Journal of Applied Mathematics and Computational Mechanics 2013, 12(3), 27-38

# ON SOME APPLICATION OF ALGEBRAIC QUASINUCLEI TO THE DETERMINANT THEORY

***Grażyna Ciecierska***

*Faculty of Mathematics and Computer Science, University of Warmia and Mazury in Olsztyn*
*Olsztyn, Poland*
*grac@matman.uwm.edu.pl*

**Abstract.** In the paper we apply the modified powers of algebraic quasinuclei to construction of determinant systems for quasinuclear perturbations of Fredholm operators. Given two pairs $(\Xi, X)$, $(\Omega, Y)$ of conjugate linear spaces, an algebraic quasinucleus $F \in an(\Omega \to \Xi, X \to Y)$ and a determinant system for the Fredholm operator $S \in op(\Omega \to \Xi, X \to Y)$, we obtain algebraic formulas for terms of a determinant system for $S + T_F$.

***Keywords:*** *determinant system, Fredholm operator, quasinucleus, quasinuclear operator*

## Introduction

Determinant systems for operators acting in infinite dimensional Banach spaces provide important tools for solving linear equations. The determinant system for linear operator $A$ gives full information on solving the equation $Ax = y_0$, where $y_0$ belongs to the range of $A$. The Sikorski's and Buraczewski's formulas [1-3] for the solution are generalizations of the famous Cramer's rule for solving finite systems of linear equations.

The first theory of determinants in arbitrary Banach spaces was developed by A.F. Ruston [4] and A. Grothendieck [5] and another one by T. Leżański [6], R. Sikorski [1] and A. Buraczewski [2]. A general approach to the theory of determinants was proposed by A. Pietsch [7], I. Gohberg, S. Goldberg and N. Krupnik [8].

The study of determinant systems leads to the study of concrete classes of Fredholm operators. In this approach we consider the class of quasinuclear perturbations of Fredholm operators. Algebraic quasinuclei play an important role in the theory of determinant systems; if $(D_n)$ is a determinant system for a Fredholm operator $S$ and $T_F$ is the quasinuclear operator determined by an algebraic quasinucleus $F$, then we can obtain effective formulas for a determinant system for the operator $S + T_F$ in Banach spaces. The purpose of this paper is to give purely algebraic formulas for terms of the mentioned determinant system. The formulas were first given by Plemelj [9] for endomorphisms of the form $I+T$, where $T$ is an integral endomorphism, in the space $C[a,b]$. These formulas were obtained on the basis of the Fredholm theory of integral equations. They were modified by Smithies [10], also in the case of endomorphisms $I+T$, where $T$ is integral. R. Sikorski [1] generalized the formulas over the endomorphisms $I+T$, where $T$ is quasinuclear. A. Buraczewski [2] made further generalization of these formulas in the case of operators of the form $S+T$, where $S$ is a fixed Fredholm operator of order zero and $T$ is quasinuclear. Later contribution was made by D.H.U. Marchetti [11] who presented an alternative to Plemelj-Smithies formulas in the case of endomorphisms $I+T$, where $T$ belongs to the trace class of endomorphisms in a separable Hilbert space. In this paper we generalize Plemelj-Smithies formulas over the operators of the form $S+T$, where $S$ is an arbitrary Fredholm operator and $T$ is quasinuclear. The result is formulated by means of the modified powers of quasinuclei.

## 1. Terminology and notation

We begin with a brief review on the terminology used in the determinant theory. We follow the notation of [1-4].

Let $(\varXi, X)$, $(\varOmega, Y)$, $(\varLambda, Z)$ denote pairs of conjugate linear spaces over $K$ $(K=R$ or $K=C)$. A bilinear functional $A:\varOmega\times X\to K$, whose value at a point $(\omega,x)\in\varOmega\times X$ is denoted by $\omega Ax$, satisfying the condition $\omega Ax=\omega(Ax)=(\omega A)x$, where $Ax\in Y$ and $\omega A\in\varXi$, is called $(\varXi,Y)$-*operator on* $\varOmega\times X$; the space of all $(\varXi,Y)$-operators on $\varOmega\times X$ is denoted by $op(\varOmega\to\varXi, X\to Y)$. For fixed non-zero elements $x_0\in X$, $\omega_0\in\varOmega$, $x_0\cdot\omega_0$ denotes the bilinear functional on $\varXi\times Y$, defined by $\xi(x_0\cdot\omega_0)y=\xi x_0\cdot\omega_0 y$ for $(\xi,y)\in\varXi\times Y$. An operator $B\in op(\varXi\to\varOmega, Y\to X)$ such that $ABA=A$, $BAB=B$ is said to be a *generalized inverse of an operator* $A\in op(\varOmega\to\varXi, X\to Y)$. The value of a $(\mu+m)$-linear functional $D:\varXi^{\mu}\times Y^{m}\to K$, $\mu,m\in N\cup\{0\}$, at a point $(\xi_1,\ldots,\ \xi_\mu, y_1,\ldots,\ y_m)\in\varXi^{\mu}\times Y^{m}$ is denoted by $D\begin{pmatrix}\xi_1,\ldots, & \xi_\mu\\ y_1,\ldots, & y_m\end{pmatrix}$. A $(\mu+m)$-
-linear functional $D$ on $\varXi^{\mu}\times Y^{m}$ is said to be *bi-skew symmetric* if it is skew symmetric in variables from both $\varXi$, and $Y$. A $(\mu+m)$-linear functional $D:\varXi^{\mu}\times Y^{m}\to K$ is said to be $(\varOmega,X)$-*functional on* $\varXi^{\mu}\times Y^{m}$, if for any fixed elements $\xi_1,\ldots,\xi_{i-1},\xi_{i+1},\ldots,\xi_\mu\in\varXi$ $(i=1,\ldots,\mu)$, $y_1,\ldots,y_m\in Y$ there exists an element $x_i\in X$ such that $\xi x_i=D\begin{pmatrix}\xi_1,\ldots, & \xi_{i-1},\xi,\xi_{i+1},\ldots, & \xi_\mu\\ y_1, & \ldots, & y_m\end{pmatrix}$ for every $\xi\in\varXi$ and

