RINGS RELATED TO FINITE POSETS

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Abstract. In this article we consider special classes of rings related to finite partially ordered sets over division rings and prime hereditary Noetherian rings. The structure and main properties of these rings are studied. These rings are closely connected with right hereditary SPSD-rings.

Introduction

One of the main goals in the ring theory is to reduce in a certain sense the description of large classes of rings to simpler classes by use of some ring theoretic constructions. The best classical example is the Wedderburn-Artin theorem describing semisimple Artinian rings in the form of direct sums of matrix rings over division rings. There are other interesting constructions of rings among which are incidence algebras.

The incidence algebra of a locally finite partial ordered set (abbreviated poset) over a field was first introduced by Rota [1]. Later this notion was extended to the case of commutative rings. The most complete information about incidence algebras over commutative rings can be found in the book of Spiegel, O'Donnell [2].

In this paper we consider special classes of rings related to posets over associative rings (not necessary commutative). These rings can be considered as some generalization of incidence algebras. In section 1 we define and study the properties of incidence rings of finite posets over division rings. The special class of right hereditary rings connected with finite posets and discrete valuation rings (not necessary commutative) are considered in section 2.

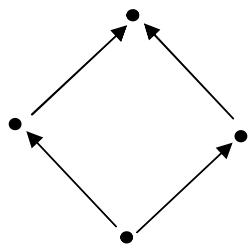
All rings considered in this paper are associative with identity and all modules are unitary. We refer to [3] and [4] for general material on theory of rings and modules.

1. Incidence rings. Rings $T(S, D)$

Consider a special class of rings related to posets. These rings are particular examples of incidence rings.

Let S = $\{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ be a finite poset with a binary ordering relation $\leq$, and *A* an associative ring with identity. A finite poset S can be represented by the **Hasse diagram** which is a directed graph with the set of vertices $\{1, 2,\ldots,n\}$ and the set of arrows given by the following way: there is an arrow σ: $i \to j$ ($i \neq j$) if and only if $\alpha_i < \alpha_j$, and moreover if $\alpha_i \leq \alpha_k \leq \alpha_j$ then either $k = i$ or $k = j$.

A poset P = {a,b,c,d | a<b<d; a<c<d} whose Hasse diagram has the following form:



(1)

is called a **rhombus**.

Denote by $\overline{\mathrm{S}}$ a non-oriented graph obtained from the Hasse diagram of S by deleting the orientation of all arrows. Then it is easy to show that a non-oriented graph $\overline{\mathrm{S}}$ is a tree if and only if S contains no subposets whose diagrams are rhombuses.

Note also the well-known fact that a finite poset S = $\{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ with an ordering relation $\leq$ can be labelled so that $\alpha_i \leq \alpha_j$ implies $i \leq j$.

Definition. The **incidence ring** of a poset S over an associative (not necessary commutative) ring *A* with identity is a subring I(S, *A*) of the generalized matrix ring $M_n(A)$ such that the (i,j)-entry of I(S, *A*) is equal to 0 if $\alpha_i \leq \alpha_j$ in S.

It is easy to show that a poset S can be labelled in such a way that a ring I(S, *A*) is isomorphic to a ring *T*(S, *A*) which is an upper triangular ring. In particular, if a poset S is a linear ordered set then $\mathrm{I}(\mathrm{S}, A) \subseteq \mathrm{T}_n(A)$.

Recall that a ring *A* is **semiperfect** if any finitely generated *A*-module has a projective cover. A ring *A* is **right hereditary** if each right ideal of *A* is projective.

The following theorem gives the main properties of the ring I(S, *A*).

Proposition 1.1. *Let* S = $\{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ *be a finite poset, and A an associative ring with identity and Jacobson radical R. Then*

1) I(S, *A*) *is semiperfect if and only if A is semiperfect*;
2) *The Jacobson radical of* I(S, *A*) *is a set of elements of* I(S, *A*) *for which the* (i,i)-*entry is in R*;
3) I(S, *A*) *is right* (*left*) *Noetherian if and only if A is right* (*left*) *Noetherian*;
4) I(S, *A*) *is right* (*left*) *Artinian if and only if A is right* (*left*) *Artinian.*

Proof.

1) follows from [3, theorem 10.3.8].

2) follows from 1) and [3, theorem 11.1.1].

