THE DISTINGUISHING FEATURES OF THE FUNDAMENTAL SOLUTION TO THE DIFFUSION-WAVE EQUATION

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Abstract. The time-fractional diffusion-wave equation with the Caputo derivative is considered. The typical features of the solution to the Cauchy problem for this equation are discussed depending on values of the order of fractional derivative.

Introduction

In recent years an increasing interest has been observed in modeling anomalous diffusion processes. Such an interest is connected with describing important physical phenomena in amorphous, colloid, glassy and porous materials, in dielectric and semiconductors, comb structures, fractals and percolation clusters, polymers, in geophysical and geological processes, in biology and medicine.

The model of anomalous transport based on fractional calculus is the subject of a considerable literature (see [1-4], among others). In this model the partial derivatives with respect to time and space in classical diffusion equation are replaced by derivatives of non-integer order. Essentials of fractional calculus can be found in [5-7]. The fundamental solution for the time-fractional diffusion-wave equation in one space-dimension was obtained by Mainardi [8]. Wyss [9] obtained the solution of the Cauchy and signaling problems in terms of *H*-functions using the Mellin transform. Schneider and Wyss [10] converted the diffusion-wave equation with appropriate initial conditions into the integrodifferential equation and found the corresponding Green functions in terms of Fox functions. Fujita [11] treated integrodifferential equation which interpolates the diffusion equation and the wave equation and exhibit properties peculiar to both these equations.

In the present paper, on the bases of numerical analysis, we examined the properties of the fundamental solution to the Cauchy problem for time-fractional diffusion-wave equation in one space-dimension. This treatment continuates the study of Mainardi [8]. The distinguishing features of the solution are discussed depending on values of the order of fractional derivative including the transition from the diffusion equation to the wave equation.

1. Statement of the problem

Consider the time-fractional diffusion-wave equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $0 < t < \infty$, $-\infty < x < \infty$, $0 < \alpha \le 2$, $\partial^\alpha u / \partial t^\alpha$ is the Caputo partial derivative of the fractional order α:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(\tau)}{\partial \tau^n} d\tau, \qquad n-1 < \alpha < n \tag{2}$$

where $\Gamma(x)$ is the gamma function. The Caputo fractional derivative has the following Laplace transform rule

$$L\left[\frac{d^\alpha u(t)}{dt^\alpha}\right] = s^\alpha L[u(t)] - \sum_{k=0}^{n-1} u^{(k)}(0^+) s^{\alpha-1-k}, \qquad n-1<\alpha<n \tag{3}$$

with s being the transform variable.

The Cauchy problem with the Dirac delta initial condition

$$t=0: \ u = p\delta(x), \qquad 0<\alpha\le 2 \tag{4}$$

is studied. In the case $1 < \alpha \le 2$ we also adopt the zero initial condition for the time-derivative of a function

$$t=0: \ \frac{\partial u}{\partial t} = 0, \qquad 1<\alpha\le 2 \tag{5}$$

The initial-value problem (1), (4), (5) was solved by Mainardi using the Laplace transform with respect to time. The solution reads

$$u = \frac{p}{2\sqrt{a} t^{\alpha/2}} M\left(\frac{|x|}{\sqrt{a} t^{\alpha/2}}; \frac{\alpha}{2}\right) \tag{6}$$

where $M(z;\beta)$ is the Mainardi function having the series representation

$$M(z;\beta) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma[-\beta n + (1-\beta)]}, \qquad 0<\beta<1 \tag{7}$$

and being a particular case of the Wright function

$$M(z;\beta) = W(-z;-\beta,1-\beta) \tag{8}$$

where the Wright function $W(z;\varphi,\psi)$ is defined by [12]

$$W(z;\varphi,\psi) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\varphi n+\psi)}, \qquad \varphi > -1, \quad \psi > 0 \tag{9}$$

We obtain the solution of the problem considered by another way using the Laplace transform with respect to time t and the exponential Fourier transform with respect to the spatial coordinate x. Denoting the transform by the bar and taking into account the Laplace transform rule (3), we get

$$\overline{u} = p\frac{s^{\alpha-1}}{s^{\alpha} + a\xi^2} \tag{10}$$

We recall that s is the Laplace transform variable, ξ is the Fourier transform variable. The inverse transforms result in

$$u = \frac{p}{2\pi}\int_{-\infty}^{\infty} E_{\alpha}\left(-a\xi^2 t^{\alpha}\right)\cos(x\xi)\,d\xi \tag{11}$$

where $E_{\alpha}(z)$ is the Mittag-Leffler function [6, 12]

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}, \qquad \alpha > 0 \tag{12}$$

