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SCENARIO BASED ANALYSIS OF LOGARITHMIC UTILITY APPROACH FOR DERIVING PRIORITY VECTORS IN ANALYTIC HIERARCHY PROCESS

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Abstract. Obviously, providing unique answers to the alternatives of a decision is a prerequisite for each authentic decision making theory. It is common knowledge that the Eigenvalue Method, usually applied in the Analytic Hierarchy Process, in a unique way captures the transitivity in matrices that are not consistent. This could lead to the conclusion that maybe the Eigenvalue Method is the only proper way to enable reliable decision making based on priority weighing during pairwise comparison judgments in a situation when inconsistency takes place. Undoubtedly, however, the Eigenvalue Method, in spite of obvious benefits, also has a few drawbacks which perhaps, should be also taken into consideration before labelling it as exceptional. That is also the reason why a relatively novel and new approach is introduced in this article. In the approach presented herein, an optimisation procedure is combined with the Eigenvalue Method, which enables the retaining of advantages of the latter, while at the same time avoiding its drawbacks.

Introduction

Plenty of methods designed for the purpose of priorities establishment on the basis of intuitive judgments can be found in the literature. Some of them are based on different statistical concepts [1-3], while others focus on constrained optimization models [4-7]. Obviously every method proposed in the literature has its own pros and cons debate and thus one can find supporters and adversaries for each of them. Comparative studies of different prioritization methods [8-11], as well as suggestions to blend various prioritization techniques for better true priority vector estimates [12], can be found as well. It seems that most of the known prioritization methods can be numbered among constrained optimization ones [13]. A few of them are briefly described in [14].

These methods can be described in the following manner. Let us presume that we have only judgments (estimates) of the relative weights of a set of activities. Then we can express them in a pairwise comparison matrix (PCM) denoted as $\boldsymbol{A} = [a_{ij}]_{nxn}$ with elements $a_{ij} = a_i/a_j$. Let us also denote $\boldsymbol{A}(\boldsymbol{w}) = [w_{ij}]_{nxn}$ as the symbol of a matrix with elements $w_{ij} = w_i/w_j$. Now, if we would like to recover the vector of weights $\boldsymbol{w} = [w_1, w_2, w_3, \ldots, w_n]^T$ which the true relative weights of a set of activities can be created from, as in the case of matrix $\boldsymbol{A}(\boldsymbol{w})$, we can apply an optimization method which seeks vector $\boldsymbol{w}$ as a solution to the following minimization problem:

$$\min D(A, A(w)) \tag{1}$$

subject to some assigned constraints such as positive coefficients and a normalization condition.

As distance function $\boldsymbol{D}$ measures an interval between matrices $\boldsymbol{A}$ and $\boldsymbol{A(w)}$, various ways of its definition lead to different prioritization concepts. It seems that the most popular one is called the *logarithmic least squares method* (LLSM), known also as the geometric mean method [2, 5, 15]. In this method, the objective function measuring the distance between $\boldsymbol{A}$ and $\boldsymbol{A(w)}$ is given by:

$$\min D(A,A(w)) = \sum_{i,j=1}^{n} (\ln a_{ij} - \ln w_i + \ln w_j)^2 \tag{2}$$

In order to receive the estimate of the priority vector, objective function (2) needs to be minimized with subjection to the following constraints:

$$\prod_{i=1}^{n} w_i = 1, \qquad w_i > 0, \qquad i = 1, \ldots, n$$

The LLSM solution also has the following closed form and is given by the normalized products of the elements in each row:

$$w_i = \frac{\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}}{\sum_{i=1}^{n}\left(\prod_{j=1}^{n} a_{ij}\right)^{1/n}} \tag{3}$$

However, there is a method that cannot be recognized as one of these characterized as constrained optimization ones. This is also the first and most commonly used prioritization method which is also a fundamental part of a mathematical theory for deriving ratio scale priority vectors (PV) from positive reciprocal matrices with entries set on the basis of pairwise comparisons. The theory is called the Analytic Hierarchy Process (AHP) and it uses the principal Eigenvalue Method (EM) to derive priority vectors [10, 11, 16, 17].

It can be described in the following manner. Let us presume that we know the relative weights of a set of activities. Then we can express them in a PCM like $\boldsymbol{A(w)}$ which was described above. Now, if we would like to recover the vector of weights $\boldsymbol{w}$ which the ratios in $\boldsymbol{A(w)}$ can be created from, we could take the matrix product of matrix $\boldsymbol{A(w)} = [w_{ij}]_{nxn}$ with vector $\boldsymbol{w}$. If we know $\boldsymbol{A(w)}$, but not $\boldsymbol{w}$, we can solve this problem for $\boldsymbol{w}$. Solving a nonzero solution for this set of equations is a very common procedure and is known as an eigenvalue problem:

$$A(w) \times w = \lambda \times w \tag{4}$$

In order to find the solution of this set of equations, in general, one needs to solve an nth order equation for λ that, in general, leads to n unique values for λ, with an associated vector $\boldsymbol{w}$ for each of the n values. However, in the case of PCM based on priority weighing, matrix $\boldsymbol{A(w)}$ has a special form, since each row is a constant multiple of the first row. In this case, matrix $\boldsymbol{A(w)}$ has only one nonzero eigenvalue and since the sum of the eigenvalues of a positive matrix is equal to the sum of its diagonal elements, the only nonzero eigenvalue in such a case equals the size of the matrix and can be denoted as $\lambda_{max} = n$. If the elements of matrix $\boldsymbol{A(w)}$ satisfy condition $w_{ij} = 1/w_{ji}$ for all $i, j = 1,\dots,n$, then matrix $\boldsymbol{A(w)}$ is said to be *reciprocal*. If its elements satisfy condition $w_{ik}w_{kj} = w_{ij}$ for all $i, j, k = 1,\dots,n$ and the matrix is reciprocal, then it is called *consistent*. Finally, matrix $\boldsymbol{A(w)}$ is said to be *transitive* if the following condition holds: if element w_{ij} is not less than element w_{ik}, then $w_{ij} \geq w_{ik}$ for $i = 1,\dots,n$.

Obviously, in real life during priority weighing we do not have $\boldsymbol{A(w)}$ but only its estimate $\boldsymbol{A}$ containing our intuitive judgments, more or less close to $\boldsymbol{A(w)}$ in accordance to our skills, experience, etc. In such a case, the consistency property does not hold and the relation between the elements of $\boldsymbol{A}$ and $\boldsymbol{A(w)}$ can be expressed in the following form:

$$a_{ij} = e_{ij} w_{ij} \tag{5}$$

where e_{ij} is a perturbation factor which should be close to 1. It has been shown that for any matrix, small perturbations in the entries imply similar perturbations in the eigenvalues, that is why in order to estimate true priority vector $\boldsymbol{w}$, one needs to solve the following matrix equation:

$$\boldsymbol{A} \times \boldsymbol{w} = \lambda_{\max} \times \boldsymbol{w} \tag{6}$$

where $\lambda_{\max}$ is the principal eigenvalue, it is not smaller than n, and other characteristic values are close to zero. The estimates of true priority vector $\boldsymbol{w}$ can be found then by normalizing the eigenvector corresponding to the largest eigenvalue in equation (6) which is simple and its existence is guaranteed by Perron's Theorem [16]. In practice, the EM solution is obtained by raising matrix $\boldsymbol{A}$ to a sufficiently large power, then summing over the rows and normalizing in order to receive $\boldsymbol{w}$. Denoting $\boldsymbol{e} = [1,1,\dots,1]$, this concept can also be delivered in the form of the following formula:

$$\boldsymbol{w} = \lim_{k \to \infty} \left(\frac{A^k \times e^T}{e \times A^k \times e^T} \right) \tag{7}$$

1. Definition of the problem

It is a prerequisite that an authentic decision making theory should provide unique answers for the alternatives of a decision. As was presented above, different methods and algorithms were devised in order to elicit true priority vectors from intuitive judgments. When judgments are rather consistent, the results of all approaches rather coincide. However, in real life, judgments are constantly inconsistent. Such a situation gives rise to different priority vectors due to the application of different methods. It was also proved that especially in multicriteria processes, even when different methods provide priority vectors that are close, both regarding criteria and alternatives, after synthesis according to a well-prescribed procedure [17] (standard AHP aggregation based on weighting and adding), the rank order of the alternatives can vary [10].

One could conclude that such a variety of results that a potential decision maker can obtain violates the uniqueness requirement mentioned above and therefore seems unacceptable. On the other hand, it is known that the EM captures transitivity in matrices that are not consistent in a unique way. That could lead to a conclusion that maybe the EM is necessary and sufficient to facilitate credible decision making based on priority weighing followed by inconsistent matrices comprising of pairwise comparison judgments.

However, let us remember that the EM, despite of its obvious advantages, also has a few disadvantages and drawbacks. First of all, it requires complex calculations connected with an iterative procedure given by equation (7). Secondly, it enforces the reciprocity of the PCM through an imposed convention concerning PCM inputs collection. Typically, PCM inputs are gathered only for the elements placed above diagonal elements of matrix $\boldsymbol{A}$. The remaining ones are entered as the inverse of the corresponding symmetric elements in relation to the diagonal elements of matrix $\boldsymbol{A}$. It is crucial to notice that such a kind of consistency imposition loses additional information which could be revealed during data collection for the lower triangle of $\boldsymbol{A}$ and in consequence may lead to worse estimates of the PV. For example, if I am supposed to judge if I like pears three times more than apples, and I believe I do, it does not necessarily have to mean that in comparison to pears, my judgment will be that I like apples three times less. Furthermore, EM results are sensitive to data outliers [4]. Additionally, in contrast to most estimation procedures, the EM does not optimise any criterion function, and that entails difficulties in the interpretation and comparison to other method calculations [2, 18]. Finally, the rank reversal phenomena is an area of criticism as well [19].