3) and 4) follows from [3, theorem 3.6.1].

Now consider a particular example of the incidence rings when a ring $A = D$ is a division ring. We denote this ring $T(S, D)$ and as mentioned above there is a numbering of S such that the ring $T(S, D) \subseteq T_n(D)$. In what follows the ring $T(S, D)$ will be always assumed to be an upper triangular ring.

From proposition 1.1 we immediately obtain the following statement.

Proposition 1.2.
1. T(S, D) *is an Artinian semiperfect ring.*
2. *The Jacobson radical R of* T(S, D) *is equal to the prime radical N of* T(S, D) *and the two-sided Peirce decomposition of R has the following form*:

$$e_{ii}Re_{ii} = 0 \text{ and } e_{ii}Re_{jj} = e_{ii}T(S, D)e_{jj}, \text{ for } i,j = 1, 2,\dots,n;\ i \neq j$$

Recall that a semiperfect ring A with Jacobson radical R is **reduced** if A/R is a direct sum of division rings. From proposition 1.2 it immediately follows that the ring $T(S, D)$ is reduced.

Proposition 1.3. *The ring* T(S, D) *possesses a classical ring of fractions which coincides with* T(S, D).

Proof. By proposition 1.2, $T(S, D)$ is an upper triangular Artinian semiperfect ring with the Jacobson radical R which coincides with the prime radical N. It is easy to show that any regular element r of $T(S, D)$ has the following form

$$r = diag(d_1, d_2,\dots,d_n) + x \tag{2}$$

where each d_i is a nonzero element of D and $x \in N$. Therefore $C(0) = C(N)$, where $C(0)$ is the set of regular elements of A, and $C(N)$ is the set of elements of A whose images are regular elements in A/N.

Therefore from [5, theorem 3], it follows that $T(S, D)$ possesses a classical ring of fractions $\tilde{T}$ which is an Artinian ring. Since any regular element of $T(S, D)$ has the form (1.2), it is invertible in $T(S, D)$, and so $\tilde{T} = T(S, D)$.

A finite poset S is said to be **connected** if the Hasse diagram of S is connected. It is obvious that the ring $T(S, D)$ is indecomposable if and only if the poset S is connected. Since $T = T(S, D)$ is an Artinian ring it is possible to construct the quiver $Q(T)$ of this ring. Recall that $Q(T) = Q(T/R^2)$, where R is the Jacobson radical of T. Let $P_i = e_{ii}T$ be a principal module of $T(S, D)$. Then the right quiver of T can be constructed by the following way. If

$$e_{ii}R/e_{ii}R^2 \cong \bigoplus_{j=1}^{n} P_j^{t_{ij}}$$

then in the quiver $Q(T)$ the vertex i is connected with the vertex j by t_{ij} arrows.

Proposition 1.4. *The quiver* Q(T) *of the ring* T(S, D) *coincides with the Hasse diagram of the poset S.*

Proof. From [3, theorem 11.1.9] we can assume that $T = T(S, D)$ is an indecomposable ring and so the Hasse diagram of S is connected. Let $\{e_{ij}\}$ be the set of all matrix units of $M_n(D)$ and $1 = e_{11} + e_{22} + \ldots + e_{nn}$ a decomposition of the identity of T in the sum of pairwise orthogonal idempotents. Consider the diagram of S and the following cases.

1. Assume that there is an arrow $i \to j$ in the diagram of S, which means that $e_{ii}Re_{jj} = D$ and $e_{ii}Re_{kk} = 0$ or $e_{kk}Re_{jj} = 0$ for any integer k. Then $e_{ii}R^2e_{jj} = \sum_{k=1}^{n} e_{ii}Re_{kk} \cdot e_{kk}Re_{jj} = 0$. Therefore there is exactly one arrow $i \to j$ in the quiver $Q(T)$.
2. Assume that $\alpha_i \leq \alpha_j$ and there is no arrow of the form $i \to j$ in the Hasse diagram of S. This means that there is a positive number k such that $e_{ii}Re_{kk} = D$ or $e_{kk}Re_{jj} = D$. Then

$$e_{ii}R^2e_{jj} = \sum_{k=1}^{n} e_{ii}Re_{kk} \cdot e_{kk}Re_{jj} = D$$

and therefore $e_{ii}Re_{jj} / e_{ii}R^2e_{jj} = 0$, i.e. in the quiver $Q(T)$ there is no arrow $i \to j$.