For α = 1, the Mittag-Leffler function turns into the exponential function, $E_1(z) = e^z$, and taking into account the following integral

$$\int_0^{\infty} \exp\left(-ax^2\right)\cos(bx)\,dx = \sqrt{\frac{\pi}{4a}}\exp\left(-\frac{b^2}{4a}\right) \tag{13}$$

we obtain the well-known fundamental solution of the diffusion equation

$$u = \frac{p}{2\sqrt{\pi a t}}\exp\left(-\frac{x^2}{4at}\right) \tag{14}$$

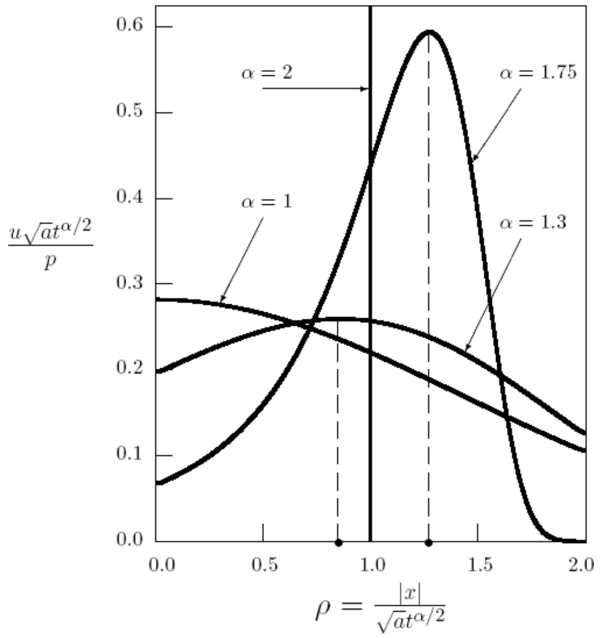

Fig. 1. Dependence of solution on the similarity variable.

For $\alpha = 2$, the Mittag-Leffler function of the negative argument turns into the cosine function $E_2\left(-x^2\right) = \cos x$, and we obtain the fundamental solution of the wave equation

$$u = \frac{p}{2}\left[\delta\left(x + \sqrt{a}t\right) + \delta\left(x - \sqrt{a}t\right)\right] \tag{15}$$

where the following integral

$$\int_0^\infty \cos(\lambda x)dx = \pi\,\delta(\lambda) \tag{16}$$

has been used.

Typical solutions (11), (14), (15) are shown in Figure 1. The vertical line represents the Dirac delta function.

2. Analysis of the solution

Fujita [11] treated integrodifferential equation, which is equivalent to the considered problem for $1 \le \alpha \le 2$ and interpolates the diffusion equation $(\alpha = 1)$ and the wave equation ($\alpha = 2$). Fujita showed that the fundamental solution to this problem in the case $1 < \alpha < 2$ takes its maximum at $x_* = \pm c_\alpha t^{\alpha/2}$ for each $t > 0$. Here c_α is the constant depending on α. Therefore, the points, where the fundamental solution takes its maximum, propagate with finite speed. We study the solution (11) using classical methods of mathematical analysis.

To find the points x_*, where the fundamental solution (11) in the case $1 \le \alpha \le 2$ takes its maximum, we differentiate (11) with respect to x and put to zero. Thus, we obtain

$$\frac{1}{\pi}\int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)\sin\left(x_*\xi\right)\xi\, d\xi = 0 \tag{17}$$

Next, we differentiate (17) with respect to time:

$$\begin{gathered} -a\alpha\, t^{\alpha-1}\int_0^\infty E_\alpha'\left(-a\xi^2 t^\alpha\right)\, \sin\left(x_*\xi\right)\xi^3\, d\xi + \\ +\frac{dx_*}{dt}\int_0^\infty E_\alpha\left(-a\xi^2 t^\alpha\right)\, \cos\left(x_*\xi\right)\xi^2\, d\xi = 0. \end{gathered} \tag{18}$$

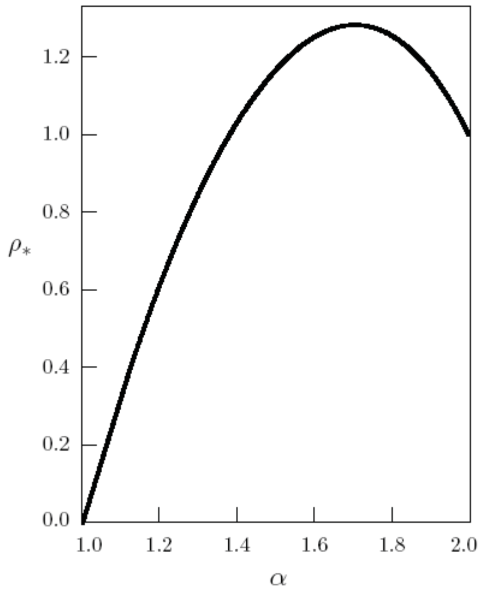

Fig. 2. Dependence of the point, where the fundamental solution takes its maximum, on the order of fractional derivative

