IDENTIFICATION OF BOUNDARY TEMPERATURE USING THE ENERGY MINIMIZATION METHOD COUPLED WITH THE 2ND SCHEME OF BEM

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Abstract. The inverse problem consisting in identification of temperature on the part of boundary limiting the domain considered is discussed. In order to solve the problem the energy minimization method coupled with the 2nd scheme of boundary element method is used. The possible disturbances of 'measured' temperatures have been taken into account. The theoretical considerations are supplemented by the examples of computations verifying the correctness of the algorithm proposed.

1. Formulation of the problem

The 2D problem analyzed in this paper could be written in following form [1, 2]

$$\begin{cases} x \in \Omega: & c\dfrac{\partial T}{\partial t} = \lambda\left(\dfrac{\partial^2 T}{\partial x_1^2} + \dfrac{\partial^2 T}{\partial x_2^2}\right) \\ x \in \Gamma_1: & T(x) = ? \\ x \in \Gamma_2: & T(x) = T_b \\ x \in \Gamma_3: & q(x) = -\lambda\dfrac{\partial T}{\partial n} = q_b \\ x \in \Gamma_4: & q(x) = -\lambda\dfrac{\partial T}{\partial n} = \alpha\left[T(x) - T^{\infty}\right] \\ \xi^i \in \Omega: & T_d(\xi^i) - \text{known},\ i=1 \ldots M \end{cases} \tag{1}$$

where λ [W/(mK)] is thermal conductivity, c [J/(m^3 K)] is the specific heat per unit of volume, α is the heat transfer coefficient [W/(m^{2}K)], T denotes temperature, T_b, q_b are the given boundary temperature and heat flux, T^{∞} is the ambient temperature, $T_d(\xi^i)$ are the known temperatures at the internal points ξ^i in the domain considered.

The aim of investigations is to determine the boundary temperature on Γ_1.

2. Method of solution

In numerical realization, the boundary Γ is divided into N constant boundary elements Γ_j [3, 4]. Additionally, we assume that N_1 nodes belong to the boundary Γ_1, the nodes $N_1+1,\ldots, N_2$ belong to Γ_2, $N_2+1,\ldots, N_3$ belong to Γ_3 and $N_3+1,\ldots, N$ belong to Γ_4 (Fig. 1).

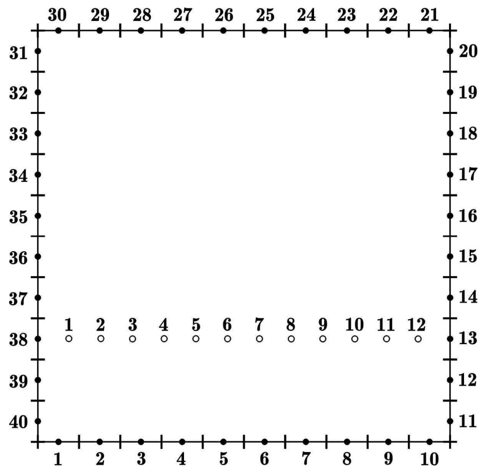

Fig. 1. Discretization and position of internal points

The boundary integral equation for problem (1) using 2nd scheme of BEM is of the form [3]:

$$
\begin{aligned}
&B(\xi)T(\xi,t^f)+\int_\Gamma q(x,t^f)g(\xi,x)\,\mathrm{d}\Gamma=\int_\Gamma T(x,t^f)h(\xi,x)\,\mathrm{d}\Gamma+\\
&+\sum_{s=1}^{f-1}\left[\int_\Gamma T(x,t^s)h^s(\xi,x)\,\mathrm{d}\Gamma-\int_\Gamma q(x,t^s)g^s(\xi,x)\,\mathrm{d}\Gamma\right]
\end{aligned}
\tag{2}
$$

