ON A CERTAIN SPECIAL PROBLEM OF THE QUASI-LINEAR HEAT CONDUCTION IN A PERIODICALLY-LAYERED CONDUCTOR

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Abstract. The aim of contribution is to investigate a quasi-linear heat conduction in the two-phased periodically-spaced multilayered rigid conductor. The analysis is based on the tolerance averaging technique. Considerations are restricted to the heat conduction in the direction normal to the interfaces.

1. Model equations

This is a continuation of earlier contribution [1] on this subject. The starting point of considerations is the following system of equations

$$\begin{aligned} &\partial_1\left(\langle k\rangle\partial_1\vartheta+\langle k\partial_1 h\rangle\psi\right)+\partial_\alpha\langle k\rangle\partial_\alpha\vartheta-\langle\rho c\rangle\dot{\vartheta}=0 \\ &\lambda^2\partial_\alpha\langle k\rangle\partial_\alpha\psi-\langle k\partial_1 h\rangle\partial_1\vartheta-\left\langle k\left(\partial_1 h\right)^2\right\rangle\psi-\lambda^2\langle\rho c\rangle\dot{\psi}=0 \end{aligned} \tag{1}$$

where $k(\cdot)$ is a heat conduction coefficient, $h(\cdot)$ is a fluctuation shape function, λ is a laminae thickness, c and ρ are specific heat and mass density, respectively. The basic unknowns are: averaged temperature $\vartheta=\vartheta(\mathbf{x},t)$ and temperature fluctuation amplitude $\psi=\psi(\mathbf{x},t)$, $\mathbf{x}\in\langle -L,L\rangle\times\langle 0,H_2\rangle\times\langle 0,H_3\rangle$, $t\in\langle 0,\infty)$. We use denotations $\mathbf{x}=(x_1,x_2,x_3)$, $\partial_1=\partial/\partial x_1$, $\partial_\alpha=\partial/\partial x_\alpha$, $\alpha=2,3$.

We recall, [1], that k is given by formula

$$k(x_1,\theta)=k_0(x_1)\left[1+\delta\left(\vartheta+h(x_1)\psi\right)\right] \tag{2}$$

where θ is the temperature field, δ is a positive constant and $k_0(x_1)$ takes the constant values k_0', k_0'' in every component material, $k_0'>0$, $k_0''>0$.

Subsequently we restrict considerations to the stationary heat conduction problems. It means that in the system of equations (1) for averaged temperature field ϑ

and temperature fluctuation amplitude ψ we have $\dot{\vartheta}=\dot{\psi}\equiv 0$. Hence, these equations take the form

$$\begin{aligned}&\partial_1\left(\langle k\rangle\partial_1\vartheta+\langle k\partial_1 h\rangle\psi\right)+\partial_\alpha\langle k\rangle\partial_\alpha\vartheta=0\\&\lambda^2\partial_\alpha\langle k\rangle\partial_\alpha\psi-\langle k\partial_1 h\rangle\partial_1\vartheta-\left\langle k\left(\partial_1 h\right)^2\right\rangle\psi=0\end{aligned}\tag{3}$$

Denoting $\langle k\partial_1 h\rangle=[k]$ and $\left\langle k\left(\partial_1 h\right)^2\right\rangle=\{k\}$, we can rewrite (3) in the form

$$\begin{aligned}&\partial_1\left(\langle k\rangle\partial_1\vartheta+[k]\psi\right)+\partial_\alpha\langle k\rangle\partial_\alpha\vartheta=0\\&\lambda^2\partial_\alpha\langle k\rangle\partial_\alpha\psi-[k]\partial_1\vartheta-\{k\}\psi=0\end{aligned}\tag{4}$$

Setting $\nu'=\lambda'/\lambda$, $\nu''=\lambda''/\lambda$ and bearing in mind (2) we obtain

$$\begin{aligned}&\langle k\rangle=\langle k_0\rangle(1+\delta\vartheta)=\left(\nu' k_0'+\nu'' k_0''\right)(1+\delta\vartheta)\\&[k]=[k_0](1+\delta\vartheta)=2\sqrt{3}\left(k_0''-k_0'\right)(1+\delta\vartheta)\\&\{k\}=\{k_0\}(1+\delta\vartheta)=12\left(\frac{k_0'}{\nu'}+\frac{k_0''}{\nu''}\right)(1+\delta\vartheta)\end{aligned}\tag{5}$$

It can be observed that coefficients $\langle k\rangle$, $[k]$, $\{k\}$ are independent on temperature fluctuation amplitude ψ.

Let us transform equations (4) by introducing dimensionless arguments $\xi_1=x_1/L$, $\xi_\alpha=x_\alpha/H_\alpha$, $\alpha=2,3$. Denoting $\sigma^2\equiv\lambda^2/L^2$ we obtain from (4) the following system of equations for the averaged temperature ϑ and the fluctuation amplitude ψ

$$\begin{aligned}&\frac{\partial}{\partial\xi_1}\left(\langle k\rangle\frac{\partial\vartheta}{\partial\xi_1}+[k]\psi\right)+\frac{\partial}{\partial\xi_\alpha}\left(\langle k\rangle\frac{\partial}{\partial\xi_\alpha}\vartheta\right)=0\\&\sigma^2\frac{\partial}{\partial\xi_\alpha}\left(\langle k\rangle\frac{\partial\psi}{\partial\xi_\alpha}\right)-\{k\}\psi-[k]\frac{\partial}{\partial\xi_1}\vartheta=0\end{aligned}\tag{6}$$

The above equations together with formulae (5) constitute the starting point for the subsequent analysis. The characteristic feature of model equations (6) is that the quasi-linearity is imposed only on averaged temperature ϑ and the problem is linear with respect to the temperature fluctuation amplitude ψ.