for any fixed elements $\xi_1,\ldots,\xi_\mu \in \Xi$, $y_1,\ldots,y_{j-1},y_{j+1},\ldots,y_m \in Y$ $(j=1,\ldots,m)$ there exists an element $\omega_j \in \Omega$ such that $\omega_j y = D\begin{pmatrix} \xi_1, & \ldots, & \xi_\mu \\ y_1,\ldots, & y_{j-1},y,y_{j+1},\ldots, & y_m \end{pmatrix}$ for every $y \in Y$.

A sequence $(D_n)_{n\in N\cup\{0\}}$ is called a *determinant system for operator* $A \in op(\Omega \to \Xi, X \to Y)$, if for $n \in N \cup \{0\}$ $D_n$ is bi-skew symmetric $(\Omega, X)$--functional on $\Xi^{\mu_n} \times Y^{m_n}$ where $\mu_n = \mu_0 + n, m_n = m_0 + n$, $min(\mu_0, m_0) = 0$, there exists $r \in N \cup \{0\}$ such that $D_r \neq 0$ and the following identities hold:

$$D_{n+1}\begin{pmatrix} \omega A, & \xi_1,\ldots, & \xi_{\mu_n} \\ y_0, & \ldots, & y_{m_n} \end{pmatrix} = \sum_{j=0}^{m_n} (-1)^j \omega y_j \cdot D_n\begin{pmatrix} \xi_1, & \ldots, & \xi_{\mu_n} \\ y_0,\ldots, & y_{j-1},y_{j+1},\ldots, & y_{m_n} \end{pmatrix},$$

$$D_{n+1}\begin{pmatrix} \xi_0, & \ldots, & \xi_{\mu_n} \\ Ax, & y_1,\ldots, & y_{m_n} \end{pmatrix} = \sum_{i=0}^{\mu_n} (-1)^i \xi_i x \cdot D_n\begin{pmatrix} \xi_0,\ldots, & \xi_{i-1},\xi_{i+1},\ldots, & \xi_{\mu_n} \\ y_1, & \ldots, & y_{m_n} \end{pmatrix},$$

where $\xi_i \in \Xi$ $(i=1,\ldots,\mu_n)$, $y_j \in Y$ $(j=1,\ldots,m_n)$, $x \in X, \omega \in \Omega$. The least $r \in N \cup \{0\}$, such that $D_r \neq 0$ is called *the order of determinant system* $(D_n)_{n\in N\cup\{0\}}$. The integer $\mu_0 - m_0$ is called the *index of determinant system* $(D_n)_{n\in N\cup\{0\}}$. If $A \in op(\Omega \to \Xi, X \to Y)$ is a Fredholm operator of order $r = min\{n', m'\}$ and index $d = n' - m'$, $B \in op(\Xi \to \Omega, Y \to X)$ is a generalized inverse of $A$, $\{z_1,\ldots,z_{n'}\}$, $\{\varsigma_1,\ldots,\varsigma_{m'}\}$ are complete systems of solutions of the homogenous equations $Ax = 0$ and $\omega A = 0$, respectively, then the sequence $(D_n)_{n\in N\cup\{0\}}$ defined by the formula

$$D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r} \\ y_1,\ldots, & y_{n+m'-r} \end{pmatrix} = \begin{vmatrix} \xi_1 B y_1 & \cdots & \xi_1 B y_{n+m'-r} & \xi_1 z_1 & \cdots & \xi_1 z_{n'} \\ \vdots & & \vdots & \vdots & & \vdots \\ \xi_{n+n'-r} B y_1 & \cdots & \xi_{n+n'-r} B y_{n+m'-r} & \xi_{n+n'-r} z_1 & \cdots & \xi_{n+n'-r} z_{n'} \\ \varsigma_1 y_1 & \cdots & \varsigma_1 y_{n+m'-r} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ \varsigma_{m'} y_1 & \cdots & \varsigma_{m'} y_{n+m'-r} & 0 & \cdots & 0 \end{vmatrix},$$

for $\xi_i \in \Xi$ $(i=1,\ldots,n+n'-r)$, $y_j \in Y$ $(j=1,\ldots,n+m'-r)$, is a determinant system for the operator $A$.