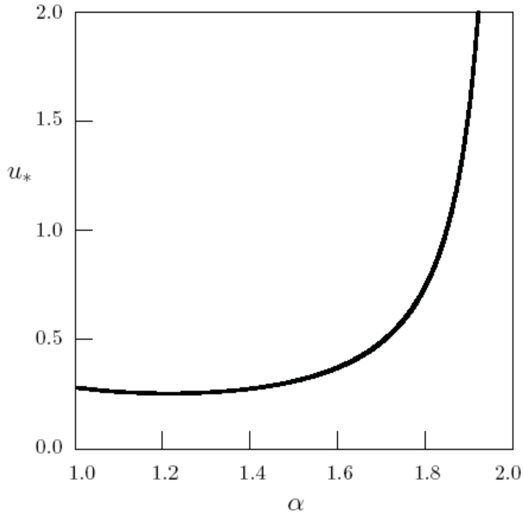

Fig. 3. Dependence of nondimensional maximum value of solution on the order of fractional derivative

Taking into account that

$$E_\alpha^{'}\left(-a\xi^2 t^\alpha\right) = -\frac{1}{2a\xi\, t^\alpha}\frac{d}{d\xi}E_\alpha\left(-a\xi^2 t^\alpha\right) \tag{19}$$

and integrating the first integral in (18) by parts, we get

$$\frac{dx_*}{dt} = \alpha\,\frac{x_*}{2t} \tag{20}$$

or after integration

$$x_* = \pm c_\alpha\, t^{\alpha/2} \tag{21}$$

where c_α is a constant of integration.

The speed of propagation of the points, where the fundamental solution takes its maximum, is defined as $v_\alpha = c_\alpha / \sqrt{a}$ and equals to the value of the similarity variable ρ_*, where the the solution takes its maximum ($v_\alpha = \rho_*$). In Figure 1 these points are marked by the dark dots. Dependence of ρ_* (or v_α) on the order of fractional derivative α is depicted in Figure 2, whereas Figure 3 shows the dependence of the maximum value of solution on α.

Now we compare the shape of the curves describing the solution for two different times. Introducing the similarity variable as $|x|/\left(\sqrt{a}\, t_0^{\alpha/2}\right)$, we present the solutions for $t/t_0 = 1$ (the firm-line curves in Figure 4) and $t/t_0 = 1.25$ (the dashed-line curves in Figure 4).

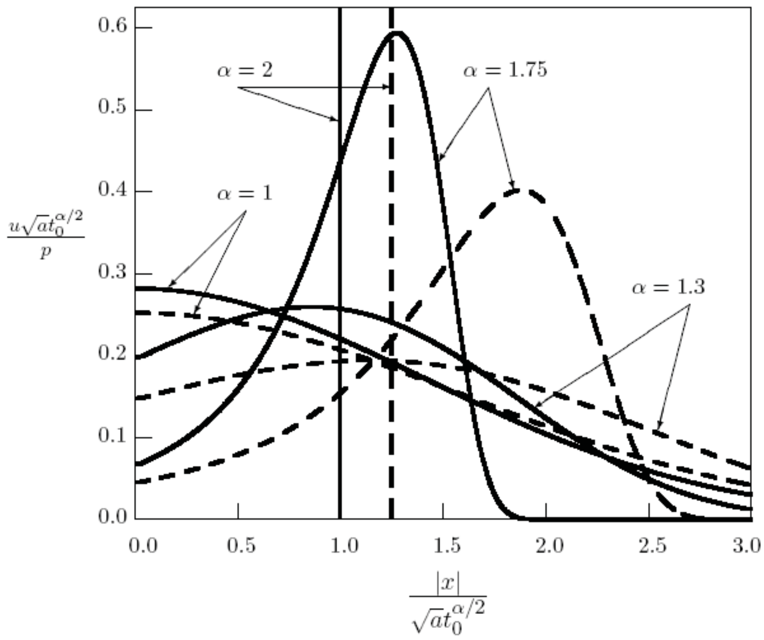

Fig. 4. Dependence of solutions on the similarity variable for different times

Conclusions

The classical theory of diffusion (or heat conduction) predicts that the effects of a disturbance will be felt instantaneously at distances infinitely far from its source. This limitation of the theory follows from the fact that the classical diffusion equation is a parabolic-type equation. The hyperbolic-type transport equation allows finite wave speed for signals. This phenomenon is known as second sound. The ballistic diffusion equation [13, 14] corresponding to $\alpha = 2$ permits propagation of waves at finite speed. Nethertheless, in the case of time-fractional diffusion equation with $1 < \alpha < 2$ the propagating peaks approximating delta function are also exhibited. The solutions for $1 < \alpha < 2$ have peaks propagating with constant speed (a feature typical for wave equation), but the strength of these peaks is decreasing with time (dissipation typical for diffusion equation). From the analysis presented above we can conclude that for $1 < \alpha < 1.4$ the solution has the features more closely resembling those of the diffusion equation, for $1.7 < \alpha < 2$ the solution resembles the solution to the wave equation, the intermediate behavior corresponds to the values $1.4 < \alpha < 1.7$.

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