The integrals in equation (2) are substituted by sum of integrals and then for constant boundary elements, taking into account the boundary conditions one obtains

$$
\begin{aligned}
&\sum_{j=1}^{N_1}G_{ij}q_j^f+\sum_{j=N_1+1}^{N_2}G_{ij}q_j^f+\sum_{j=N_2+1}^{N_3}G_{ij}q_b+\sum_{j=N_3+1}^{N}G_{ij}\alpha\left(T_j^f-T^\infty\right)=\\
&=\sum_{j=1}^{N_1}H_{ij}T_j^f+\sum_{j=N_1+1}^{N_2}H_{ij}T_b+\sum_{j=N_2+1}^{N_3}H_{ij}T_j^f+\sum_{j=N_3+1}^{N}H_{ij}T_j^f+\\
&+\sum_{s=1}^{f-1}\left(\sum_{j=1}^{N}H_{ij}^sT_j^s-\sum_{j=1}^{N}G_{ij}^sq_j^s\right)
\end{aligned}
\tag{3}
$$

where:

$$\xi^i \in \Gamma: \quad G_{ij} = \int_{\Gamma_j} T^*(\xi^i, x, t^f, t)\,\mathrm{d}\Gamma_j, \quad H_{ij} = \begin{cases} \int_{\Gamma_j} q^*(\xi^i, x, t^f, t)\,\mathrm{d}\Gamma_j, & i \neq j \\ -0.5, & i = j \end{cases} \tag{4}$$

while

$$\xi^i \in \Gamma: \quad G_{ij}^s = \int_{\Gamma_j} T^*(\xi^i, x, t^s, t)\,\mathrm{d}\Gamma_j, \quad H_{ij}^s = \begin{cases} \int_{\Gamma_j} q^*(\xi^i, x, t^s, t)\,\mathrm{d}\Gamma_j, & i \neq j \\ -0.5, & i = j \end{cases} \tag{5}$$

In (4) and (5) T^* and q^* are fundamental solution and heat flux resulting from fundamental solution, respectively, $B(\xi) \in (0, 1)$ [3, 4].
The well ordered system of equations has the form

$$\mathbf{A}_1 \mathbf{Y}^f = \mathbf{A}_2 \mathbf{P} + \mathbf{M}^f \tag{6}$$

where:

$$\mathbf{A}_1 = \begin{bmatrix} G_{11} & \dots & G_{1N_2} & -H_{1N_2+1} & \dots & -H_{1N_3} & (\alpha G_{1N_3+1} - H_{1N_3+1}) & \dots & (\alpha G_{1N} - H_{1N}) \\ & \dots & & & \dots & & & \dots & \\ G_{N1} & \dots & G_{NN_2} & -H_{NN_2+1} & \dots & -H_{NN_3} & (\alpha G_{NN_3+1} - H_{NN_3+1}) & \dots & (\alpha G_{NN} - H_{NN}) \end{bmatrix} \tag{7}$$

$$\mathbf{Y}^f = \begin{bmatrix} q_1^f & \dots & q_{N_1}^f & q_{N_1+1}^f & \dots & q_{N_2}^f & T_{N_2+1}^f & \dots & T_{N_3}^f & T_{N_3+1}^f & \dots & T_N^f \end{bmatrix}^{\mathrm{T}} \tag{8}$$

$$\mathbf{A}_2 = \begin{bmatrix} H_{11} & \dots & H_{1N_2} & -G_{1N_2+1} & \dots & -G_{1N_3} & \alpha G_{1N_3+1} & \dots & \alpha G_{1N} \\ & \dots & & & \dots & & & \dots & \\ H_{N1} & \dots & H_{NN_2} & -G_{NN_2+1} & \dots & -G_{NN_3} & \alpha G_{NN_3+1} & \dots & \alpha G_{NN} \end{bmatrix} \tag{9}$$

$$\mathbf{P} = \begin{bmatrix} T_1 & \dots & T_{N_1} & T_b & \dots & T_b & q_b & \dots & q_b & T^\infty & \dots & T^\infty \end{bmatrix}^{\mathrm{T}} \tag{10}$$

$$\mathbf{M}^f = \sum_{s=1}^{f-1} M_i^s = \sum_{s=1}^{f-1} \left(\sum_{j=1}^{N} H_{ij}^s T_j^s - \sum_{j=1}^{N} G_{ij}^s q_j^s \right) \tag{11}$$