2. Conception of problem solution

It has already been deduced [14, 20] that instead of solving eigenvalue equation (6), one may seek a vector $\boldsymbol{w}$ which best estimates equation (4). In order to satisfy equation (4) as perfectly as possible, we propose to estimate the PV by solving the following optimization problem:

$$\min \sum_{i=1}^{n} \left(\ln \left(\sum_{j=1}^{n} \frac{a_{ij} w_j}{n w_i} \right) \right)^2 \tag{8}$$

subject to:

$$\sum_{i=1}^{n} w_i = 1, \qquad w_i > 0, \qquad i, j = 1, \ldots, n$$

We call the above proposed method of PVs deriving as the *Logarithmic Squared Deviations Minimization Method* (denoted LSDM).

In contrast to the EM, the above proposed method does not suffer from the drawbacks already mentioned. It can be easily applied to reciprocal and nonreciprocal matrices as well. The computations performed during PVs deriving procedure are considerably easier than in the case of the EM and, what is more, they can be easily performed with the application of standard office software packages commonly available. Additionally, what is quite important, LSDM deals positively with rank reversal phenomena.

3. Inconsistency issue

Obviously, along with the PVs deriving method it seems imperative to also deliver a measure of our intuitive judgments inconsistency. It is obvious that even the best method of PVs estimation is useless until information about the scale of PCM inconsistency is provided. The proposed method, however, gives rise to a very simple measure of inconsistency which can be a square root of the objective function minimum, divided by *n*. Obtained in this way, the index (denoted as inconsistency index - INCI) has an intuitive interpretation because it can be given in percentage points and be perceived as a standard deviation from zero, an optimum value which denotes a perfect consistency. We can also take another path in order to establish a measure of inconsistency for the proposed method and instead of taking the INCI indication directly, we could divide it by its equivalent for a random matrix of the same size. Then we could decide for example that the value of such a quotient, denoted as the inconsistency ratio (INCR) cannot exceed the level of 10%. This concept basically would be then a mirror copy of an idea successfully applied together with the EM, however, with one obvious exception: the EM procedure uses differently defined indices, and in the case of nonreciprocal matrices, their values are negative and therefore inexplicable. Which path to choose as the best one, for the time being, is the dilemma which we leave pending until further studies.

4. Scenario based analysis

In this section of the article, we provide LSDM efficacy analysis based on an already published case study. We will analyze if the LSDM preserves the *intensity* of preferences (rank reversal phenomena) in the case where the EM fails. However, in order to do so, we must first clarify the meaning of the order preservation condition formulated by Bana e Costa et al. [19]. They provide the following definition: for all alternatives a1, a2, a3, a4 such that a1 dominates a2 and a3 dominates a4, and the extent to which a1 dominates a2 is greater than the extent to which a3 dominates a4, we have not only w1 > w2 and w3 > w4 but also w1/w2 > w3/w4 for the derived PV. Now we analyze the scenario provided in [19] to verify the efficacy of LSDM. Let the PCM be as follows:

$$\begin{bmatrix} 1 & 2 & 3 & 5 & 9 \\ 1/2 & 1 & 2 & 4 & 9 \\ 1/3 & 1/2 & 1 & 2 & 8 \\ 1/5 & 1/4 & 1/2 & 1 & 7 \\ 1/9 & 1/9 & 1/8 & 1/7 & 1 \end{bmatrix}$$

As we can see here, according to a common linguistic interpretation for the AHP, a1 *strongly* dominates a4 (a1/a4 = 5), and a4 *very strongly* dominates a5 (a4/a5 = 7). That implies a1/a4 < a4/a5. However, the PV derived from the EM provides $[0.4262, 0.2809, 0.1652, 0.1008, 0.0269]^T$ and yields the ratios a1/a4 = = 4.218 > a4/a5 = 3.741 which violate the COP.

Let us now apply the method just proposed in this article, i.e. LSDM. The PV derived from the LSDM provides $[0.434659, 0.282449, 0.163602, 0.097671, 0.021620]^T$ and yields the ratios a1/a4 = 4.450245<a4/a5 = 4.517668 which, contrary to the EM, satisfy the COP. We recapitulate now with the following conclusions.

Conclusions

To summarize, there are other valid methods for deriving priority vectors from pairwise comparison matrices, particularly when the matrices are inconsistent, that are equally satisfying as the eigenvalue method and sometimes they are even better. There is at least one such method herein presented. It is so, because this method can be applied to both reciprocal and nonreciprocal matrices, it is computationally simpler and what is most important it prevails the rank reversal phenomena (condition of order preservation). Facing these facts we deem it reasonable to emphasize them in the form of the following statement.

STATEMENT. The *logarithmic squared deviations minimization method is probably equally as good as the eigenvalue method for true priority vectors deri-*

ving procedure based on inconsistent pairwise comparison judgments and sometimes it is even better.

Obviously, the statement needs to be strengthed, and further studies and analysis are necessary in order to make it happen. Certainly, they have already commenced.

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