A linear functional $F : op(\Xi \to \Omega, Y \to X) \to K$ is said to be an *algebraic quasinucleus on* $op(\Xi \to \Omega, Y \to X)$, if there exists $T_F \in op(\Omega \to \Xi, X \to Y)$ such that $F(x \cdot \omega) = \omega T_F x$ for $(\omega, x) \in \Omega \times X$. $T_F$ is called a *quasinuclear operator* *determined by F*. The space of all algebraic quasinuclei on $op(\varXi \to \varOmega, Y \to X)$ is denoted by $an(\varOmega \to \varXi, X \to Y)$. If $y_0 \in Y, \xi_0 \in \varXi$ are fixed, then algebraic quasinucleus $\xi_0 \otimes y_0 \in an(\varOmega \to \varXi, X \to Y)$ such that $(\xi_0 \otimes y_0)(B) = \xi_0 B y_0$ for $B \in op(\varXi \to \varOmega, Y \to X)$ is called *one-dimensional*. Every finite sum $\sum_{i=1}^{n} \xi_i \otimes y_i$ of one-dimensional quasinuclei is called *finitely dimensional quasinucleus*. By the *trace of an algebraic quasinucleus* $F \in an(\varXi \to \varXi, X \to X)$ we understand the number $TrF = F(I)$. For $F \in an(\varOmega \to \varXi, X \to Y)$ and $C \in op(\varLambda \to \varOmega, Y \to Z)$ we define $CF \in an(\varLambda \to \varXi, X \to Z)$:

$$(CF)(A) = F(AC) \text{ for } A \in op(\varXi \to \varLambda, Z \to X). \tag{1}$$

Let $D$ be a bi-skew symmetric $(\varOmega, X)$-functional on $\varXi^{\mu} \times Y^{m}$, $\mu, m \in N$, and $F \in an(\varOmega \to \varXi, X \to Y)$. Fixing all the variables $\xi_2, \ldots \xi_\mu \in \varXi$, $y_2, \ldots, y_m \in Y$ and interpreting $D$ as the function of variables $\xi_1, y_1$ only, we define $(\mu + m - 2)$-linear functional $F_{\xi_1 y_1} D$ on $\varXi^{\mu-1} \times Y^{m-1}$ by

$$\left(F_{\xi_1 y_1} D\right)\begin{pmatrix} \xi_2, \ldots, & \xi_\mu \\ y_2, \ldots, & y_m \end{pmatrix} = F(A),$$

where

$$\xi_1 A y_1 = D\begin{pmatrix} \xi_1, & \xi_2, \ldots, & \xi_\mu \\ y_1, & y_2, \ldots, & y_m \end{pmatrix} \text{ for } \xi_1 \in \varXi, y_1 \in Y.$$

We can iterate the procedure $k$ times, $k = min\{\mu, m\}$, provided

$$F_{\xi_1 y_1} D, \quad F_{\xi_2 y_2} F_{\xi_1 y_1} D, \; \ldots \; F_{\xi_k y_k} \ldots F_{\xi_2 y_2} F_{\xi_1 y_1} D$$

are $(\varOmega, X)$-functionals [12]. By a reasoning similar to that in [6], since $F_{\xi'_{\tau_k} y'_{\tau_k}} \ldots F_{\xi'_{\tau_1} y'_{\tau_1}} D$ does not depend on the choice of permutation $\tau$ of integers $1, \ldots, k$, we denote by $\underbrace{F\square \ldots \square F\square}_{k-times} D$ the common value of all $F_{\xi'_{\tau_k} y'_{\tau_k}} \ldots F_{\xi'_{\tau_1} y'_{\tau_1}} D$.

We also use the notation suitable for the formulation and the proof of the main theorem of the paper. A matrix $M = [a_{ij}]_{\substack{1 \le i \le \mu \\ 1 \le j \le m}}$ over the field $K$, is denoted by $(M_1, \ldots, M_\mu)^T$, where $M_i = [a_{i1}, \ldots, a_{im}]$, $(i = 1, \ldots, \mu)$, i.e. $M_i$ is the i-th row of $M$.

## 2. Algebraic formulas for terms of determinant systems for quasinuclear perturbations of Fredholm operators

We present the theorem, which gives a generalization of Plemelj-Smithies formulas for operators of the form $S+T$, where $S$ is Fredholm and $T$ is quasi-nuclear.

**Theorem**. *Let* $S \in op(\Omega \to \Xi, X \to Y)$ *be a Fredholm operator of order* $r = min\{n', m'\}$, *index* $d = n' - m' \geq 0$ *and determinant system* $(D_n)_{n \in N \cup \{0\}}$. *Suppose that* $U \in op(\Xi \to \Omega, Y \to X)$ *is a generalized inverse of* $S$ *and* $\{z_1, \ldots, z_{n'}\}$, $\{\varsigma_1, \ldots, \varsigma_{m'}\}$ *are complete systems of solutions of the homogenous equations* $Sx = 0$ *and* $\omega S = 0$, *respectively. Then for any* $F \in an(\Omega \to \Xi, X \to Y)$ *which determines* $T_F = T \in op(\Omega \to \Xi, X \to Y)$ *the following formulas hold:*

$$\underbrace{F\square\ldots\square F\square}_{k-times} D_{n+k} = \begin{vmatrix} T_n^0 & k & 0 & \ldots & 0 & 0 \\ T_n^1 & \sigma_1 & k-1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ T_n^{k-1} & \sigma_{k-1} & \sigma_{k-2} & \ldots & \sigma_1 & 1 \\ T_n^k & \sigma_k & \sigma_{k-1} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix} \tag{2}$$