Conversely, suppose that there is an arrow $i \to j$ in the quiver $Q(T)$. This means that there is an exact sequence

$$\bigoplus_{j=1}^{n} P_j \to e_{ii}R \to 0$$

Therefore $e_{ii}Re_{jj} \neq 0$ which means that $\alpha_i \leq \alpha_j$, and there is an arrow $i \to j$ in the diagram of S.

Proposition 1.5. *The ring $T(S, D)$ is two-sided hereditary if and only if the Hasse diagram of S is a tree, i.e. a poset* S *contains no subposets which diagrams are rhombuses.*

Proof. Let $T = T(S, D)$ be a hereditary ring. Assume the diagram of S contains subposet which diagram is a rhombus. This means that T contains an idempotent e such that the ring $eTe = B$, where

$$B = \begin{pmatrix} D & D & D & D \\ 0 & D & 0 & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix}$$

This is impossible, since the ring B is not right hereditary, and any minor of a right hereditary ring is right hereditary.

Conversely, suppose the two-sided Peirce decomposition of T does not contain the minors of the form B. Then the diagram of S is an acyclic simply laced quiver with no extra arrows such that its underlying graph $\overline{S}$ (obtaining from S by deleting the orientation of the arrows) is a tree. From proposition 1.4 it follows that the ring T can be considered as a path algebra corresponding to the graph $\overline{S}$ over a division ring D. Therefore T is a hereditary ring, by [3, theorem 2.3.4].

Theorem 1.6. *The ring $T = T(\mathrm{S}, D)$ is an Artinian semidistributive piecewise domain.*

Proof. Since for any primitive pairwise orthogonal idempotents $e, f \in T$ the ring $(e+f)T(e+f)$ is either of the form $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$ or $\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$, T is a semidistributive ring, by [3, theorem 14.2.1]. Denote $P_i = e_{ii}T$, $i = 1,\ldots,n$. Let φ: $P_i \to P_j$ be a nonzero homomorphism. Then $\varphi(e_{ii}\, a) = \varphi(e_{ii})\, a = e_{jj}a_0e_{ii}\, a$, where a_0, $a \in T$ and $e_{jj}a_0e_{ii}$ is a nonzero element from $e_{jj}\mathrm{T}e_{ii} = D$. Thus $d_0 = e_{jj}a_0e_{ii}$ defines a monomorphism. Therefore the ring T is a piecewise domain.

Theorem 1.7. *If any subdiagram of the diagram of a poset S contains no rhombuses then T(S, D) is a two-sided Artinian hereditary semidistributive ring.*

Proof. This follows immediately from theorem 1.6 and proposition 1.5.

Proposition 1.8. *The ring $T = T(\mathrm{S}, D)$ is serial if and only if S is a disconnected union of linearly ordered sets.*

Proof. *Since the ring T is indecomposable if and only if the poset S is connected, one can assume that T is indecomposable. If S is a chain, then $T = \mathrm{T}_n(D)$ and the statement is obvious.*

Conversely, suppose that T is a serial indecomposable ring. Then by proposition 1.6 the quiver of the ring T coincides with the diagram of the poset S and, by [3, theorem 12.1.2], it is a chain. Thus, S is a linearly ordered poset.

2. Right hereditary rings $A(\mathrm{S}, O)$

Let O be a discrete valuation ring with a division ring of fractions D and the Jacobson radical M. By [3, corollary 10.2.2], O is a local Noe-therian hereditary ring which is a right and left principal ideal domain (PID) and M is its unique maximal right and left ideal.

Consider the ring

$$H_n(O) = \begin{pmatrix} O & O & \cdots & O \\ M & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ M & M & \cdots & O \end{pmatrix} \qquad (3)$$

which is a subring in the matrix ring $M_n(D)$. Clearly, $H_n(O)$ is a Noetherian serial prime hereditary ring. And so, by the Goldie theorem, it has a classical ring of fractions, which is $M_n(D)$.

