ON THE MODELLING OF FUNCTIONALLY GRADED LAMINATED STRUCTURES

Jowita Rychlewska, Jolanta Szymczyk, Czesław Woźniak

Institute of Mathematics and Informatics, Czestochowa University of Technology

Abstract. The object of analysis is a functionally graded material (FGM). The aim of this paper is threefold. Firstly, we outline and recall basic technological motivations and physical foundations of FGM concept. Secondly, we propose a certain special kind of FGM, which on the microstructural level are made of very thin laminae (a functionally graded laminate-FGL). Thirdly, a new approach to the formulation of averaged models for the analysis of dynamic problems in FGL is proposed. Example of applications of the proposed models will be discussed separately.

1. Functionally graded materials - review of literature

Functionally graded materials are usually regarded as heterogeneous composite elements having effective (macroscopic) properties varying smoothly in space. These new materials was proposed by Japanese, cf. [1, 2]. It is known that if two dissimilar materials are bonded together, there is a very strong possibility that debonding will occur at some extreme loading conditions. Moreover large deformations at the interface may trigger the initation and propagation of cracks in material. For the overcoming these adverse effects functionally graded materials are employed. Following Ref. [3] we can say that "functionally graded materials are ideal candidates for applications involving severe thermal gradients, ranging from thermal structures in advanced aircraft and aerospace engines to computer circuit boards." However, an exact description of the geometry of graded composite microstructures is usually not possible. We can describe only volume fraction distribution. Hence, following Ref. [4] we can state that "evaluation of thermomechanical response and local stresses in graded materials must rely on analysis of micromechanical models with idealized geometries. While such idealizations may have much in common with those that have been developed for analysis of macroscopically homogeneous composites, there are significant differences between the analytical models for the two classes of materials." The response of FGM can be obtained by applying certain averaged thermoelastic moduli evaluated for an arbitrary but fixed representative volume element. It is important that such representative element is not easily defined for structures with variable phase volume fraction. However a variety of methods, proposed to analyse macroscopically homogeneous composites, are applied to analysis the overall behaviour of FGM. The aforementioned modelling methods formulated in the recent literature make it possible to determine the effective properties at a given point within FGM. Certain basic methods used to estimate properties of FGM are discussed in [5] where the wide list of references is included. A detailed review and description of the full generalization of a new Cartesian-coordinate-based higher order theory for FGM is presented in [3]. This theory is based on volumetric averaging of the various field quantities, together with imposition of boundary and interfacial conditions in an average sense. In Ref. [4] have been performed detailed finite element studies of discrete models and compared the obtain results with these computed from homogenized models in which effective properties were derived by applying the Mori-Tanaka or the self-consistent method. Rule-of-mixture approaches have been used in the series of works, for example in elastic systems in [6]. In Ref. [7] have been estimated elastic moduli and averages of the Hashin-Sthrikman bounds for statistically homogeneous systems.

In the present contribution the object of investigation is a special class of FGM which on the microstructural level are made of two kinds of laminae and will be referred to as the functionally graded laminates (FGL). A fragment of FGL on the macro and micro levels is shown in Figure 1.