*for* $n, k \in N \cup \{0\}$, *where*

$$\sigma_m = Tr\left[(UT)^{m-1} UF\right] \quad (m = 1, \ldots k) \tag{3}$$

$$T_n^0 = D_n \text{ and } T_n^m \text{ is the } (2n + n' + m' - 2r)\text{-linear functional} \tag{4}$$

$$T_n^m \begin{pmatrix} \xi_1, \ldots, & \xi_{n+n'-r} \\ y_1, \ldots, & y_{n+m'-r} \end{pmatrix} =$$

$$= \sum \begin{vmatrix} \xi_1 (UT)^{i_1} Uy_1 & \ldots & \xi_1 (UT)^{i_1} Uy_{n+m'-r} & \xi_1 (UT)^{i_1} z_1 & \ldots & \xi_1 (UT)^{i_1} z_{n'} \\ \vdots & & \vdots & \vdots & & \vdots \\ \xi_{n+n'-r} (UT)^{i_{n+n'-r}} Uy_1 & \ldots & \xi_{n+n'-r} (UT)^{i_{n+n'-r}} Uy_{n+m'-r} & \xi_{n+n'-r} (UT)^{i_{n+n'-r}} z_1 & \ldots & \xi_{n+n'-r} (UT)^{i_{n+n'-r}} z_{n'} \\ \varsigma_1 (TU)^{i_{n+n'-r+1}} y_1 & \ldots & \varsigma_1 (TU)^{i_{n+n'-r+1}} y_{n+m'-r} & \varsigma_1 [T(UT)]^{(i_{n+n'-r+1})} z_1 & \ldots & \varsigma_1 [T(UT)]^{(i_{n+n'-r+1})} z_{n'} \\ \vdots & & \vdots & \vdots & & \vdots \\ \varsigma_{m'} (TU)^{i_{n+n'-r+m'}} y_1 & \ldots & \varsigma_{m'} (TU)^{i_{n+n'-r+m'}} y_{n+m'-r} & \varsigma_{m'} [T(UT)]^{(i_{n+n'-r+m'})} z_1 & \ldots & \varsigma_{m'} [T(UT)]^{(i_{n+n'-r+m'})} z_{n'} \end{vmatrix},$$

*where for* $s = 1, \ldots m'$, $t = 1, \ldots n'$

$$\varsigma_s\left[T(UT)\right]^{(i_{n+n'-r+s})} z_t = \begin{cases} 0 & if \quad i_{n+n'-r+s} = 0 \\ \varsigma_s T(UT)^{i_{n+n'-r+s}-1} z_t & if \quad i_{n+n'-r+s} = 1,\dots m \end{cases}$$

*and* $\sum$ *is extended over all finite sequences of non-negative integers* $i_1,\dots i_{n+n'-r+m'}$, *such that* $i_1+\dots+i_{n+n'-r+m'} = m$; *i.e.*

$$T_n^m\begin{pmatrix} \xi_1,\dots, & \xi_{n+n'-r} \\ y_1,\dots, & y_{n+m'-r} \end{pmatrix} =$$

$$= \sum_{i_1+\dots+i_{n+d+r}=m} \overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\dots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}} / \varsigma_1\left[T(UT)\right]^{(i_{n+n'-r+1})},\dots, & \varsigma_{m'}\left[T(UT)\right]^{(i_{n+n'-r+m'})} \\ y_1,\dots, & y_{n+m'-r} / z_1,\dots, & z_{n'} \end{pmatrix}$$

*where*

$$\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\dots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}} / \varsigma_1\left[T(UT)\right]^{(i_{n+n'-r+1})},\dots, & \varsigma_{m'}\left[T(UT)\right]^{(i_{n+n'-r+m'})} \\ y_1,\dots, & y_{n+m'-r} / z_1,\dots, & z_{n'} \end{pmatrix} =$$

$$= D_n\begin{pmatrix} \xi_1(UT)^{i_1},\dots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}} \\ y_1,\dots, & y_{n+m'-r} \end{pmatrix} \; if \;\; i_1+\dots+i_{n+n'-r} = m,$$

*and if* $i_1+\dots+i_{n+n'-r} < m$, *then*

$$\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\dots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}} / \varsigma_1\left[T(UT)\right]^{(i_{n+n'-r+1})},\dots, & \varsigma_{m'}\left[T(UT)\right]^{(i_{n+n'-r+m'})} \\ y_1,\dots, & y_{n+m'-r} / z_1,\dots, & z_{n'} \end{pmatrix} = det\left(D_1^{(n)},\dots,D_{n+n'-r+m'}^{(n)}\right)^T,$$

*where*

$$D_j^{(n)} = \left[\xi_j(UT)^{i_j}Uy_1,\dots,\xi_j(UT)^{i_j}Uy_{n+m'-r},\xi_j(UT)^{i_j}z_1,\dots,\xi_j(UT)^{i_j}z_{n'}\right] \; (j=1,\dots,n+n'-r)$$

$$D_{n+n'-r+j}^{(n)} = \left[\varsigma_j y_1,\dots,\varsigma_j y_{n+m'-r},\underbrace{0,\dots,0}_{n'}\right] \; if \; i_{n+n'-r+j} = 0 \;\; (j=1,\dots,m'),$$

$$D_{n+n'-r+j}^{(n)} = \left[\varsigma_j(TU)^{i_{n+n'-r+j}}y_1,\dots,\varsigma_j(TU)^{i_{n+n'-r+j}}y_{n+m'-r},\varsigma_jT(UT)^{i_{n+n'-r+j}-1}z_1,\dots,\varsigma_jT(UT)^{i_{n+n'-r+j}-1}z_{n'}\right]$$

*if* $i_{n+n'-r+j} = 1,\dots,m \; (j=1,\dots,m')$.