Let $\{O_i\}$ be a family of discrete valuation rings (not necessary commutative) with Jacobson radicals M_i and a common division ring of fractions D, for i = 1, 2,…,k; $S = \{\alpha_1, \alpha_2,\dots,\alpha_n\}$ a finite poset with a partial order $\leq$; $S_0 = \{\alpha_1, \alpha_2,\dots,\alpha_k\}$ a subset of minimal points of S ($k \leq n$), and $S = S_0 \cup S_1$, where $S_1 = \{\alpha_{k+1}, \alpha_{k+2},\dots,\alpha_n\}$.

According with this partition of S consider the **poset** S **with weights** so that the point i has the weight $H_{n_i}(O_i)$, $i = 1,2,\dots,k$; $n_i \in \mathbf{N}$; and all other points j have the weight D.

Construct the ring $A = A(S, S_0, S_1, O_1,\dots,O_k, D, n_1, n_2, \dots,n_k)$ (or $A(S,O)$, in short) which is a subring of $M_s(D)$, $s = n_1+n_2+\dots+n_k+(n-k)$ by the following way. Let the identity of A be decomposed into a sum of pairwise orthogonal idempotents $1 = f_1 + f_2 +\dots+ f_n$ and the two-sided Peirce decomposition have the following form:

$$A = \bigoplus_{i,j=1}^{n} f_i A f_j$$

where $f_i A f_i = H_{n_i}(O_i)$ for $i = 1, 2,\dots,k$; $fAf = T(S_1)$ for $f = f_{k+1} + \dots + f_n$; and $A_{ij} = f_iAf_j$ is an (A_{ii}, A_{jj})-bimodule, for $i,j = 1, 2,\dots, n$. Moreover, $A_{ij} = 0$ if $\alpha_i \leq \alpha_j$ in S. If $\alpha_i \leq \alpha_j$ in S and $\alpha_i \in S_0$, $\alpha_j \in S_1$, then $eAf_j = D$ for any $e \in f_i$. So the two-sided Peirce decomposition of A has the following form:

$$A = \begin{pmatrix} H_{n_1}(O_1) & \cdots & O & M_1 \\ \vdots & \ddots & \vdots & \vdots \\ O & \cdots & H_{n_k}(O_k) & M_k \\ O & \cdots & O & T(S_1) \end{pmatrix} \qquad (4)$$

where M_i is a $(H_{n_i}(O_i), T(S_1))$-bimodule for $i = 1, 2,\dots,k$; $T(S_1)$ is the incidence ring of a poset S_1 over a division ring D. These rings were first considered in [6].

Proposition 2.1. *Let $\{O_i\}$ be a family of discrete valuation rings with a common division ring of fractions D. Then $A(S,O)$ is a right Noetherian semiperfect ring.*

Proof. Since the identity of $A = A(S,O)$ is decomposed into a sum of a finite number of pairwise orthogonal local idempotents, A is semiperfect, by [3, theorem 10.3.8]. From the facts that all $H_{n_i}(O_i)$ are Noetherian rings, $T(S, D)$ is an Artinian ring, all M_i are finite dimension vector spaces over D, it follows that A is a right Noetherian ring, by [3, theorem 3.6.1].

Let N be the prime radical of a semiperfect ring $A(S,O)$. Then the two-sided Peirce decomposition of N has the following form

$$N = \begin{pmatrix} \mathbf{O} & \cdots & \mathbf{O} & M_1 \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \cdots & \mathbf{O} & M_k \\ \mathbf{O} & \cdots & \mathbf{O} & N_1 \end{pmatrix}, \tag{5}$$

where N_1 is the prime radical of an Artinian ring $T(S_1)$ which coincides with its Jacobson radical by [3, theorem 10.3.8]. Since N_1 is nilpotent, by [3, proposition 3.5.1], N is nilpotent, as well.

Denote $A_0 = A/N$ and $W = N/N^2$. Then $A = A_0 \oplus N$ as a direct sum of two Abelian subgroups and

$$A_0 \equiv \overline{A}_1 \times \overline{A}_2 \times \ldots \times \overline{A}_k \times \overline{T}$$

where $\overline{A}_i = H_{n_i}(O_i)$ for $i = 1,\ldots, k$; $\overline{T} \cong T(S_1)/R_1 = A_{k+1} \times \cdots \times A_n$; $A_j = D$ for $i = k+1,\ldots,n$. Thus we obtain the following statement:

Proposition 2.2. *The prime radical of the ring $A(S,O)$ is nilpotent, and A/N is a finite direct sum of prime rings.*

Let $\overline{1} = \bar{f}_1 + \ldots + \bar{f}_n$ be the corresponding decomposition of the identity of A_0 in the sum of pairwise orthogonal idempotents. Set the correspondence between the idempotents $\bar{f}_1, \ldots, \bar{f}_n$ and vertices $1,\ldots,n$ connecting the vertex i with the vertex j by an arrow if and only if $\bar{f}_i W \bar{f}_j \neq 0$. The obtained finite directed graph PQ(A) is the prime quiver of A (see [3, section 11.4]).

