k) - R_{\tilde{n},1}(s)\Big(f_{(\tilde{n})}(s,k,m) +$$
$$+ \sum_{i=2}^{m-\tilde{n}+1} f_{(\tilde{n}-1+i)}(s,k,m)\Bigg(\mathcal{R}_{\tilde{n},i-1}(s) - f_{\tilde{n}}(s) \sum_{k=0}^{i-2} p_{\tilde{n},k}(s) \mathcal{R}_{\tilde{n}+k,i-1-k}(s)\Bigg)\Bigg) =$$

$$= \left( \mathcal{R}_{\tilde{n}-1,m-\tilde{n}+1}(s) - \sum_{i=1}^{m-\tilde{n}+1} \mathcal{R}_{\tilde{n}-1,i}(s) f_{\tilde{n}-1+i}(s) \overline{p}_{\tilde{n}-1+i,m-\tilde{n}+1-i}(s) \right) \Phi_m(s,k) -$$

$$- \sum_{i=1}^{m-\tilde{n}+1} \mathcal{R}_{\tilde{n}-1,i}(s) f_{(\tilde{n}-1+i)}(s,k,m).$$

The obtained expression is identical to (6) at $n = \tilde{n} - 1$. So, equations (6) are proved.

Putting $n = 0$ in (6) and using the boundary condition (4), we arrive at the formula (7). The theorem is proved.

If the system starts functioning when the first group of customers arrives, then for $1 \le k \le m+1$ by using the formula of total probability we obtain the equalities

$$\int_0^\infty e^{-st} \mathbf{P}\{\xi(t) = k,\ \tau(m) > t\} dt = \sum_{i=1}^{m} a_i \Phi_i(s,k) + \overline{a}_{m+1} \Phi_{m+1}(s,k). \tag{8}$$

Taking into account that

$$\varphi_{m+1}(t,k) = \int_0^t \varphi_m(t-x,k) dF_{m+1}(x) + I\{k = m+1\} \overline{F}_{m+1}(t),$$

$$\Phi_{m+1}(s,k) = \Phi_m(s,k) + I\{k = m+1\} \frac{1 - f_{m+1}(s)}{s},$$

and using the relations (6), we can write the right side of (8) as follows:

$$\int_0^\infty e^{-st} \mathbf{P}\{\xi(t) = k,\ \tau(m) > t\} dt =$$

$$= \left( \sum_{i=1}^{m} a_i \left( \mathcal{R}_{i,m-i}(s) - \sum_{j=1}^{m-i} \mathcal{R}_{ij}(s) f_{i+j}(s) \overline{p}_{i+j,m-i-j}(s) \right) + \overline{a}_{m+1} \right) \Phi_m(s,k) - \tag{9}$$

$$- \sum_{i=1}^{m-1} a_i \sum_{j=1}^{m-i} \mathcal{R}_{ij}(s) f_{(i+j)}(s,k,m) + \overline{a}_{m+1} I\{k = m+1\} \frac{1 - f_{m+1}(s)}{s}.$$

To obtain a representation for $\int_0^\infty e^{-st} \mathbf{P}\{\tau(m) > t\} dt$, we should pass in equalities (9) to summation over $k$ from 1 to $m+1$. Taking into account definition of $f_{(n)}(s,k,m)$ and $q_{ni}(s)$, we see that

$$\sum_{k=1}^{m+1} f_{(n)}(s,k,m) = \sum_{k=0}^{\infty} q_{nk}(s) = \frac{1 - f_n(s)}{s}, \qquad 1 \le n \le m.$$

Thus, from (9) we obtain the following statement.

**Theorem 2.** The Laplace transform on the distribution function of the busy period for the $M^\theta/G_1,\ldots,G_{m+1}/1/m$ system has the form

$$\int_0^\infty e^{-st}\mathbf{P}\{\tau(m)>t\}dt=\left(\sum_{i=1}^{m}a_i\left(\mathcal{R}_{i,m-i}(s)-\sum_{j=1}^{m-i}\mathcal{R}_{ij}(s)f_{i+j}(s)\overline{p}_{i+j,m-i-j}(s)\right)+\overline{a}_{m+1}\right)\times$$
$$\times\frac{\sum_{i=1}^{m}\mathcal{R}_{0i}(s)\frac{1-f_i(s)}{s}}{\mathcal{R}_{0m}(s)-\sum_{i=1}^{m}\mathcal{R}_{0i}(s)f_i(s)\overline{p}_{i,m-i}(s)}-\sum_{i=1}^{m-1}a_i\sum_{j=1}^{m-i}\mathcal{R}_{ij}(s)\frac{1-f_{i+j}(s)}{s}+\overline{a}_{m+1}\frac{1-f_{m+1}(s)}{s}.$$

Note that the formulas for the average duration of the busy period and the stationary distribution of the number of customers in the $M^\theta/G_1,\ldots,G_{m+1}/1/m$ system are found in [1].

## Conclusions

In this paper we show that our approach based on the potential method, allows us to find not only the stationary distribution of the number of customers in the queuing system, but also allows us to get the results for the transient operation of the system, namely, to study the busy period for the semi-Markov-type systems with single server and service time dependent on the queue length.

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