**Proof**. We shall use an induction argument on $k$ to establish the formulas. If $k=0$, then $D_n = T_n^0$ for any $n \in N \cup \{0\}$. Let $k$ be any fixed positive integer. Assume that for any $n \in N \cup \{0\}$ the formulas (2) hold. For fixed $n$

$$\Big(\underbrace{F\square\ldots\square F\square}_{(k+1)-times}D_{n+k+1}\Big)\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}=F_{\xi'_{k+1}y'_{k+1}}\Big(\underbrace{F\square\ldots\square F\square}_{k-times}D_{n+k+1}\Big)\begin{pmatrix}\xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r}\\ y'_{k+1},y_1,\ldots, & y_{n+m'-r}\end{pmatrix}.$$

Since the formula is valid for $k$ and $n+1$, expanding the determinant in the formula (2) in terms of its first column, we obtain

$$\Big(\underbrace{F\square\ldots\square F\square}_{(k+1)-razy}D_{n+k+1}\Big)\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}= \tag{5}$$

$$=F_{\xi'_{k+1}y'_{k+1}}\left[A_0D_{n+1}\begin{pmatrix}\xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r}\\ y'_{k+1},y_1,\ldots, & y_{n+m'-r}\end{pmatrix}+\sum_{m=1}^{k}(-1)^m A_m T_{n+1}^m\begin{pmatrix}\xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r}\\ y'_{k+1},y_1,\ldots, & y_{n+m'-r}\end{pmatrix}\right],$$

where $(-1)^m A_m$ $(m=0,\ldots,k)$ is the cofactor of the element $T_{n+1}^m$ of the matrix

$$\begin{bmatrix} T_{n+1}^0 & k & 0 & \ldots & 0 & 0\\ T_{n+1}^1 & \sigma_1 & k-1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ T_{n+1}^{k-1} & \sigma_{k-1} & \sigma_{k-2} & \ldots & \sigma_1 & 1\\ T_{n+1}^{k} & \sigma_{k} & \sigma_{k-1} & \ldots & \sigma_2 & \sigma_1 \end{bmatrix},$$

i.e.

$$A_0=\begin{vmatrix}\sigma_1 & k-1 & \ldots & 0 & 0\\ \sigma_2 & \sigma_1 & \ldots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \sigma_k & \sigma_{k-1} & \ldots & \sigma_2 & \sigma_1\end{vmatrix},\quad A_k=\begin{vmatrix}k & 0 & \ldots & 0 & 0\\ \sigma_1 & k-1 & \ldots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \sigma_{k-1} & \sigma_{k-2} & \ldots & \sigma_1 & 1\end{vmatrix},$$

$$A_m=\begin{vmatrix} k & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0\\ \sigma_1 & k-1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{m-1} & \sigma_{m-2} & \sigma_{m-3} & \ldots & k-m+1 & 0 & 0 & \ldots & 0 & 0\\ \sigma_{m+1} & \sigma_m & \sigma_{m-1} & \ldots & \sigma_2 & \sigma_1 & k-m-1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_k & \sigma_{k-1} & \sigma_{k-2} & \ldots & \sigma_{k-m+1} & \sigma_{k-m} & \sigma_{k-m-1} & \ldots & \sigma_2 & \sigma_1 \end{vmatrix}$$

$(m=1,\ldots k-1)$.

By the Laplace expansion of the determinant $D_{n+1}\begin{pmatrix} \xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r} \\ y'_{k+1},y_1,\ldots, & y_{n+m'-r} \end{pmatrix}$ along its first column, bearing in mind that $T$ is the operator determined by the algebraic quasinucleus $F$,

$$F_{\xi'_{k+1}y'_{k+1}} D_{n+1}\begin{pmatrix} \xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r} \\ y'_{k+1},y_1,\ldots, & y_{n+m'-r} \end{pmatrix} = F_{\xi'_{k+1}y'_{k+1}}\Bigg[\xi'_{k+1}Uy'_{k+1}D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r} \\ y_1,\ldots, & y_{n+m'-r} \end{pmatrix}+ \tag{6}$$

$$+\sum_{i=1}^{n+n'-r}(-1)^i\xi_i Uy'_{k+1}D_n\begin{pmatrix} \xi'_{k+1},\xi_1,\ldots, & \xi_{i-1},\xi_{i+1},\ldots, & \xi_{n+n'-r} \\ y_1, & \ldots, & y_{n+m'-r} \end{pmatrix}+$$

