DIFFERENT VARIANTS OF THE BOUNDARY ELEMENT METHOD FOR PARABOLIC EQUATIONS

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Abstract. In the paper the different algorithms of boundary element method for parabolic equation are presented, this means the 1st and 2nd scheme of the BEM, the BEM using discretization in time and the BEM using Laplace transform.

1. Introduction

The transient temperature field in 2D domain oriented in Cartesian co-ordinate system is described by the equation

$$x \in \Omega: \quad \frac{\partial T(x,\ t)}{\partial t} = a \nabla^2 T(x,t) \tag{1}$$

where $a = \lambda/c$ (c is the volumetric specific heat, λ is the thermal conductivity), $x = \{x_1, x_2\}$ denotes spatial co-ordinates, t is the time. The equation (1) is supplemented by following boundary-initial conditions:

$$\begin{aligned} x \in \Gamma_1: &\quad T(x,t) = T_b \\ x \in \Gamma_2: &\quad q(x,t) = -\lambda \frac{\partial T(x,t)}{\partial n} = q_b \\ t = 0: &\quad T(x,t) = T_0(x) \end{aligned} \tag{2}$$

where T_b is the known boundary temperature, $\partial T(x,t)/\partial n$ is the normal derivative at the boundary point x, q_b is the given boundary heat flux, T_0 is the initial temperature.

In order to solve the problem discussed, the following variants of the boundary element method can be used:
– 1st scheme of the BEM [1-3],
– 2nd scheme of the BEM [1, 2, 4],

– the BEM using discretization in time [1, 2, 5, 6],
– the BEM using Laplace transform [1].

2. 1st and 2nd schemes of the BEM

Initially, we formulate the weighted residual criterion for the problem analyzed

$$\int_{t^0}^{t^F}\int_{\Omega}\left[a\nabla^2 T(x,t)-\frac{\partial T(x,t)}{\partial t}\right]T^*(\xi,x,t^F,t)\,\mathrm{d}\Omega\,\mathrm{d}t=0 \tag{3}$$

where $[t^0, t^F]$ is the time interval, T^* is a fundamental solution and it is a function of the form [1-4]

$$T^*(\xi,x,t^F,t)=\frac{1}{4\pi a(t^F-t)}\exp\left[-\frac{r^2}{4a(t^F-t)}\right] \tag{4}$$

where ξ denotes a point in which the concentrated heat source is applied, while r is the distance from the considered point x to the point ξ.
The normal heat flux of fundamental solution should be found in analytic way

$$q^*(\xi,x,t^F,t)=-\lambda\frac{\partial T^*(\xi,x,t^F,t)}{\partial n}=\frac{\lambda d}{8\pi a^2(t^F-t)^2}\exp\left[-\frac{r^2}{4a(t^F-t)}\right] \tag{5}$$

where

$$d=(x_1-\xi_1)\cos\alpha_1+(x_2-\xi_2)\cos\alpha_2 \tag{6}$$

while $\cos\alpha_1$, $\cos\alpha_2$ are the directional cosines of the normal boundary vector.

In order to construct the integral equation corresponding to the problem considered, the 2nd Green's formula is applied for the first component of equation (2), while the second component of the equation (2) is integrated by parts. Next, using the properties of fundamental solution [1-4] one obtains following integral equation

$$\begin{gathered}B(\xi)\,T(\xi,t^F)+\frac{1}{c}\int_{t^0}^{t^F}\int_{\Gamma}T^*(\xi,x,t^F,t)\,q(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t=\\=\frac{1}{c}\int_{t^0}^{t^F}\int_{\Gamma}q^*(\xi,x,t^F,t)\,T(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t+\int_{\Omega}T^*(\xi,x,t^F,t^0)\,T(x,t^0)\,\mathrm{d}\Omega\end{gathered} \tag{7}$$

where $B(\xi)\in(0,\ 1]$. The numerical approximation of integral equation (6) consists in the discretization of considered interval of time $[t^0, t^F]$, namely

$$t^0 = 0 < t^1 < \ldots < t^{f-1} < t^f < t^{f+1} < \ldots < t^F < \infty \tag{8}$$

with constant step $\Delta t = t^f - t^{f-1}$.