The temperatures at internal nodes ξ^i are calculated using the formula [3, 4]

$$T^f(\xi^i)=\sum_{j=1}^{N_1}H_{ij}^wT_j^f+\sum_{j=N_1+1}^{N_2}H_{ij}^wP_j+\sum_{j=N_2+1}^{N_3}H_{ij}^wT_j^f+\sum_{j=N_3+1}^{N}H_{ij}^wT_j^f-$$
$$-\sum_{j=1}^{N_1}G_{ij}^wq_j^f-\sum_{j=N_1+1}^{N_2}G_{ij}^wq_j^f-\sum_{j=N_2+1}^{N_3}G_{ij}^wP_j-\sum_{j=N_3+1}^{N}G_{ij}^w\alpha\left(T_j^f-P_j\right)+ \tag{12}$$
$$+\sum_{s=1}^{f-1}\left(\sum_{j=1}^{N}H_{ij}^{ws}T_j^s-\sum_{j=1}^{N}G_{ij}^{ws}q_j^s\right)$$

Equation (6) can be written in the form

$$\mathbf{Y}^f=\mathbf{A}_1^{-1}\mathbf{A}_2\mathbf{P}+\mathbf{A}_1^{-1}\mathbf{M}^f \tag{13}$$

or

$$\mathbf{Y}^f=\mathbf{UP}+\mathbf{BM}^f \tag{14}$$

From the system of equation (14) results that

$$q_j^f=\sum_{k=1}^{N_1}U_{jk}T_k+\sum_{k=N_1+1}^{N}U_{jk}P_k+\sum_{k=1}^{N}B_{jk}M_k^f,\quad j=1...N_2 \tag{15}$$

$$T_j^f=\sum_{k=1}^{N_1}U_{jk}T_k+\sum_{k=N_1+1}^{N}U_{jk}P_k+\sum_{k=1}^{N}B_{jk}M_k^f,\quad j=N_2+1...N \tag{16}$$

$$q_j^f=\alpha\left(T_j^f-T^\infty\right)=\alpha\left(T_j^f-P_j\right),\quad j=N_3+1...N \tag{17}$$

Putting (15), (16) and (17) into (12) one has [1, 2]

$$T^f(\xi^i)=\sum_{j=1}^{N_1}\left[H_{ij}^w-\sum_{k=1}^{N_2}G_{ik}^wU_{kj}+\sum_{k=N_2+1}^{N_3}H_{ik}^wU_{kj}+\sum_{k=N_3+1}^{N}\left(H_{ik}^w-\alpha G_{ik}^w\right)U_{kj}\right]T_j+$$
$$+\sum_{j=N_1+1}^{N}\left[-\sum_{k=1}^{N_2}G_{ik}^wU_{kj}+\sum_{k=N_2+1}^{N_3}H_{ik}^wU_{kj}+\sum_{k=N_3+1}^{N}\left(H_{ik}^w-\alpha G_{ik}^w\right)U_{kj}\right]P_j+$$
$$+\sum_{j=N_1+1}^{N_2}H_{ij}^wP_j-\sum_{j=N_2+1}^{N_3}G_{ij}^wP_j+\sum_{j=N_3+1}^{N}\alpha G_{ij}^wP_j+ \tag{18}$$
$$+\sum_{j=1}^{N}\left[-\sum_{k=1}^{N_2}G_{ik}^wB_{kj}+\sum_{k=N_2+1}^{N_3}H_{ik}^wB_{kj}+\sum_{k=N_3+1}^{N}\left(H_{ik}^w-\alpha G_{ik}^w\right)B_{kj}\right]M_j^f+$$
$$+\sum_{s=1}^{f-1}\left(\sum_{j=1}^{N}H_{ij}^{ws}T_j^s-\sum_{j=1}^{N}G_{ij}^{ws}q_j^s\right)$$

We introduce following denotations:

$$D_{ij} = -\sum_{k=1}^{N_2} G_{ik}^{w} U_{kj} + \sum_{k=N_2+1}^{N_3} H_{ik}^{w} U_{kj} + \sum_{k=N_3+1}^{N} \left(H_{ik}^{w} - \alpha G_{ik}^{w} \right) U_{kj} \tag{19}$$

$$W_{ij} = H_{ij}^{w} + D_{ij} \tag{20}$$

$$E_i = \sum_{j=N_1+1}^{N_2} H_{ij}^{w} P_j - \sum_{j=N_2+1}^{N_3} G_{ij}^{w} P_j + \sum_{j=N_3+1}^{N} \alpha G_{ij}^{w} P_j \tag{21}$$