$$+\sum_{j=1}^{m'}(-1)^{n+n'-r+j}\varsigma_j y'_{k+1}\overline{D}_n\begin{pmatrix} \xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r}/\varsigma_1,\ldots, & \varsigma_{j-1},\varsigma_{j+1},\ldots, & \varsigma_{m'} \\ y_1,\ldots, & y_{n+m'-r}/z_1, & \ldots, & z_{n'} \end{pmatrix}\Bigg] =$$

$$= F(U)D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r} \\ y_1,\ldots, & y_{n+m'-r} \end{pmatrix}+\sum_{i=1}^{n+n'-r}(-1)^i D_n\begin{pmatrix} \xi_i UT,\xi_1,\ldots, & \xi_{i-1},\xi_{i+1},\ldots, & \xi_{n+n'-r} \\ y_1, & \ldots, & y_{n+m'-r} \end{pmatrix}+$$

$$+\sum_{j=1}^{m'}(-1)^{n+n'-r+j}\overline{D}_n\begin{pmatrix} \varsigma_j T,\xi_1,\ldots, & \xi_{n+n'-r}/\varsigma_1,\ldots, & \varsigma_{j-1},\varsigma_{j+1},\ldots, & \varsigma_{m'} \\ y_1,\ldots, & y_{n+m'-r}/z_1, & \ldots, & z_{n'} \end{pmatrix} = F(U)D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r} \\ y_1,\ldots, & y_{n+m'-r} \end{pmatrix}+$$

$$-\sum_{i=1}^{n+n'-r}D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{i-1},\xi_i UT,\xi_{i+1},\ldots, & \xi_{n+n'-r} \\ y_1, & \ldots, & y_{n+m'-r} \end{pmatrix}-\sum_{j=1}^{m'}\overline{D}_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r}/\varsigma_1,\ldots, & \varsigma_{j-1},\varsigma_j T,\varsigma_{j+1},\ldots, & \varsigma_{m'} \\ y_1,\ldots, & y_{n+m'-r}/z_1, & \ldots, & z_{n'} \end{pmatrix}=$$

$$= F(U)D_n\begin{pmatrix} \xi_1,\ldots, & \xi_{n+n'-r} \\ y_1,\ldots, & y_{n+m'-r} \end{pmatrix}-\sum_{i_1+\ldots+i_{n+n'-r+m'}=1}\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1 T^{i_{n+n'-r+1}},\ldots, & \varsigma_1 T^{i_{n+n'-r+m'}} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}.$$

Setting $m=1,\ldots k$, by the induction hypothesis,

$$F_{\xi'_{k+1}y'_{k+1}}T_{n+1}^m\begin{pmatrix} \xi'_{k+1},\xi_1,\ldots, & \xi_{n+n'-r} \\ y'_{k+1},y_1,\ldots, & y_{n+m'-r} \end{pmatrix} \tag{7}$$

can be presented as the sum of elements

$$F_{\xi'_{k+1}y'_{k+1}}\overline{D}_n\begin{pmatrix} \xi'_{k+1}(UT)^{i_0},\xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y'_{k+1},y_1,\ldots & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}$$

where $i_0+\ldots+i_{n+n'-r+m'}=m$.

Therefore, (7) can be expressed by the sum of elements

$$F_{\xi'_{k+1}y'_{k+1}}\overline{D}_n\begin{pmatrix} \xi'_{k+1}(UT)^{l},\xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y'_{k+1},y_1,\ldots & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}$$

where $l=0,\ldots m$ and $i_1+\ldots+i_{n+n'-r+m'}=m-l$.

By expanding

$$\overline{D}_n\begin{pmatrix} \xi'_{k+1}(UT)^l,\xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y'_{k+1},y_1,\ldots & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}$$

along the first column, we obtain

$$\xi'_{k+1}(UT)^l Uy'_{k+1}\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}+$$
$$+\sum_{j=1}^{n+n'-r}(-1)^j\xi_j(UT)^{i_j}Uy'_{k+1}\times$$
$$\times\overline{D}_n\begin{pmatrix} \xi'_{k+1}(UT)^l,\xi_1(UT)^{i_1},\ldots,\xi_{j-1}(UT)^{i_{j-1}},\xi_{j+1}(UT)^{i_{j+1}},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}+$$
$$+\sum_{j=1}^{m'}(-1)^{n+n'-r+j}\varsigma_j(TU)^{i_{n+n'-r+j}}y'_{k+1}\times$$
$$\overline{D}_n\begin{pmatrix} \xi'_{k+1}(UT)^l,\ldots & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots\varsigma_{j-1}[T(UT)]^{(i_{n+n'-r+j-1})},\varsigma_{j+1}[T(UT)]^{(i_{n+n'-r+j+1})},\ldots & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}.$$

It follows from the definition of $F$, that (7) can be transformed into the form:

$$\sum_{l=0}^{m}\sum_{i_1+\ldots+i_{n+n'-r+m'}=m-l}\Bigg[F\big[(UT)^lU\big]\times\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}+$$
$$+\sum_{j=1}^{n+n'-r}(-1)^j\times$$
$$\times\overline{D}_n\begin{pmatrix} \xi_j(UT)^{i_j+l+1},\xi_1(UT)^{i_1},\ldots,\xi_{j-1}(UT)^{i_{j-1}},\xi_{j+1}(UT)^{i_{j+1}},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}+$$
$$+\sum_{j=1}^{m'}(-1)^{n+n'-r+j}\times$$
$$\times\overline{D}_n\begin{pmatrix} \varsigma_j[T(UT)]^{i_{n+n'-r+j}+l+1},\ldots\xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots\varsigma_{j-1}[T(UT)]^{(i_{n+n'-r+j-1})},\varsigma_{j+1}[T(UT)]^{(i_{n+n'-r+j+1})},\ldots\varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots y_{n+m'-r}/z_1,\ldots z_{n'} \end{pmatrix}\Bigg].$$

