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USE OF LAGRANGE MULTIPLIER FORMALISM TO SOLVE TRANSVERSE VIBRATIONS PROBLEM OF STEPPED BEAMS ACCORDING TO TIMOSHENKO THEORY

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Abstract. In this paper Lagrange multiplier formalism has been used to find a solution to a free transverse vibrations problem of stepped beams. The beams have been circumscribed according to the Timoshenko theory. The sample numerical calculations for a cantilever two-stepped beam have been carried out to illustrate the validity and accuracy of the present method.

Introduction

Free vibrations of a non-uniform beam according to the Bernoulli-Euler theory have been the subject of research of many authors [1-6]. However, the classical Bernoulli-Euler theory of flexural behaviour of an elastic beam is inadequate for the vibration of higher modes or for short beams. In this case, the effect of shear deformation and rotary inertia should be taken into account. Therefore Timoshenko [7, 8] modified the classical Bernoulli-Euler beam theory with the above--considered influence. Among the papers devoted to vibrations of beams with non-uniform inertia, cross-section and mass distribution according to the Timoshenko theory, publications [9-15] are worth listing.

In this paper, the free vibrations problem of stepped beams has been formulated and solved with the help of Lagrange multiplier formalism [16, 17]. The beams have been circumscribed according to Timoshenko theory. Exemplary numerical calculations have been carried out and compared to the results of other authors.

1. Formulation and solution of the problem

Kinetic (T_b) and potential (V_b) energy of a free homogeneous Timoshenko beam without additional elements can expressed as [18]:

$$T_b(t)=\frac{1}{2}\int_0^L\left[\frac{\partial y(x,t)}{\partial t}\right]^2\rho A(x)\mathrm{d}x+\frac{1}{2}\int_0^L\left[\frac{\partial \psi(x,t)}{\partial t}\right]^2\rho I(x)\mathrm{d}x \tag{1}$$

$$V_b\left(t\right)=\frac{1}{2}\int_0^L EI\left(x\right)\left[\frac{\partial\psi\left(x,t\right)}{\partial x}\right]^2 \mathrm{d}x+\frac{1}{2}\int_0^L k'GA\left(x\right)\left[\frac{\partial y\left(x,t\right)}{\partial x}-\psi\left(x,t\right)\right]^2 \mathrm{d}x \tag{2}$$

where: $y(x,t)$ is the total deflection of the beam at a point x, $\psi(x,t)$ is the angle of rotation due to bending, ρ is the mass density, $A(x)$ is the cross sectional area, $I(x)$ is the moment of inertia, E is the modulus of elasticity, G is the shear modulus and k' is a numerical factor depending on the shape of the cross-section.

Within the pondered theory, displacements $y(x,t)$ and $\psi(x,t)$ can be expressed (on basis of the solution of free vibrations beam problem without additional elements) as:

$$y(x,t)=\sum_{i=1}^{m} Y_i(x)\xi_i(t) \tag{3}$$

$$\psi(x,t)=\sum_{i=1}^{m} \Psi_i(x)\xi_i(t) \tag{4}$$

at the same time: $Y_i(x)$ and $\Psi_i(x)$ denote the i-th transverse and rotational vibrational mode respectively, $\xi_i(t)$ are the time functions.

Substituting equations (3) and (4) into equations (1) and (2), one obtains:

$$T_b(t)=\frac{1}{2}\sum_{i=1}^{m} M_i\dot{\xi}_i^2 \tag{5}$$

$$V_b(t)=\frac{1}{2}\sum_{i=1}^{m} K_i\xi_i^2 \tag{6}$$

where:

$$M_i=\int_0^L Y_i^2\left(x\right)\rho A\left(x\right)dx+\int_0^L \Psi_i^2\left(x\right)\rho I\left(x\right)dx \tag{7}$$

$$K_i=\int_0^L EI\left(x\right)\Psi_i'^2\left(x\right)dx+\int_0^L k'GA\left(x\right)\left[Y_i'\left(x\right)-\Psi_i\left(x\right)\right]^2 dx \tag{8}$$

Considering the vibrations of stepped beams (Fig. 1), the beam can come down to a system of N segments. Each segment is described according to the Timoshenko theory and has constant parameters ρ, $A(x)$, $I(x)$, E, G and k'.