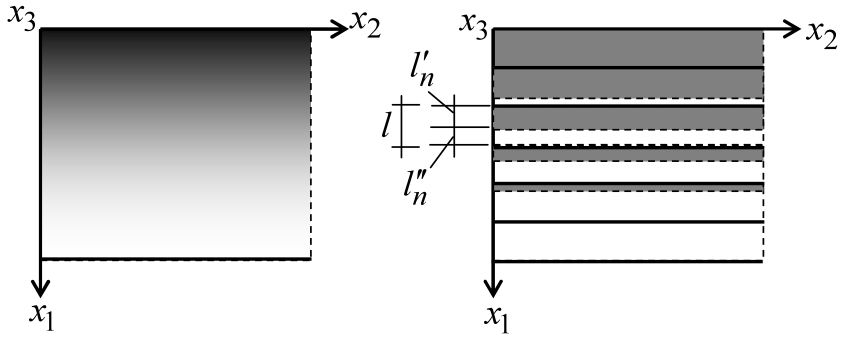

Fig. 1. A fragment of FGL on the macro- and microstructural level

The proposed modelling approach, in contrast to the most computational strategies presently formulated in literature, explicitly couples micro- and macrostructural effects. This technique was used also in the modelling of elastodynamics of functionally graded laminated plates, [8], functionally graded laminated shells, [9], and functionally graded laminates with interlaminar microcracks, [10]. In the subsequent sections we are to outline some of results of the studies on the mathematical modelling of functionally graded materials which are recently carried out at the Institute of Mathematics and Computer Science, [5, 11].

2. Basic notions and concepts

The considerations will be restricted to the dynamic behavior of functionally graded laminates (FGL). These solids are made of a large number of layers with a constant thickness and every layer consists of two homogeneous, linear-elastic laminae. Moreover, thicknesses of laminae in pertinent layers are different so that the macroscopic material characteristics of FGL can be treated as continuously varying in the direction normal to the layering. It is assumed that the pertinent laminae are perfectly bonded. For the sake of simplicity it is also assumed that lamina materials have elastic symmetry planes parallel to the lamina interfaces.

To make our considerations self consistent we recall some of the denotations and basic concepts which are presented in [5, 11].

Small bold-face letters stand for vectors and points in 3-space, large bold-face letters denote second-order tensors and block letters are used for the third- and fourth-order tensors. The scalar and double-scalar products of these objects are denoted by the dot or the double dot between letters. By $0x_1,x_2,x_3$ we denote the orthogonal Cartesian coordinate system in the physical space. Let $\Omega=\left(0,L\right)\times\Pi$, $\Pi\subset R^2$, be the region in this space occupied by FGL in its natural configuration. The Ox_1-axis is normal to the lamina interfaces. We introduce the gradient operators $\nabla=\left(\partial_1,\partial_2,\partial_3\right)$ and $\overline{\nabla}=\left(0,\partial_2,\partial_3\right)$ where ∂_k, k = 1,2,3, stand for the partial derivative with respect to the Cartesian coordinate x_k. The time derivatives are denoted by the overdot.

Let m be the number of layers of FGL solid. The thickness of every layer is the same and denoted by l; hence $L=ml$ is the thickness of FGL, $m^{-1}<<1$. Thicknesses of lamina in the n-th layer, n = 1,..., m, are denoted by l_n', l_n''; it follows that $\nu_n'=l_n'/l$, $\nu_n''=l_n''/l$ are volume fractions of lamina materials in this layer. Mass densities and tensors of elastic moduli in every pair of adjacent laminae will be denoted by ρ', ρ'' and $\mathbb{C}'$, $\mathbb{C}''$, respectively.

By $\nu'\left(\cdot\right)$, $\nu''\left(\cdot\right)=1-\nu'\left(\cdot\right)$ we denote slowly varying functions defined on $\left[0,L\right]$ such that $\nu'\left(0\right)=1$, $\nu'\left(L\right)=0$, these functions represent mean volume fractions of lamina materials. Let $\tilde{\nu}\equiv\sqrt{\nu'\nu''}$; $\tilde{\nu}\left(\cdot\right)$ will be referred to as the phase transition function. Let $g:\left[0,L\right]\to\mathrm{R}$ be a continuous function the diagram of which in an arbitrary interval $\left[\left(n-1\right)l,nl\right]$, n = 1,...,m, is shown in Figure 2.

This function will be referred to as the fluctuation shape function. Now, we introduce certain notions which enable us applying the tolerance averaging technique to the modeling of functionally graded laminate under considerations. Let us denote by $\Delta\left(x_1\right)=\left[x_1-l/2,x_1+l/2\right]$, $x_1\in\left[0,L\right]$ a certain "idealized" (or virtual) cell which consists of two subcells of lengths $\nu'\left(x_1\right)l$, $\nu''\left(x_1\right)l$.

