THE AXIOM SYSTEM OF SIMILARITY GEOMETRY THE PROBLEM OF INDEPENDENCY OF AXIOMS

Bogusława Waligóra

Institute of Mathematics and Computer Science, Czestochowa University of Technology

1. Introduction

There are the axiomatics of the central geometry of similarities in papers [2], [3] and [4]. The primitive notions of these axiomatics are the set E, the distinguished point O, the operation of the addition of points and the quaternary relation of similarity P. The paper [2] contains an axiom system of 2-dimensional geometry. The paper [3] contains an axiom system of the central geometry of similarities for dimensions > 2. These axiomatics have been studying in order to attain the solution to the problem of independency of the axioms. This paper contains the models of independence for some axioms and the proofs of dependence for another axioms. This paper contains plain form of some axioms too.

2. The axiom system of central similarity geometry

Now we present the axiom system in the form of paper [2].

We consider the structure (E, +, 0, P), where $+: E^2 \to E$, $0 \in E$, $P \subset E^4$, satisfying the following axioms:

A00: (E, +, 0) is the abelian group with at least two elements,

A01: $P(x,y,z,u) \Rightarrow x \neq 0$,

A02: $x \neq 0 \Rightarrow P(x,y,x,y)$,

A03: $P(x,y,0,y') \Rightarrow y' = 0$,

A04: $P(x,y,x',y') \Rightarrow P(x,y,-x',-y')$,

A05: $\forall x,x',y\ [x \neq 0 \Rightarrow \exists y'\ (P(x,y,x',y') \wedge P(x,y,x+x',y+y'))]$,

A06: $P(x,y,x',y') \wedge P(x,y,x'',y'') \wedge x' \neq 0 \Rightarrow P(x',y',x'',y'')$,

A07: $x \neq 0 \Rightarrow P(x,x,y,y)$,

A08: $P(x,0,y,y') \Rightarrow y' = 0$,

A09: $P(x,x',y,y') \Rightarrow P(x,-x',y,-y')$,

A010: $P(x,x',y,y') \Rightarrow P(x, x+x',y, y+y')$,

A011: $\forall x,x',x'',y,y'\ [x' \neq 0 \wedge P(x,x',y,y') \Rightarrow \exists y''\ (P(x,x'',y,y'') \wedge P(x',x'',y',y''))]$,

A012: $\forall x,x'',y\ [x \neq 0 \Rightarrow \exists y'\ (P(x,y,x',y') \wedge P(x,x',y,y')]$,

A013: $P(x,y,x',y') \wedge P(y,y',y',y) \Rightarrow P(x,x',x',x)$,

We denote by **A0** = {A00,...,A013}.

Now, we will present certain model of axiom system **A0** in order to develop geometrical intuitions.

MODEL I

Let **F** = (F, +, •, 0, 1) be an arbitrary commutative field. We define the structure:

M_F^2 = (E_F, $+_F$, 0_F, P_F); $E_F = F^2$, $0_F = (0,0)$, $(x_1, x_2) +_F (y_1,y_2) = (x_1+y_1,x_2+y_2)$,

$P_F(x,y,z,u) \Leftrightarrow [x \neq 0 \wedge \exists\, \lambda \in F\ (z \circ z = \lambda(x \circ x) \wedge u \circ u = \lambda(y \circ y) \wedge u \circ z = \lambda(y \circ x))]$

where $x \circ y = x_1y_1 + x_2y_2$ for $x = (x_1,x_2)$, $y = (y_1,y_2)$.

It can be shown that M_F^2 is the model for **A0.**

MODEL II

Interpreting E as the set of vectors of the Euclidean real plane which are fixed at the point O, operation + as the addition of vectors (Fig. 1) and relation P as the relation of similarity of triangles with common vertex (Fig. 2) we obtain the model of **A0** axiomatic.

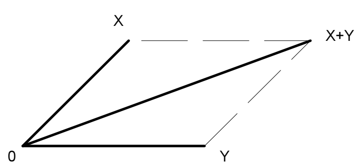

Fig. 1

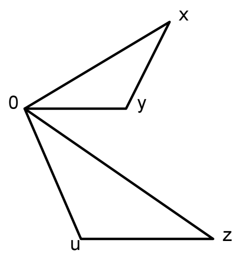

Fig. 2

Using the notation introduced in the model I, we see that the model II is isomorphic with M_R^2.

Two axioms of axiom system **A0** are dependence on another axioms. We will proof this in the following lemmas.

LEMMA 1.

$x \neq 0 \Rightarrow P(x,x,y,y)$.

Proof: From the assumption by A0.12 $\exists\, t\ P(x,x,y,t)$ by A0.9, A0.10, A0.8 we get:

$$P(x,x,y,t) \Rightarrow P(x,-x,y,-t) \Rightarrow P(x,0,y,y-t) \Rightarrow y = t. \square$$

On the basis of these axioms it is quite simple to prove:

LEMMA 2.

$x \neq 0 \Rightarrow P(x,y,x,y)$.

Now, we add the following axioms to the axiom system **A0.**

A2.14: $\forall x \neq 0\ \exists y \neq x\ P(x,y,y,-x)$

A2.15: $P(x,y,y,-x) \wedge P(x',y',y',-x') \Rightarrow P(x,y,y',x')$, A2.16: $\forall x,y \neq 0\ \exists z\ [P(y,z,z,y) \wedge \forall z'\ (P(x,z,x,z') \Rightarrow z = z')]$.