In the case of stepped beams consisting of N segments, relationships (3) and (4) can be rewritten as:

$$y_n\left(x_n,t\right)=\sum_{i=1}^{m} Y_{n_i}\left(x_n\right)\xi_{n_i}\left(t\right) \tag{9}$$

$$\psi_n\left(x_n,t\right)=\sum_{i=1}^{m} \Psi_{n_i}\left(x_n\right)\xi_{n_i}\left(t\right) \tag{10}$$

while kinetic (5) and potential (6) energy as:

$$T_b(t)=\frac{1}{2}\sum_{n=1}^{N}\sum_{i=1}^{m}M_{n_i}\dot{\xi}_{n_i}^2 \tag{11}$$

$$V_b(t)=\frac{1}{2}\sum_{n=1}^{N}\sum_{i=1}^{m}K_{n_i}\xi_{n_i}^2 \tag{12}$$

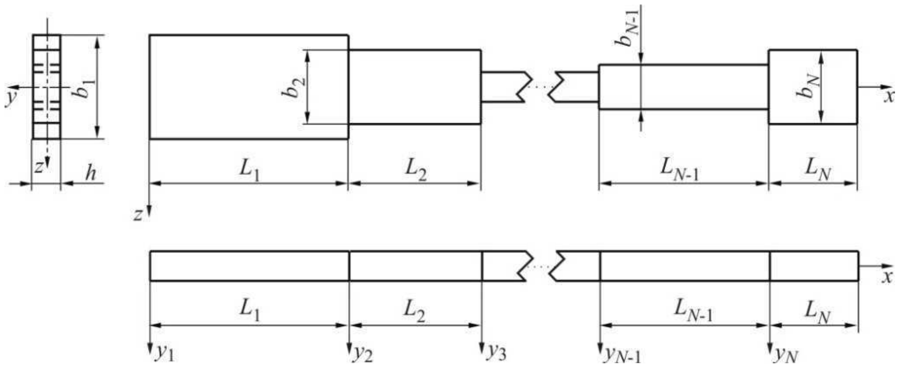

Fig. 1. Scheme of stepped beam

In formulas (11) and (12), the following denotations have been introduced:

$$M_{n_i}=\int_0^{L_n}Y_{n_i}^2(x_n)\,\rho A_n(x_n)\mathrm{d}x_n+\int_0^{L_n}\Psi_{n_i}^2(x_n)\,\rho I_n(x_n)\mathrm{d}x_n \tag{13}$$

$$K_{n_i}=\int_0^{L_n}EI_n(x_n)\Psi_{n_i}'^2(x_n)\mathrm{d}x_n+\int_0^{L_n}k'GA_n(x_n)\left[Y_{n_i}'(x_n)-\Psi_{n_i}(x_n)\right]^2\mathrm{d}x_n \tag{14}$$

In the mathematical model, the joints of all the segments have been realized by introducing the constraint functions:

$$\begin{gathered}
f_1\equiv y_1(L_1,t)-y_2(0,t)=0\\
f_2\equiv \psi_1(L_1,t)-\psi_2(0,t)=0\\
f_3\equiv y_2(L_2,t)-y_3(0,t)=0\\
f_4\equiv \psi_2(L_2,t)-\psi_3(0,t)=0\\
\ldots,\\
f_{2(N-1)-1}\equiv y_{N-1}(L_{N-1},t)-y_N(0,t)=0\\
f_{2(N-1)}\equiv \psi_{N-1}(L_{N-1},t)-\psi_N(0,t)=0
\end{gathered} \tag{15}$$

The Lagrangian for the above complex system has the form:

$$L = T - V + \sum_{r=1}^{2(N-1)} \lambda_r f_r \tag{16}$$

where λ_r are the Lagrange multipliers, whose numbers depend directly on the number of segments.

Substituting relationships (11), (12), (15) into (16) and using the Lagrange equation, the system of motion equations can be formulated:

$$M_{1_i}\ddot{\xi}_{1_i} + K_{1_i}\xi_{1_i} - \sum_{r=1}^{2}\lambda_r b_{1_{i,r}} = 0$$

$$M_{2_i}\ddot{\xi}_{2_i} + K_{2_i}\xi_{2_i} + \sum_{r=1}^{2}\lambda_r b_{2_{i,r}} - \sum_{r=3}^{4}\lambda_r b_{2_{i,r}} = 0$$

$$\ldots, \tag{17}$$

$$M_{(N-1)_i}\ddot{\xi}_{(N-1)_i} + K_{(N-1)_i}\xi_{(N-1)_i} + \sum_{r=2(N-1)-3}^{2(N-1)-2}\lambda_r b_{(N-1)_{i,r}} - \sum_{r=2(N-1)-1}^{2(N-1)}\lambda_r b_{(N-1)_{i,r}} = 0$$

$$M_{N_i}\ddot{\xi}_{N_i} + K_{N_i}\xi_{N_i} + \sum_{r=2N-3}^{2N-2}\lambda_r b_{N_{i,r}} = 0$$