Since A_0 is a semiprime Noetherian ring, by the Goldie theorem it has a classical ring of fractions which has the following form:

$$\tilde{A}_0 = \tilde{A}_1 \times \ldots \times \tilde{A}_n = M_{n_1}(D) \times \cdots \times M_{n_k}(D) \times D \times \cdots \times D$$

Lemma 2.3. *Let O be a discrete valuation ring with a division ring of fractions D. Then the ring $A(S,O)$ = $\begin{pmatrix} H_n(O) & M \\ 0 & T(S_1) \end{pmatrix}$, where M is a $(H_n(O),T(S_1))$-bimodule, has the right classical ring of fractions which is an Artinian ring and has the following form*

$$\tilde{A} = \begin{pmatrix} M_n(D) & \tilde{M} \\ \mathbf{O} & T(S_1) \end{pmatrix} \tag{6}$$

where $\tilde{M} = \tilde{H}M \cong \tilde{H} \otimes_{\tilde{H}} M$, and $\tilde{H} = M_n(D)$.

Proof. Let $1 = e_1 + e_2$ be a decomposition of the identity of A into a sum of orthogonal idempotents such that $e_1Ae_1 = A_1 = H_n(O)$ and $e_2Ae_2 = A_2 = T(S_1)$. Then any regular element of A has the following form $r = \begin{pmatrix} r_1 & x \\ 0 & r_2 \end{pmatrix}$, where r_i is a regular element of A_i for $i = 1, 2$ and $x \in M$. So $C(0) = C(N)$, where N is the prime radical of A. Therefore from [5, theorem 3] and proposition 2.2 it follows that A has a classical ring of fraction $\tilde{A}$ which is an Artinian ring. From the representation of a regular element of A it follows that there exist $r_1^{-1} \in M_n(O)$ and $r_2^{-1} \in T(S_1)$ such that $r^{-1} = \begin{pmatrix} r_1^{-1} & y \\ 0 & r_2^{-1} \end{pmatrix}$, where $y = -r_1^{-1}xr_2^{-1} \in \tilde{M}$, which shows that $\tilde{A}$ has the form (6).

Proposition 2.4. *Let $\{O_i\}$ be a family of discrete valuation rings with a common division ring of fractions D. Then the ring $A(S,O)$ possesses a right classical ring of fractions $\tilde{A}$ which is an Artinian ring and has the form:*

$$\tilde{A} = \begin{pmatrix} M_{n_1}(D) & \cdots & \mathbf{O} & \tilde{M}_1 \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{O} & \cdots & M_{n_k}(D) & \tilde{M}_k \\ \mathbf{O} & \cdots & \mathbf{O} & T(S_1) \end{pmatrix}$$

where $\tilde{M}_i = \tilde{H}_i \otimes_{\tilde{H}_i} M_i$, and $\tilde{H}_i = M_{n_i}(D)$ for $i = 1, 2, \ldots, k$.

Proof. In accordance with (4), the two-sided decomposition of A can be written in the following form $A = \begin{pmatrix} H & M \\ \mathbf{O} & T \end{pmatrix}$, where

$$H = \begin{pmatrix} H_{n_1}(O_1) & \cdots & \mathbf{O} \\ \vdots & \ddots & \vdots \\ \mathbf{O} & \cdots & H_{n_k}(O_k) \end{pmatrix},$$

$T = T(S_1)$ for some subposet $S_1 \subseteq S$, and M is an (H, T)-bimodule.