Hence, (7) can be written as the sum

$$\sum_{l=0}^{m}\sum_{i_1+\ldots+i_{n+n'-r+m'}=m-l}F\big[(UT)^lU\big]\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}+$$
$$-(m+1)\sum_{i_1+\ldots+i_{n+n'-r+m'}=m+1}\overline{D}_n\begin{pmatrix} \xi_1(UT)^{i_1},\ldots, & \xi_{n+n'-r}(UT)^{i_{n+n'-r}}/\varsigma_1[T(UT)]^{(i_{n+n'-r+1})},\ldots, & \varsigma_{m'}[T(UT)]^{(i_{n+n'-r+m'})} \\ y_1,\ldots, & y_{n+m'-r}/z_1,\ldots, & z_{n'} \end{pmatrix}.$$

Since $(UT)^{m-1}U\in op(\Xi\to\Omega,Y\to X)$ for $m=1,\ldots k$, in view of (1), $(UT)^{m-1}UF\in an(\Xi\to\Xi,X\to X)$ and $\sigma_m=F\big[(UT)^{m-1}U\big]=Tr\big[(UT)^{m-1}UF\big]$. By (5), (6), (7), bearing in mind (3), we obtain

$$\Big(\underbrace{F\square\ldots\square F\square}_{(k+1)-razy} D_{n+k+1}\Big)\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}= \tag{8}$$

$$= A_0\sigma_1 D_n\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix} - A_0 T_n^1\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix} - \sum_{m=1}^{k}(-1)^m(m+1)A_m T_n^{m+1}\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}+$$

$$+\sum_{m=1}^{k}(-1)^m A_m\sum_{l=0}^{m}\sigma_{l+1}T_n^{m-l}\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix} = \sum_{m=0}^{k}(-1)^m A_m\sigma_{m+1}D_n\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}+$$

$$+\sum_{p=1}^{k}(-1)^p\left(pA_{p-1}+\sum_{m=p}^{k}(-1)^{m+p}A_m\sigma_{m-p+1}\right)T_n^p\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}+(-1)^{k+1}(k+1)A_k T_n^{k+1}\begin{pmatrix}\xi_1,\ldots, & \xi_{n+n'-r}\\ y_1,\ldots, & y_{n+m'-r}\end{pmatrix}.$$

Expanding the determinant

$$\begin{vmatrix} T_{n,k+1}^0 & k+1 & 0 & \ldots & 0 & 0\\ T_{n,k+1}^1 & \sigma_1 & k & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ T_{n,k+1}^k & \sigma_k & \sigma_{k-1} & \ldots & \sigma_1 & 1\\ T_{n,k+1}^{k+1} & \sigma_{k+1} & \sigma_k & \ldots & \sigma_2 & \sigma_1\end{vmatrix} \tag{9}$$

in terms of its first column, we get

$$T_n^0\left(\sum_{m=0}^{k}(-1)^m\sigma_{m+1}A_m\right)+T_n^{k+1}\left[(-1)^{k+1}(k+1)A_k\right]+\sum_{p=1}^{k}T_n^p\left[(-1)^p(k+1)A_{p-1}\right]. \tag{10}$$

Moreover, for $p=1,\ldots k$

$$(k+1)A_{p-1}=pA_{p-1}+\frac{k!}{(k-p)!}\begin{vmatrix}\sigma_1 & k-p & 0 & \ldots & 0 & 0\\ \sigma_2 & \sigma_1 & k-p-1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{k-p} & \sigma_{k-p-1} & \sigma_{k-p-2} & \ldots & \sigma_1 & 1\\ \sigma_{k-p+1} & \sigma_{k-p} & \sigma_{k-p-1} & \ldots & \sigma_2 & \sigma_1\end{vmatrix}. \tag{11}$$

Since

$$\begin{vmatrix}\sigma_1 & k-p & 0 & \ldots & 0 & 0\\ \sigma_2 & \sigma_1 & k-p-1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{k-p} & \sigma_{k-p-1} & \sigma_{k-p-2} & \ldots & \sigma_1 & 1\\ \sigma_{k-p+1} & \sigma_{k-p} & \sigma_{k-p-1} & \ldots & \sigma_2 & \sigma_1\end{vmatrix}=\sigma_1\begin{vmatrix}\sigma_1 & k-p-1 & 0 & \ldots & 0 & 0\\ \sigma_2 & \sigma_1 & k-p-2 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{k-p} & \sigma_{k-p-1} & \sigma_{k-p-2} & \ldots & \sigma_2 & \sigma_1\end{vmatrix}+$$