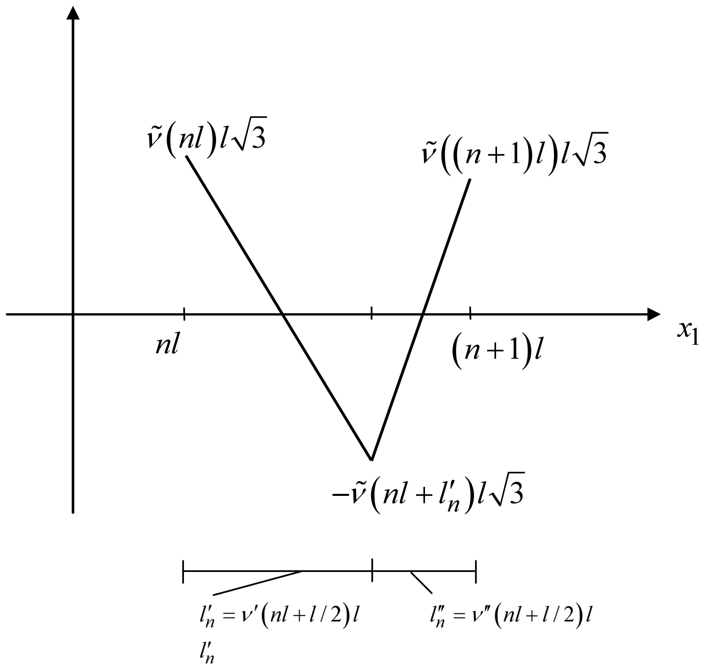

Fig. 2. A diagram of the fluctuation shape function in an arbitrary interval [nl, $(n+1)l$]

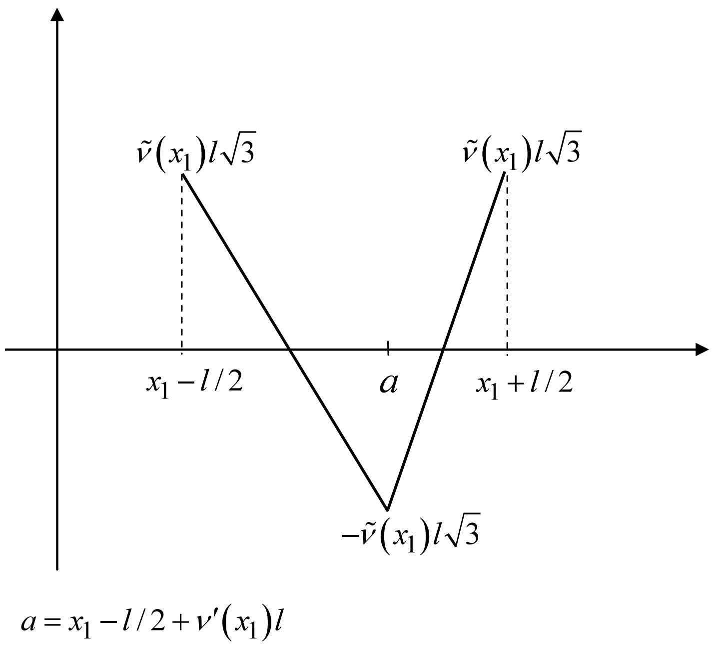

Fig. 3. A diagram of the l-periodic approximation of the fluctuation shape function in the idealized cell

Moreover, let $\mathbb{C}(y),\ \rho(y),\ y\in\Delta(x_1),\ x_1\in[0,L]$ be the functions which attains constant values $\mathbb{C}',\rho'$ and $\mathbb{C}'',\rho''$ in the pertinent subcells. It can be seen that functions $\mathbb{C}(y),\ \rho(y)$ have the physical interpretations provided that $y\in[0,L]$. For an arbitrary integrable function $f(y),\ y\in\Delta(x_1)$ we define

$$\langle f\rangle(x_1)=\frac{1}{l}\int\limits_{x_1-l/2}^{x_1+l/2} f(y)\,dy \tag{1}$$

By $g^*(y)$, $y\in\Delta(x_1)$ we denote l-periodic approximation of the fluctuation shape function $g(\cdot)$. A diagram of $g^*(y)$ for an arbitrary, but fixed $x_1\in(0,L)$ is shown in Figure 3.