Since H is isomorphic to a finite direct product of serial Noetherian rings, it possesses the classical ring of fractions $\tilde{H}$, by [3, theorem 13.2.2]. Moreover,

$$\tilde{H} = \begin{pmatrix} M_{n_1}(D) & \cdots & \mathbf{O} \\ \vdots & \ddots & \vdots \\ \mathbf{O} & \cdots & M_{n_k}(D) \end{pmatrix}$$

The ring T possesses the classical ring of fractions $\tilde{T} = T$, by proposition 1.3.

Let $1 = e_1 + e_2 + \ldots + e_k + e_{k+1}$ be a decomposition of the identity of A into a sum of orthogonal idempotents such that $e_iAe_i = A_i = H_{n_i}(O_i)$ for $i = 1, 2,\ldots,k$ and $e_{k+1}Ae_{k+1} = A_{k+1} = T(S_1)$. It is easy to show that any regular element of A has the following form

$$r = \begin{pmatrix} r_1 & 0 & \cdots & 0 & x_1 \\ 0 & r_2 & \cdots & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & r_k & x_k \\ 0 & 0 & \cdots & 0 & r_{k+1} \end{pmatrix} \tag{8}$$

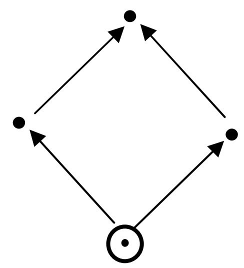
where each element r_i is regular in A_i, $i = 1,\ldots,k+1$, and $x_i \in M_i$ for $i = 1, 2, \ldots,k$. So $C(0) = C(N)$, where N is a prime radical of A. Therefore from [5, theorem 3] and proposition 2.2 it follows that A has a classical ring of fractions $\tilde{A}$ which is an Artinian ring. Taking into account the form of regular elements of A from lemma 2.3 it follows that $\tilde{A}$ has the form (7).

The diagram of a poset $S = \{\alpha_1, \alpha_2,\ldots,\alpha_n\}$ is a quiver $Q(S)$ with a set of vertices $VS = \{1, 2,\ldots,n\}$ and a set of arrows AS which contains an arrow σ_{ij} with the start at the vertex i and the end at the vertex j if and only if $\alpha_i < \alpha_j$ and there is no an element α_k such that $\alpha_i < \alpha_k < \alpha_j$. Now consider the quiver $Q(S)$ with weights, namely, suppose that each vertex $i \in VS$ has the weight corresponding to the weight α_i. Since S is a poset, the quiver $Q(S)$ is acyclic and has no multiplied arrows, i.e. it is an acyclic simply laced quiver. (Recall that a quiver Q is called **acyclic** if it does not contain oriented cycles and it is called **simply laced** if it does not contain multiple arrows and multiple loops). Write a vertex with a weight $H_{n_i}(O_i)$ by $\odot$ and a vertex with a weight D by $\bullet$.

Proposition 2.5. *The ring $A(\mathrm{S},O)$ corresponding to a poset* S *with weights is right hereditary if and only it does not contain the minors of the following forms*:

$$B = \begin{pmatrix} D & D & D & D \\ 0 & D & 0 & D \\ 0 & 0 & D & D \\ 0 & 0 & 0 & D \end{pmatrix} \quad \textit{and} \quad C = \begin{pmatrix} H_n(O) & C_2 & C_3 & C_4 \\ \mathbf{O} & D & 0 & D \\ \mathbf{O} & 0 & D & D \\ \mathbf{O} & 0 & 0 & D \end{pmatrix}$$

where C_i are uniserial left $H_n(O)$-modules and one-dimensional right D-vector spaces for i = 2, 3, 4, *i.e. any subdiagram of a poset* S *contains no rhombuses and diagrams of the following form*:



(9)

Proof. 1. Recall that, by [3, corollary 10.4.14], any finitely generated projective module over a semiperfect ring is a finite direct sum of principal modules. Suppose the ring $A = A(\mathrm{S},O)$ is right hereditary and contains the minor B. Since B is a right Noetherian ring, any submodule of a finite generated right B-module is finitely generated. Since the right B-submodule $(0\ D\ D\ D)$ of a projective right B-module $P_1 = e_{11}B$ is not projective over B, the ring B is not right hereditary. Analogously the right C-module $(0\ C_2\ C_3\ C_4)$ is not projective over C, and therefore the ring C is not right hereditary. Since each minor of a right hereditary ring is right hereditary itself, $A(\mathrm{S},O)$ contains neither B nor C.