$$R_{ij} = -\sum_{k=1}^{N_2} G_{ik}^{w} B_{kj} + \sum_{k=N_2+1}^{N_3} H_{ik}^{w} B_{kj} + \sum_{k=N_3+1}^{N} \left(H_{ik}^{w} - \alpha G_{ik}^{w} \right) B_{kj} \tag{22}$$

$$V_i = \sum_{s=1}^{f-1} \left(\sum_{j=1}^{N} H_{ij}^{ws} T_j^{s} - \sum_{j=1}^{N} G_{ij}^{ws} q_j^{s} \right) \tag{23}$$

$$Z_i = \sum_{j=N_1+1}^{N} D_{ij} P_j + \sum_{j=1}^{N} R_{ij} M_j^{f} + E_i + V_i \tag{24}$$

and then the equation (18) can be written in the form [1]

$$T^{f}(\xi^{i}) = \sum_{j=1}^{N_1} W_{ij} T_j^{f} + Z_i \tag{25}$$

In order to solve the problem considered, the energy minimization method is applied which resolves itself into seek of minimum of functional [3, 5]

$$J = -\frac{1}{2\lambda} \int_{\Gamma} T q \, \mathrm{d}\Gamma \tag{26}$$

with the following restrictions

$$\left| T^{f}(\xi^{i}) - T_d(\xi^{i}) \right| \le \varepsilon \tag{27}$$

Because $T^{f}(\xi^{i})$ in equation (27) is expressed by (25) then restrictions take a form

$$\left| \sum_{j=1}^{N_1} W_{ij} T_j^{f} - F(\xi^{i}) \right| \le \varepsilon \tag{28}$$

where

$$F(\xi^{i}) = T_d(\xi^{i}) - Z_i \tag{29}$$

After the discretization, the functional (26) can be expressed as follows

$$J(T_1, T_2, ..., T_{N_1}) = -\frac{1}{2\lambda}\sum_{j=1}^{N}\int_{\Gamma_j} \mathbf{T}^{\mathrm{T}}\mathbf{q}\,\mathrm{d}\Gamma_j \tag{30}$$

where

$$\begin{aligned}
&\mathbf{T}^{\mathrm{T}}\mathbf{q} = \begin{bmatrix} T_1 & \dots & T_{N_1} & T_b & \dots & T_b & T_{N_2+1} & \dots & T_{N_3} & T_{N_3+1} & \dots & T_N \end{bmatrix}^{\mathrm{T}} \cdot \\
&\cdot \begin{bmatrix} q_1 & \dots & q_{N_1} & q_{N_1+1} & \dots & q_{N_2} & q_b & \dots & q_b & q_a & \dots & q_a \end{bmatrix} = \\
&= \begin{bmatrix} T_1 & \dots & T_{N_1} & T_b & \dots & T_b & q_b & \dots & q_b & q_a & \dots & q_a \end{bmatrix} \cdot \\
&\cdot \begin{bmatrix} q_1 & \dots & q_{N_1} & q_{N_1+1} & \dots & q_{N_2} & T_{N_2+1} & \dots & T_{N_3} & T_{N_3+1} & \dots & T_N \end{bmatrix}^{\mathrm{T}} = \\
&= \mathbf{S}^{\mathrm{T}}\mathbf{Y}^f = \mathbf{S}^{\mathrm{T}}\left(\mathbf{UP} + \mathbf{BM}^f\right)
\end{aligned} \tag{31}$$

while q_a is known boundary heat flux on Γ_4.
Finally, the energy minimization method requires the solution of following problem

$$\begin{cases} \min J(T_1, T_2, ..., T_{N_1}) = \min\left[-\mathbf{S}^{\mathrm{T}}\left(\mathbf{UP} + \mathbf{BM}^f\right)\right] \\ \left|\sum_{j=1}^{N_1} W_{ij}T_j - F(\xi^i)\right| \le \varepsilon, \; i = 1, 2, ..., M \end{cases} \tag{32}$$

3. Examples of computations

The square of dimension 0.1 × 0.1 m has been considered. The thermal conductivity $\lambda = 1$ W/mK, specific heat per unit of volume $c = 1$ J/(m^3 K).