3. Modeling technique

In this section we are going to describe a dynamic behavior of the linear-elastic FGL by means of PDEs with smooth functional coefficients. Moreover, the proposed model has to describe this behavior also on the microstructural level by taking into account the effect of lamina thickness on the averaged characteristics of FGL. At last, the model equations should have a relatively simple form which makes it possible to obtain also analytical solution to selected benchmark problems.

The mathematical model of the functionally graded laminate under consideration will be formulated using the tolerance averaging technique [12]. We restrict ourselves to problems in which at every time t the displacement field $\mathbf{w}=\mathbf{w}(\mathbf{x},t)$, $\mathbf{x}=(x_1,x_2,x_3)\in\Omega$ across the thickness of every lamina can be approximated by a linear function of $x^1\in[0,L]$. This is the first kinematic modeling assumption. The second modeling assumption states that displacements in every interval $[x_1,x_1+l],\ x_1\in[0,L-l]$, can be approximated (within a certain tolerance) by l-periodic functions of argument x_1. For the detailed discussion of the above assumptions the reader is referred to [12]. In this case the aforementioned approximations lead to the following decomposition of the displacement field

$$\mathbf{w}(\mathbf{x},t)=\mathbf{u}(\mathbf{x},t)+g(x_1)\,\mathbf{v}(\mathbf{x},t) \tag{2}$$

where $\mathbf{u}$ and $\mathbf{v}$ are slowly-varying functions of argument x_1 (related to the period l and a certain tolerance parameter ε, $0<\varepsilon<<1$). It means that increments of $\mathbf{u}$ and $\mathbf{v}$ in every interval $[x_1,x_1+l],\ x_1\in[0,L-l]$ can be neglected as small when compared to the supreme values of their modulae. Functions **u** and **v** in (1) are basic kinematic unknowns where **u** is the averaged displacement field and **v** is referred to as the fluctuation amplitude field. Bearing in mind (1) we obtain

$$\langle\rho\rangle(x_1)=\nu'(x_1)\rho'+\nu''(x_1)\rho'',$$

$$\langle\mathbb{C}\rangle(x_1)=\nu'(x_1)\mathbb{C}'+\nu''(x_1)\mathbb{C}'',\qquad x_1\in[0,L]$$

Setting $\mathbf{e}=(1,0,0)$ we also denote

$$[\mathbb{C}]=\left\langle\frac{dg^*}{dy}\mathbb{C}\right\rangle\cdot\mathbf{e}=2\sqrt{3}\tilde{\nu}(x_1)(\mathbb{C}''-\mathbb{C}')\cdot\mathbf{e},\ [\mathbb{C}]^T=2\sqrt{3}\mathbf{e}\cdot(\mathbb{C}''-\mathbb{C}')\tilde{\nu}(x_1)$$

$$\{\mathbf{C}\}(x_1)=\mathbf{e}\cdot\left\langle\left(\frac{dg^*}{dy}\right)^2\mathbb{C}\right\rangle\cdot\mathbf{e}=12\mathbf{e}\cdot\left(\nu'(x_1)\mathbb{C}''+\nu''(x_1)\mathbb{C}'\right)\cdot\mathbf{e}$$

Taking into account results derived in [12] it can be shown that the governing equations for **u** and **v** are

$$\begin{aligned}&\langle\rho\rangle\ddot{\mathbf{u}}-\nabla\cdot\left(\langle\mathbb{C}\rangle:\nabla\mathbf{u}+[\mathbb{C}]\cdot\mathbf{v}\right)=\mathbf{0}\\&l^2\tilde{\nu}^2\langle\rho\rangle\ddot{\mathbf{v}}-l^2\tilde{\nu}^2\overline{\nabla}\cdot\left(\langle\mathbb{C}\rangle:\overline{\nabla}\mathbf{v}\right)+\{\mathbf{C}\}\cdot\mathbf{v}+[\mathbb{C}]^T:\nabla\mathbf{u}=\mathbf{0}\end{aligned}\tag{3}$$