In this place two approaches can be taken into account, i.e. the 1st or the 2nd scheme of the BEM. The idea of the 1st scheme consists in the treatment of transition from t^{f-1} to t^f as a certain separate problem with adequate pseudo-initial condition. In the case of the 2nd scheme f the BEM the integration process starts from $t = t^0$ and then the knowledge of successive pseudo-initial conditions is needless, but temporary values of boundary temperatures and heat fluxes for $t = t^0$, $t = t^1$, ..., $t = t^{f-1}$ must be 'registered'. So, if the 1st scheme of the BEM is used, then the boundary integral equation (7) takes a form [1-3]

$$\begin{gathered} B(\xi)\, T(\xi, t^f) + \frac{1}{c}\int_{t^{f-1}}^{t^f}\int_\Gamma T^*(\xi, x, t^f, t)\, q(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t = \\ = \frac{1}{c}\int_{t^{f-1}}^{t^f}\int_\Gamma q^*(\xi, x, t^f, t)\, T(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_\Omega T^*(\xi, x, t^f, t^{f-1})\, T(x, t^{f-1})\,\mathrm{d}\Omega \end{gathered} \tag{9}$$

while in the case of the 2nd scheme of the BEM one has [1, 2, 4]

$$\begin{gathered} B(\xi)\, T(\xi, t^f) + \frac{1}{c}\sum_{s=1}^{f}\int_{t^{s-1}}^{t^s}\int_\Gamma T^*(\xi, x, t^f, t)\, q(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t = \\ = \frac{1}{c}\sum_{s=1}^{f}\int_{t^{s-1}}^{t^s}\int_\Gamma q^*(\xi, x, t^f, t)\, T(x,t)\,\mathrm{d}\Gamma\,\mathrm{d}t + \int_\Omega T^*(\xi, x, t^f, t^0)\, T(x, t^0)\,\mathrm{d}\Omega \end{gathered} \tag{10}$$

3. BEM using discretization in time

The boundary element method using discretization in time consists in the substitution of time derivative $\partial T / \partial t$ for $t \in [t^{f-1},\ t^f]$ by the adequate differential quotient [1, 2, 5, 6] and then the equation (1) can be written in the form

$$t \in [t^{f-1},\ t^f]: \quad \frac{T(x, t^f) - T(x, t^{f-1})}{\Delta t} = a \nabla^2 T(x, t) \tag{11}$$

or

$$\nabla^2 T(x, t^f) - \frac{1}{a\Delta t} T(x, t^f) + \frac{1}{a\Delta t} T(x, t^{f-1}) = 0 \tag{12}$$

Using the weighted residual criterion one has

$$\int_{\Omega}\left[\nabla^2 T(x,t^f)-\frac{1}{a\Delta t}T(x,t^f)+\frac{1}{a\Delta t}T(x,t^{f-1})\right]T^*(\xi,x)\,\mathrm{d}\Omega=0 \tag{13}$$

where $T^*(\xi,x)$ is the fundamental solution and for domain oriented in Cartesian co-ordinate system it is a function of the form [2, 5, 6]

$$T^*(\xi,x)=\frac{1}{2\pi}\mathrm{K}_0\left(\frac{r}{\sqrt{a\Delta t}}\right) \tag{14}$$

where K_0 is the modified Bessel function of zero order. The heat flux resulting from the fundamental solution is the following

$$q^*(\xi,x)=-\lambda\frac{\partial T^*(\xi,x)}{\partial n}=\frac{\lambda d}{2\pi r\sqrt{a\Delta t}}\mathrm{K}_1\left(\frac{r}{\sqrt{a\Delta t}}\right) \tag{15}$$

where K_1 is the modified Bessel function of first order, d is defined in formula (5).

The boundary integral equation resulting from transformation of equation (12) can be expressed as follows [2, 5, 6]

$$\begin{gathered}B(\xi)\,T(\xi,t^f)+\frac{1}{\lambda}\int_{\Gamma}T^*(\xi,x)q(x,t^f)\,\mathrm{d}\Gamma=\\=\frac{1}{\lambda}\int_{\Gamma}q^*(\xi,x)T(x,t^f)\,\mathrm{d}\Gamma+\frac{1}{a\Delta t}\int_{\Omega}T(x,t^{f-1})\,T^*(\xi,x)\,\mathrm{d}\Omega\end{gathered} \tag{16}$$

4. BEM using Laplace transform

We introduce the Laplace transform [1]

$$\mathrm{L}\left[T(x,t)\right]=U(x,s)=\int_0^{\infty}T(x,t)\,\mathrm{e}^{-st}\mathrm{d}t \tag{17}$$

where $s \geq 0$ (real number) is the transformed parameter. Because

$$\mathrm{L}\left[\frac{\partial T(x,t)}{\partial t}\right]=s\,U(x,s)-T(x,t^0) \tag{18}$$

so the equation (1) can be transformed as follows

$$s\,U(x,s)-T(x,t^0)=a\nabla^2 U(x,s) \tag{19}$$

while the boundary conditions take a form:

$$\begin{aligned} x \in \Gamma_1 &: \quad U\left(x, s\right) = U_b = T_b / s \\ x \in \Gamma_2 &: \quad Q\left(x, s\right) = Q_b = q_b / s \end{aligned} \tag{20}$$