The boundary has been divided into 40 constant boundary elements. In order to solve the inverse problem, it is assumed that the values of temperature are known at 12 selected points from interior of the domain (Fig. 1). These values have been obtained from direct problem (1) solution under the assumption that the temperature on the bottom surface equals 50°C.
In computations two different values of parameters ε have been assumed (0.1 and 0.5). Time step: $\Delta t = 0.5$ s.

Table 1

Variants of boundary conditions

	Variant 1	Variant 2
Left side	$T_b = 50$°C	$q_b = 0$
Top side	$T_b = 100$°C	$q_b = 0$
Right side	$T_b = 100$°C	$T_b = 100$°C

Table 2

Solution of inverse problem (Variant 1, ε = 0.1)

Node	0.5 s	10 s	20 s	30 s	40 s	50 s
1	49.968	49.992	49.985	49.980	49.976	49.974
2	49.963	49.982	49.977	49.973	49.971	49.968
3	49.749	49.756	49.755	49.754	49.753	49.752
4	49.596	49.591	49.592	49.594	49.595	49.596
5	50.211	50.202	50.204	50.205	50.206	50.207
6	51.853	51.852	51.851	51.851	51.851	51.850
7	51.710	51.718	51.717	51.716	51.715	51.714
8	45.556	45.558	45.559	45.559	45.559	45.560
9	43.669	43.658	43.659	43.660	43.660	43.661
10	73.412	73.416	73.413	73.411	73.409	73.408
mean	51.56868	51.5724	51.57109	51.5702	51.56952	51.56898

Table 3

Solution of inverse problem (Variant 1, ε = 0.5)

Node	0.5 s	10 s	20 s	30 s	40 s	50 s
1	50.381	50.409	50.401	50.395	50.390	50.386
2	51.182	51.215	51.205	51.198	51.193	51.189
3	51.574	51.603	51.595	51.589	51.584	51.580
4	51.324	51.344	51.338	51.334	51.331	51.329
5	50.169	50.175	50.173	50.172	50.172	50.171
6	47.911	47.904	47.907	47.908	47.909	47.910
7	45.009	44.996	45.000	45.003	45.005	45.007
8	44.474	44.464	44.466	44.468	44.469	44.470
9	54.138	54.137	54.136	54.135	54.135	54.134
10	81.376	81.390	81.385	81.382	81.379	81.377
mean	52.75389	52.7637	52.76055	52.75839	52.75675	52.75543

Table 4

Solution of inverse problem (Variant 2, ε = 0.1)

Node	0.5 s	10 s	20 s	30 s	40 s	50 s
1	53.809	55.408	55.491	55.517	55.675	55.824
2	51.559	53.151	53.347	53.449	53.572	53.677
3	48.125	49.484	49.711	49.840	49.875	49.889
4	44.528	45.477	45.670	45.782	45.719	45.640
5	42.317	42.782	42.900	42.971	42.864	42.748
6	43.104	43.115	43.142	43.160	43.091	43.021
7	48.219	47.892	47.834	47.801	47.826	47.855
8	58.346	57.852	57.732	57.661	57.763	57.874
9	73.206	72.736	72.591	72.502	72.599	72.708
10	90.768	90.496	90.375	90.299	90.312	90.336
mean	55.39814	55.83918	55.8794	55.8982	55.92957	55.9572

Table 5

Solution of inverse problem (Variant 2, ε = 0.5)

Node	0.5 s	10 s	20 s	30 s	40 s	50 s
1	41.595	43.322	43.424	43.460	43.477	43.485
2	42.159	43.843	44.052	44.161	44.233	44.287
3	43.256	44.649	44.882	45.013	45.104	45.174
4	45.323	46.240	46.428	46.538	46.616	46.677
5	48.772	49.148	49.252	49.316	49.363	49.399
6	54.004	53.891	53.900	53.908	53.915	53.921
7	61.297	60.838	60.762	60.719	60.689	60.666
8	70.691	70.086	69.951	69.871	69.814	69.769
9	81.922	81.380	81.224	81.129	81.060	81.006
10	93.857	93.559	93.435	93.356	93.298	93.252
mean	58.28742	58.69547	58.73094	58.74714	58.75694	58.76366

In Table 1 two variants of boundary conditions are presented, while in the Tables 2-5 the results of computations for different values ε and variants 1 and 2 are shown.

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