The model equations (3) together with formula (2) represent what is called the tolerance model of the functionally graded laminate under considerations. It has to be emphasized that functionally coefficients in (3) are slowly varying functions of argument x_1. Equations (3) have to be considered with initial conditions for **u** and **v**, boundary condition for **u** on $\partial\Omega=((0,L)\times\partial\Pi)\cup(\{0\}\times\Pi)\cup(\{L\}\times\Pi)$ and boundary condition for **v** on $(0,L)\times\partial\Pi.$ The main feature of the above model is the coupling between macro- and micro-mechanics of a laminated medium, represented by the presence of the microstructure length parameter l in the second from equations (3). Equations (2), (3) constitute the basis for the subsequent analysis.

Let us observe that tensor field $\{\mathbf{C}\}$ represents the non-singular linear transformation. Hence from (3), after the formal limit passage $l\to 0,$ we obtain

$$\mathbf{v}=-\{\mathbf{C}\}^{-1}\cdot[\mathbb{C}]^T:\nabla\mathbf{u}\tag{4}$$

where $\{\mathbf{C}\}^{-1}$ is the inverse of $\{\mathbf{C}\}$. Introducing the following fourth-order tensor

$$\mathbb{C}^\circ\equiv\langle\mathbb{C}\rangle-[\mathbb{C}]\cdot\{\mathbf{C}\}^{-1}\cdot[\mathbb{C}]^T\tag{5}$$

and using the first from equations (3) we obtain

$$\langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\mathbb{C}^{\circ} : \nabla \mathbf{u} \right) = 0 \tag{6}$$

It can be shown that $\mathbb{C}^{\circ}$ coincides with the tensor of effective elastic moduli well known in the homogenization theory of elastic laminates. Equations (6) and (4) together with formula (2) represent *the homogenized* (or effective) *model* of the linear-elastic functionally graded laminate under consideration.

Now, we present the equations (3) in the alternative form. To this end, instead of fluctuation amplitude **v**, we introduce a new kinematic unknown $\mathbf{r}(\cdot)$ defined by

$$\mathbf{r} = \{\mathbf{C}\}^{-1} \cdot [\mathbb{C}]^{T} : \nabla \mathbf{u} + \mathbf{v} \tag{7}$$

Let us observe that for the homogenized model we obtain **r** = **0**. That is why **r** will be referred to as the intrinsic fluctuation amplitude, i. e., the amplitude independent of the averaged displacement field **u**. At the same time from (2) and (7) we obtain

$$\mathbf{w}(\mathbf{x},t) = \mathbf{u}(\mathbf{x},t) - g(x_1)\{\mathbb{C}\}^{-1} \cdot [\mathbb{C}]^{T} : \nabla \mathbf{u}(\mathbf{x},t) + g(x_1)\mathbf{r}(\mathbf{x},t)$$

where g**r** will be called the intrinsic fluctuation of displacement. In order to formulate equations for functions $\mathbf{u}(\cdot)$ and $\mathbf{r}(\cdot)$ in the simple form we introduce the following differential operators

$$A\mathbf{u} \equiv \langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\mathbb{C}^{\circ} : \nabla \mathbf{u} \right)$$

$$D\mathbf{r} \equiv l^2 \tilde{v}^2 \left[\langle \rho \rangle \ddot{\mathbf{r}} - \overline{\nabla} \cdot \left(\langle \mathbb{C} \rangle : \overline{\nabla} \mathbf{r} \right) \right] + \{\mathbf{C}\} \cdot \mathbf{r}$$

$$F\mathbf{u} \equiv l^2 \tilde{v}^2 \left[\langle \rho \rangle \{\mathbb{C}\}^{-1} \cdot [\mathbb{C}]^{T} : \nabla \ddot{\mathbf{u}} - \overline{\nabla} \cdot \left(\langle \mathbb{C} \rangle : \overline{\nabla} \cdot \left(\{\mathbb{C}\}^{-1} \cdot [\mathbb{C}]^{T} : \nabla \mathbf{u} \right) \right) \right]$$