2. Conversely, assume the ring $A = A(\mathrm{S},O)$ satisfies the condition of the proposition. For any $i = 1,\ldots,k$ a ring $A_i = H_{n_i}(O_i)$ has the classical ring of fractions $\tilde{A}_i = M_{n_i}(D)$ which is a simple Artinian ring. Obviously, $A_{ij} \cong \tilde{A}_i \otimes_{A_i} A_{ij}$, for any $i,j =1, 2,\ldots,n$. By proposition 2.4, A has the classical ring of fractions $\tilde{A}$ which is an Artinian ring and has the form (2.5). For any $i = 1,\ldots,k$ consider a ring $B_i = \begin{pmatrix} M_{n_i}(D) & \tilde{M}_i \\ \mathbf{O} & T(\mathrm{S}_1) \end{pmatrix}$, which is Morita equivalent to the ring $T_i = \begin{pmatrix} D & X_i \\ \mathbf{O} & T(\mathrm{S}_1) \end{pmatrix}$, where X_i is a right $T(\mathrm{S}_1)$-module and a left one-dimension vector space over D.

Since A does not contain subrings isomorphic B and C, the rings B_i and T_i are hereditary by proposition 1.5, for any $i = 1,\dots,k$. Then from theorem [7, theorem 2] it follows that each $\overline{A}_{ij} = \overline{A}_{ij} / \sum\limits_{i<s<j} A_{is}A_{sj}$ is a projective right D-module and all $\mu^0_{isj} : \overline{A}_{is} \otimes_{A_s} A_{sj} \to \overline{A}_{ij}$ induced by the multiplication in A are monomorphisms for all i, s, $j = 1, 2,\dots,n$.

Since, by proposition 2.1, the ring A is right Noetherian, all conditions of [7, theorem 2] are fulfilled. So A is also a right hereditary ring.

Recall that a module M is called **distributive** if $K\cap(L+N) = K\cap L+K\cap N$ for all submodules K, L, N. A module is called **semidistributive** if it is a direct sum of distributive modules. A ring A is called **right** (**left**) **semidistributive** if the right (left) regular module A_A ($_AA$) is semidistributive. A right and left semidistributive ring is called **semidistributive**. Semiperfect semidistributive rings (SPSD-rings, in short) were first considered by A.A.Tuganbaev in [8] and [9].

Theorem 2.6. *Let* $\{O_i\}$ *be a family of discrete valuation rings with a common division ring of fractions* D, *and let any subdiagram of a poset* S *contain no rhombuses and diagrams of the form* (2.9). *Then* A(S,O) *is a right Noetherian right hereditary* SPSD-*ring*.

Proof. The ring $A = A$(S,O) is a semiperfect right Noetherian ring due to proposition 2.1 and a right hereditary ring due to proposition 2.5. Moreover, for any two primitive idempotents e and f a ring $(e+f)A(e+f)$ has one of the forms: $\begin{pmatrix} O_i & 0 \\ 0 & O_j \end{pmatrix}$, $\begin{pmatrix} O_i & O_i \\ M_i & O_i \end{pmatrix}$, $\begin{pmatrix} O_i & 0 \\ 0 & D \end{pmatrix}$, $\begin{pmatrix} O_i & D \\ 0 & D \end{pmatrix}$, $\begin{pmatrix} D & D \\ 0 & D \end{pmatrix}$, or $\begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix}$, where D is a common division ring of fractions of O_i and O_j. It is obviously, that each of these rings is a semidistributive ring. So A is also semidistributive, by [3, theorem 14.2.1].

Corollary 2.7. *The right classical ring of fractions* $\tilde{A}$ *of the right hereditary ring* A(S,O) *is a right Artinian right hereditary* SPSD-*ring and the prime radical of* A(S,O) *coincides with the Jacobson radical of* $\tilde{A}$.

Corollary 2.8. *The prime quiver of the right hereditary ring* A(S,O) *coincides with the quiver of the right classical ring of fractions* $\tilde{A}$ *and coincides with the diagram of the poset* S.

From the results proved in [10] we obtain the following statement.

Theorem 2.9. *Any indecomposable right hereditary reduced SPSD-ring is exactly the right hereditary ring* A(S,O) *for some finite poset* S *which diagram contains neither rhombuses nor subdiagrams of the form* (9).

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