2. Heat conduction across laminae

Let us assume that $\vartheta = \vartheta(\xi_1)$, $\psi = \psi(\xi_1)$ and $\xi_1 \in \langle 0,1 \rangle$. It means that we shall deal with the heat conduction in the direction normal to the interfaces between adjacent laminae. Denoting

$$
\begin{aligned}
k^h &= \langle k \rangle - \frac{[k]^2}{\{k\}} = \frac{k'k''}{\nu'k'' + \nu''k'} = k^0 (1 + \delta\vartheta) \\
k^0 &= \frac{k_0' k_0''}{\nu' k_0'' + \nu'' k_0'},
\end{aligned}
\tag{7}
$$

we obtain from (6) the following system of equations

$$
\begin{aligned}
&\partial_1 \left(k^0 (1 + \delta\vartheta) \partial_1 \vartheta \right) = 0. \\
&\psi = -\frac{[k]}{\{k\}} \partial_1 \vartheta
\end{aligned}
\tag{8}
$$

The general solution to equation $(8)_1$ has the form

$$
\vartheta + \frac{\delta}{2} \vartheta^2 = C\xi_1 + D \tag{9}
$$

where C and D are constants. Assuming boundary conditions $\vartheta(0) = \vartheta_0$ and $\vartheta(1) = \vartheta_1$ we obtain

$$
\begin{aligned}
C &= \vartheta_1 - \vartheta_0 + \frac{\delta}{2} \left[\vartheta_1^2 - \vartheta_0^2 \right] \\
D &= \vartheta_0 + \frac{\delta}{2} \vartheta_0^2
\end{aligned}
\tag{10}
$$

It has to be underlined that $\vartheta(\xi_1) \geq 0$ for every $\xi_1 \in \langle 0,1 \rangle$, so that $C\xi_1 + D \geq 0$.

The boundary value problem for ϑ depends on parameter δ. We shall deal with three following solutions to equation $(8)_1$

$$
\begin{aligned}
&1^\circ \text{ If } \delta = 0 \text{ then } \vartheta = (\vartheta_1 - \vartheta_0)\xi_1 + \vartheta_0 \\
&2^\circ \text{ If } \delta > 0 \text{ then } \vartheta = \frac{1}{\delta}\left(-1 + \sqrt{1 + 2\delta (C\xi_1 + D)} \right) \\
&3^\circ \text{ If } \delta \to \infty \text{ then } \vartheta \to \sqrt{\left[\vartheta_1^2 - \vartheta_0^2 \right] \xi_1 + \vartheta_0^2}
\end{aligned}
\tag{11}
$$

It can be seen that the effect of quasi-linearity of equation $(8)_1$ results in increasing of the values of the averaged temperature ϑ while $\delta \to \infty$, cf. Figure 1.

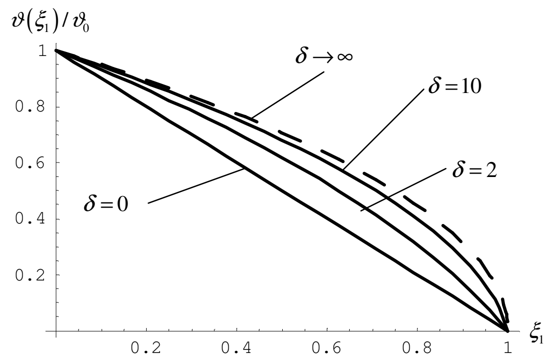

Fig. 1. The effect of quasi-linearity on the averaged temperature of the heat conduction across laminae

3. Final remarks

Let us remind that in the problem under consideration the quasi-linearity of the heat conduction equations reduces to the averaged temperature field ϑ while the problem is linear with respect to the temperature fluctuation amplitude ψ.

The conclusion is that the quasi-linearity of the heat conduction in a laminated medium increases values of the averaged temperature inside the conductor.

More detailed analysis of the problem under consideration can be found in [1].

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References

[1] Woźniak C., Siedlecka U., On a certain quasi-linear heat conduction problems in a two-phased laminated medium, Scientific Research of the Institute of Mathematics 2008, 1(7) 181-184.

[2] Woźniak C. at al (eds.), Mathematical modeling and analysis in continuum mechanics of microstructured media, Wydawnictwo Politechniki Śląskiej, Gliwice, Poland, Part II, Section 7 (in the course of publication).

[3] Woźniak C., Michalak B., Jędrysiak J. (eds.), Thermomechanics of microheterogeneous solids and structures, Tolerance averaging approach, Wydawnictwo Politechniki Łódzkiej, Łódź 2008.