Combining equations (3) with formula (7) we obtain the system of model equations for **u** and **v**

$$\begin{aligned} A\mathbf{u} &= [\mathbb{C}]^{T} : \nabla \mathbf{r} \\ D\mathbf{r} &= F\mathbf{u} \end{aligned} \tag{8}$$

which is equivalent to model equations (3). A dynamic behavior of the functionally graded laminate described by the averaged displacement **u** will be referred to as a macro-dynamic while that described by the intrinsic fluctuation **r** will be called a micro-dynamic.

4. Conclusions and perspectives

New results obtained in this contribution and some perspectives of the research are presented below:

1. The mathematical model for investigations of the linear-elastic behavior of FGL was derived on the basis of a certain heuristic assumption. The kinematics of FGL in the framework of this model is described by two vector fields u and v which satisfy general model equations (3). Fields $\mathbf{u}(\cdot)$ and $\mathbf{v}(\cdot)$ describe the averaged and fluctuational parts of the displacement field, respectively. The model equations make it possible to analyze the effect of microstructure size of FGL on its overall behavior, in contrast to the local homogenized model used in the recent literature.
2. After decomposition fluctuations of displacement into the part uniquely determined by the gradient of averaged displacement field and the intrinsic part representing micro-dynamic behavior of FGL the analysis of micro- and macro-behavior of FGL can be independently carried out. This analysis is based on the model equations (8) and will be presented in the forthcoming paper [13].
3. The proposed theory which explicitly couples micro- and macro-structural effects will be developed and applied to functionally graded materials with spatially varying microstructures in three orthogonal directions.

References

[1] Hasselman D.P.H., Youngblood G.E., Enhanced Thermal Stress Resistance of Structural Ceramics with Thermal Conductivity Gradient, Journal of the American Ceramic Society 1978, 61(1,2), 49-53.

[2] Koizumi M., The concept of FGM, Ceramic Transactions, Functionally Graded Materials 1993, 34, 3-10.

[3] Aboudi J., Pindera M.-J., Arnold S.M., Higher-order theory for functionally graded materials, Composites: Part B-Engineering 1999, 30, 777-832.

[4] Reiter T., Dvorak G. J., Tvergaard V., Micromechanical models for graded composite materials, J. Mech. Phys. Solids 1997, 45, 1281-1302.

[5] Suresh S., Mortensen A., Fundamentals of functionally graded materials, The University Press, Cambridge 1998.

[6] Markworth A.J., Saunders J.H., A model of structure optimization for a functionally graded material, Mater. Lett. 1995, 22, 103-107.

[7] Miller D.P., Lannutti J.J., Fabrication and properties of functionally graded NiAl/Al_2O_3 composites, J. Mater. Res. 1993, 8, 2004-2013.

[8] Jędrysiak J., Rychlewska J., Woźniak C., Microstructural 2D-models of functionally graded laminated plates, Proc. of 8th Conference Shell Structures Theory and Applications, Gdańsk-Jurata 2005.

[9] Woźniak C., Rychlewska J., Wierzbicki E., Modelling and analysis of functionally graded laminated shells, Proc. of 8th Conference Shell Structures Theory and Applications, Gdańsk-Jurata 2005.

[10] Woźniak C., Rychlewska J., Elastodynamics of functionally graded laminates with interlaminar microcracks, Proc. of III-th Sympozjum Mechaniki Zniszczenia Materiałów i Konstrukcji, Augustów 2005, 437-442.

[11] Rychlewska J., Woźniak C., Boundary layer phenomena in elastodynamics of functionally graded laminates (in the course of publication).

[12] Woźniak C., Wierzbicki E., Averaging Techniques in Thermomechanics of Composite Solids, Wyd. Politechniki Częstochowskiej, Częstochowa 2000.

[13] Szymczyk J., Woźniak C., Higher-order approximations in continuum modeling of laminated media (in the course of publication).