The introduced denotations $b_{n_{i,r}}$ in (17) represent the i-th translational and rotational vibrational modes of n-th beam segments without additional elements:

$$b_{n_{i,r}} = \begin{cases} Y_{n_i}\left(x_{n,r}\right) \text{for } r = 1,3,5,\ldots,2N-3,\ n = 1,2,\ldots,N \\ \Psi_{n_i}\left(x_{n,r}\right) \text{for } r = 2,4,6,\ldots,2N-2,\ n = 1,2,\ldots,N \end{cases} \tag{18}$$

where:

$$x_{n,r} = \begin{cases} 0 \text{ for } r = \begin{cases} 2n-3 \\ 2n-2 \end{cases} \\ L_n \text{ for } r = \begin{cases} 2n-1 \\ 2n \end{cases} \end{cases} \tag{19}$$

In order to receive the solution of the system of equations (17), the harmonic motion has been assumed:

$$\xi_{n_i} = A_{n_i}\sin\omega t \ \text{ for } \ n = 1 \div N\ ,\ i = 1 \div m \tag{20a}$$

$$\lambda_r = \Lambda_r \sin\omega t \ \text{ for } \ r = 1 \div (2N-2) \tag{20b}$$

By substituting relationships (20) into the equations of motion (17), the values of A_{ni} determined in depending on the values of the Lagrange multipliers amplitudes:

$$A_{1_i}=\frac{\sum_{r=1}^{2}\Lambda_r b_{1_{i,r}}}{K_{1_i}-\omega^2 M_{1_i}},\ A_{2_i}=\frac{-\sum_{r=1}^{2}\Lambda_r b_{2_{i,r}}+\sum_{r=3}^{4}\Lambda_r b_{2_{i,r}}}{K_{2_i}-\omega^2 M_{2_i}},\ \ldots,$$

$$A_{(N-1)_i}=\frac{-\sum_{r=2(N-2)-1}^{2(N-2)}\Lambda_r b_{(N-1)_{i,r}}+\sum_{r=2(N-1)-1}^{2(N-1)}\Lambda_r b_{(N-1)_{i,r}}}{K_{(N-1)_i}-\omega^2 M_{(N-1)_i}},\ A_{N_i}=\frac{-\sum_{r=2(N-1)-1}^{2(N-1)}\Lambda_r b_{N_{i,r}}}{K_{N_i}-\omega^2 M_{N_i}} \tag{21}$$

Inserting relationships (21) into constraint functions (15), the following system of equations in the matrix form has been obtained:

$$\mathbf{C}\boldsymbol{\Lambda}=0 \tag{22}$$

where:

$$\boldsymbol{\Lambda}=\left[\Lambda_1,\Lambda_2,\ldots,\Lambda_{2(N-1)-1}\right]^T \tag{23}$$

is the vector of Lagrange multipliers and band matrix **C** has the form:

$$\mathbf{C}=\begin{bmatrix}\mathbf{C}_{1,1} & \mathbf{C}_{1,2} & 0 & 0 & 0 & \ldots & 0 & 0 & 0\\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} & \mathbf{C}_{2,3} & 0 & 0 & \ldots & 0 & 0 & 0\\ 0 & \mathbf{C}_{n,n-1} & \mathbf{C}_{n,n} & \mathbf{C}_{n,n+1} & 0 & \ldots & 0 & 0 & 0\\ 0 & 0 & \mathbf{C}_{n+1,n} & \mathbf{C}_{n+1,n+1} & \mathbf{C}_{n+1,n+2} & \ldots & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{C}_{n+2,n+1} & \mathbf{C}_{n+2,n+2} & \ldots & 0 & 0 & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & 0 & 0 & 0 & \ldots & \mathbf{C}_{N-2,N-2} & \mathbf{C}_{N-2,N-1} & 0\\ 0 & 0 & 0 & 0 & 0 & \ldots & \mathbf{C}_{N-1,N-2} & \mathbf{C}_{N-1,N-1} & \mathbf{C}_{N-1,N}\\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & \mathbf{C}_{N,N-1} & \mathbf{C}_{N,N}\end{bmatrix} \tag{24}$$