Using the weighted residual one has

$$\int_{\Omega} \left[\nabla^2 U(x,s) - \frac{s}{a} U(x,s) + \frac{1}{a} T(x,t^0) \right] U^*(\xi, x, s)\, \mathrm{d}\Omega = 0 \tag{21}$$

where $U^*(\xi, x, s)$ is the fundamental solution and for 2D domain oriented in Cartesian co-ordinate system it is a function of the form [1]

$$U^*(\xi, x, s) = \frac{1}{2\pi} \mathrm{K}_0 \left(\sqrt{\frac{s}{a}}\, r \right) \tag{22}$$

The heat flux resulting from the fundamental solution is the following

$$Q^*(\xi, x, s) = -\lambda \frac{\partial U^*(\xi, x, s)}{\partial n} = \frac{\lambda d}{2\pi r} \sqrt{\frac{a}{s}}\, \mathrm{K}_1 \left(\sqrt{\frac{s}{a}}\, r \right) \tag{23}$$

After a certain mathematical manipulations we obtain the boundary integral equation

$$\begin{aligned} & B(\xi)\, U(\xi, s) + \frac{1}{\lambda} \int_{\Gamma} U^*(\xi, x, s) Q(x, s)\, \mathrm{d}\Gamma = \\ & = \frac{1}{\lambda} \int_{\Gamma} Q^*(\xi, x, s) U(x, s)\, \mathrm{d}\Gamma + \frac{1}{a} \int_{\Omega} T(x, t^0)\, U^*(\xi, x, s)\, \mathrm{d}\Omega \end{aligned} \tag{24}$$

5. Results of computations

The square domain of dimensions 0.1 m × 0.1 m has been considered. The following thermophysical parameters have been assumed: thermal conductivity $\lambda = 330$ W/mK, volumetric specific heat $c = 3.7464 \cdot 10^6$ J/m^3 K, initial temperature T_0 = 1000°C. On the right and upper surfaces the Dirichlet condition T_b = 500°C has been accepted, on the left surface the Robin condition $q(x, t) = \alpha(T - T^{\infty})$, where α = 1500 W/m^2 K, T^{∞} = 30°C has been taken into account. On the lower surface the no-heat flux condition has been assumed.

The boundary of the domain considered has been divided into 40 constant boundary elements, while the interior has been divided into 100 constant internal cells. Time step: $\Delta t = 2$ s. The details concerning numerical realization of different variants of the BEM can be found in [1-5].

In Figures 1 and 2 the temperature field for times 4 and 8 s is presented. The solutions have been obtained both applying the 1st scheme of the BEM as well as the BEM using discretization in time. The differences between these solutions are very small.

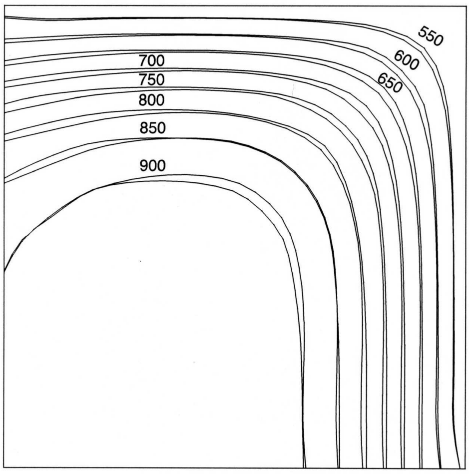

Fig. 1. Temperature field for $t = 4$ s

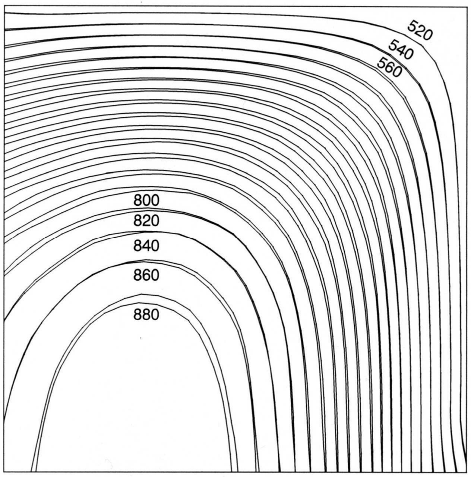

Fig. 2. Temperature field for $t = 8$ s

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