$$+\sum_{m=1}^{k-p}(-1)^m\sigma_{m+1}\begin{vmatrix} k-p & 0 & \dots & 0 & 0\\ \sigma_1 & k-p-1 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \sigma_{m-1} & \sigma_{m-2} & \dots & 0 & 0\\ \sigma_{m+1} & \sigma_m & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ \sigma_{k-p} & \sigma_{k-p-1} & \dots & \sigma_2 & \sigma_1\end{vmatrix}+(-1)^{k-p}(k-p)!\,\sigma_{k-p+1},$$

by (11) and algebraic properties of a determinant, (10) can be transformed in the following way:

$$T_n^0\left(\sum_{m=0}^{k}(-1)^m\sigma_{m+1}A_m\right)+T_n^{k+1}\left[(-1)^{k+1}(k+1)A_k\right]+$$

$$+\sum_{p=1}^{k}T_n^p\left\{(-1)^p\left[pA_{p-1}+(-1)^{k-p}\sigma_{k-p+1}A_k+\sigma_1\begin{vmatrix} k & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \sigma_1 & k-1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{p-1} & \sigma_{p-2} & \dots & \sigma_1 & k-p+1 & 0 & 0 & \dots & 0 & 0\\ \sigma_{p+1} & \sigma_p & \dots & \sigma_3 & \sigma_2 & \sigma_1 & k-p-1 & \dots & 0 & 0\\ \sigma_{p+2} & \sigma_{p+1} & \dots & \sigma_4 & \sigma_3 & \sigma_2 & \sigma_1 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_k & \sigma_{k-1} & \dots & \sigma_{k-p+2} & \sigma_{k-p+1} & \sigma_{k-p} & \sigma_{k-p-1} & \dots & \sigma_2 & \sigma_1\end{vmatrix}+\right.\right.$$

$$\left.\left.+\sum_{m=1}^{k-p}(-1)^m\sigma_{m+1}\begin{vmatrix} k & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \sigma_1 & k-1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{p-1} & \sigma_{p-2} & \dots & \sigma_1 & k-p+1 & 0 & 0 & \dots & 0 & 0\\ \sigma_p & \sigma_{p-1} & \dots & \sigma_2 & \sigma_1 & k-p & 0 & \dots & 0 & 0\\ \sigma_{p+1} & \sigma_p & \dots & \sigma_3 & \sigma_2 & \sigma_1 & k-p-1 & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_{p+m-1} & \sigma_{p+m-2} & \dots & \sigma_{m+1} & \sigma_m & \sigma_{m-1} & \sigma_{m-2} & \dots & 0 & 0\\ \sigma_{p+m+1} & \sigma_{p+m} & \dots & \sigma_{m+3} & \sigma_{m+2} & \sigma_{m+1} & \sigma_m & \dots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots\\ \sigma_k & \sigma_{k-1} & \dots & \sigma_{k-p+2} & \sigma_{k-p+1} & \sigma_{k-p} & \sigma_{k-p-1} & \dots & \sigma_2 & \sigma_1\end{vmatrix}\right]\right\}=$$

$$=T_n^0\left(\sum_{m=0}^{k}(-1)^m\sigma_{m+1}A_m\right)+T_n^{k+1}\left[(-1)^{k+1}(k+1)A_k\right]+\sum_{p=1}^{k}T_n^p(-1)^p\left[pA_{p-1}+\sum_{m=0}^{k-p}(-1)^m\sigma_{m+1}A_{p+m}\right].$$

Hence, in view of (8), $\underbrace{F\square\dots\square F\square}_{(k+1)-times}D_{n+k+1}$ is of the form (9). This proves the induction thesis and completes the proof.

## References


[1] Sikorski R., The determinant theory in Banach spaces, Colloq. Math. 1961, 8, 141-198.

[2] Buraczewski A., The determinant theory of generalized Fredholm operators, Studia Math. 1963, 22, 265-307.

[3] Buraczewski A., Sikorski R., Analytic formulae for determinant systems in Banach spaces, Studia Math. 1980, 67, 85-101.

[4] Ruston A.F., On the Fredholm theory of integral equations for operators belonging to the trace class of a general Banach space, Proc. Lond. Math. Soc. III. Ser. (2) 1951, 53, 109-124.

[5] Grothendieck A., La theorie de Fredholm, Bull. Soc. Math. Fr. 1956, 84, 319-384.

[6] Leżański T., The Fredholm theory of linear equations in Banach spaces, Stud. Math. 1953, 13, 244-276.

[7] Pietsch A., Eigenvalues and S-numbers, Cambridge Studies in Advanced Math. Vol. 13, Cambridge Univ. Press, Cambridge 1987.

[8] Gohberg I., Goldberg S., Krupnik N., Traces and determinants of linear operators, Integr. Equat. Oper. Th. 1996, 26, 136-187.

[9] Plemelj J., Zur Theorie der Fredholmschen Funktionalgleichung, Monat. Math. Phys. 1904, 15, 93-128.

[10] Smithies F., The Fredholm theory of integral equations, Duke Math. J. 1941, 107-130.

[11] Marchetti D.H.U., An alternative to Plemelj-Smithies formulas on infinite determinants, J. Funct. Anal. 1993, 117, 360-376.

[12] Ciecierska G., On some property of the modified power of an algebraic nucleus, Scientific Research of the Institute of Mathematics and Computer Science, Czestochowa University of Technology 2008, 2 (7), 15-21.