The sub-matrices on the diagonal have the form:

$$\mathbf{C}_{n,n}=\begin{bmatrix}C_{n_{2n-1,2n-1}}+C_{n+1_{2n-1,2n-1}} & C_{n_{2n-1,2n}}+C_{n+1_{2n-1,2n}}\\ C_{n_{2n,2n-1}}+C_{n+1_{2n,2n-1}} & C_{n_{2n,2n}}+C_{n+1_{2n,2n}}\end{bmatrix} \tag{25}$$

and the sub-matrices above and below the diagonal have the form:

$$\mathbf{C}_{n,n+1}=\mathbf{C}^T_{n+1,n}=\begin{bmatrix}-C_{n+1_{2n-1,2n+1}} & -C_{n+1_{2n-1,2(n+1)}}\\ -C_{n+1_{2n,2n+1}} & -C_{n+1_{2n,2(n+1)}}\end{bmatrix} \tag{26}$$

Coefficients $C_{nk,r}$ have been defined as:

$$C_{n_{k,r}} = \sum_{i=0}^{m} \frac{b_{n_{i,k}} b_{n_{i,r}}}{K_{n_i} - \omega^2 M_{n_i}} \tag{27}$$

and they characterize the dynamic properties of separate segments of the beam.

Equation set (22) yields the eigenvalue equation:

$$\det \mathbf{C} = 0 \tag{28}$$

which enables one to calculate the free vibration frequency values ω_i of the system.

2. Sample results

In order to check the present method, a numerical program has been worked out for a cantilever two-stepped beam (Fig. 2). This combined system can be treated like a free-ends beam (in accordance with formulation of the problem) with additional elements [16, 17]. Therefore matrix (24) has to be modified by introducing an additional support against the beam translation and rotation.

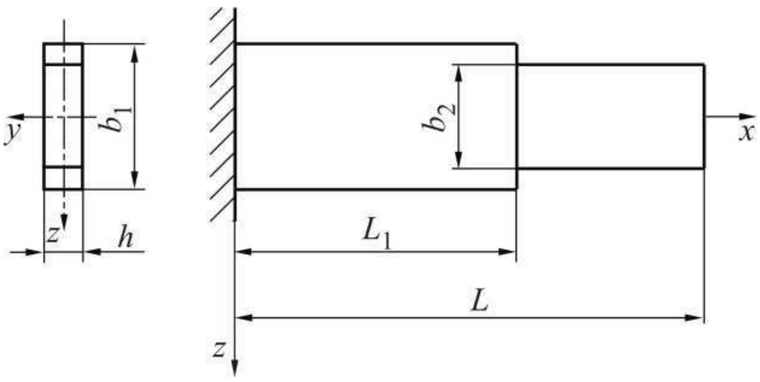

Fig. 2. A cantilever two-stepped beam

The numerical calculations have been completed for the following data:

$b_2/b_1 = 0.8$, $L_1/L = 2/3$, $k' = 5/6$, $\nu = 0.3$, $\Omega_i = \sqrt{\rho A_1 L^4 / EI_1}\,\omega_i$, $r_1^2 = I_1 / A_1 L^2$.

The first five dimensionless natural frequencies obtained by taking into account ten, twenty and thirty terms of coefficients $C_{n_{k,r}}$ (27) are compared to the results of other authors [9, 10, 14] and shown in Table 1.

On the basis of the carried out numerical calculations, one can state that the present method shows good correspondence to the research results of other authors, when thirty terms of coefficients $C_{n_{k,r}}$ have been taken into account.

Table 1

Frequency coefficient of a cantilever two-stepped beam

r_1	Source		Frequencies				
			Ω_1	Ω_2	Ω_3	Ω_4	Ω_5
0.0133	Present method	m = 10	4.096	24.061	62.58	120.563	205.215
		m = 20	3.952	22.846	59.333	115.507	192.861
		m = 30	3.903	22.412	58.214	113.987	188.847
	Reference [9]		3.82	21.35	55.04	107.50	173.62
	Reference [10]		3.8219	21.3540	55.0408	107.4993	173.6322
	Reference [14]		3.8244	21.3546	55.0445	107.5079	173.6228

Summary

In this paper, the free vibrations problem of stepped Timoshenko beams has been formulated and solved on the basis of Lagrange multiplier formalism. In the consideration, the beam has been treated like a free-free beam without additional elements. Only during examination of the cantilever stepped beam was the system modified by the addition of two elements substituting the clamp.

The presented sample numerical calculations compared to the results of other authors indicate a good agreement. However, if the results have to be received with the demanded precision, then the number of terms of coefficients $C_{n_{k,r}}$ have to be appropriately selected.

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