A2.17: $[P(x,x',x,x'') \wedge P(y,x',y,x'') \wedge P(y',x',y',x'') \wedge P(x,y,x,y') \wedge x'' \neq x'] \Rightarrow y = y'$.

Let **A2** = {A0.0,...,A0.13,A2.14,…,A2.17}. This is the axiom system of similarity geometry of dimension 2.

Now, we discuss the problem of independency of axioms.

We introduce independence models for some axioms of axiom system **A2**. In order to we construct the following model.

MODEL III

Let **F** = (F,+,· ,0,1) be an arbitrary commutative field and

$P_F(x,y,z,u) \Leftrightarrow [x \neq 0 \wedge u = yx^{-1}z]$.

It is easy to prove that M_F = (F,+,0,P_F) is a model for **A0**. We present this in two lemmas.

LEMMA 3

$P_F(x,y,x',y') \wedge P_F(x,y,x'',y'') \wedge x' \neq 0 \Rightarrow P_F(x',y',x'',y'')$.

Proof: The assumptions and the definition of the relation P_F imply $y' = yx^{-1}x'$, $y'' = yx^{-1}x''$ and $x' \neq 0$ so exist $(x')^{-1}$. By the properties of the field we get $y'(x')^{-1} = yx^{-1}x'(x')^{-1} = yx^{-1}1 = yx^{-1}$ hence $y'' = y'(x')^{-1}x''$. □

LEMMA 4

$\forall x,x',x'',y,y'\ [x' \neq 0\ \wedge P_F(x,x',y,y') \Rightarrow \exists y''\ (P_F(x,x'',y,y'') \wedge P_F(x',x'',y',y''))]$.

Proof: From the assumptions we get $y' = x'x^{-1}y$ and exist $(x')^{-1}$. Let $y'' = x''x^{-1}y$. By the properties of the field and the assumptions we get $y'(x')^{-1} = yx^{-1}x'(x')^{-1} = yx^{-1}1 = yx^{-1}$ and $y'' = x''x^{-1}y = x''yx^{-1} = x''y'(x')^{-1} = x''(x')^{-1}y'$. □

The proofs for the remaining axioms are similar.

We introduce independence models for some axioms of axiom system **A2**.

The independence model for A2.14:

We assume F = R in the model III. The relation P_R does not satisfy A2.14 because for $x \neq 0$ and $y \neq x$ $\neg(-x = yx^{-1}y)$.

The remaining axioms from **A2** are satisfied.

The independence model for A2.15:

We assume F = C in the model III and x = 1, y = i, x' = -i y' = 1.

We get $-1 = i \cdot 1 \cdot i$ hence $P_C(x,y,y,-x)$ and $i = 1 \cdot \frac{-1}{i} \cdot 1$ hence $P_C(x',y',y',-x')$ but $-i \neq i \cdot 1 \cdot 1$ then $\neg P_C(x,y,y',x')$.

The remaining axioms from **A2** are satisfied.

The independence model for A2.16:

Let F = Q(i) , + - the addition in this field and

$P_Q(x,y,z,u) \Leftrightarrow [x \neq 0 \wedge (xu = yz \vee x\bar{u} = y\bar{z})]$

where if $z = a+bi$ then $\bar{z} = a-bi$.

We assume x = 1, y = 1+i. In order to satisfy the axiom A2.16 there must exist $z = \sqrt{2}$ or $z = \sqrt{-2}$ but $\sqrt{\pm 2} \notin Q(i)$.
The remaining axioms from **A2** are satisfied.

MODEL IV

Let **F** = (F, +, •, 0, 1) be an arbitrary commutative field. We define the structure:

$M_F^n = (E_F, +_F, 0_F, P_F)$; $E_F = F^n$, $0_F = (0,...,0)$,
$(x_1,..., x_n) +_F (y_1,...,y_n) = (x_1+y_1,...,x_n+y_n)$,
$P_F(x,y,z,u) \Leftrightarrow [x \neq 0 \wedge \exists\, \lambda\in F\, (z\,o\,z = \lambda(x\,o\,x) \wedge u\,o\,u = \lambda(y\,o\,y) \wedge u\,o\,z = \lambda(y\,o\,x))]$
where $x\,o\,y = x_1y_1+...+ x_ny_n$ for $x = (x_1,...,x_n)$, $y = (y_1,...,y_n)$.

On the base of the model IV it can be prove that A0.5 is dependent of remaining axioms of axiom system **A0.**

We assume x = (1,0,0), y = (0.1,0), x' = (0,0,1), y' = (a,b,c).
$P_F(x,y,x',y') \Rightarrow \exists\, \lambda\in F\, (1 = \lambda \wedge a^2+b^2+c^2 = \lambda \wedge c = 0)$.
$P_F(x,y,x+x',y+y') \Rightarrow \exists\, \mu\in F\, (2 = \mu \wedge a^2+(b+1)^2+c^2 = \mu \wedge a+c = 0)$.
Then $a^2+b^2 = 1 \wedge a^2+(b+1)^2 = 2 \wedge a = 0$ i.e. $(b+1)^2 = 2 \wedge b^2 = 1$, which is contradictory.
The remaining axioms from **A0** are satisfied.

We replace the axiom A0.5 in the axiom system **A0** by
A3.5: $\forall x,x',y\, [x \neq 0 \Rightarrow \exists y'\, (P(x,y,x,y') \wedge P(x,y',z,u) \wedge P(x,y',x+z,y'+u))]$
We denote by **A01** = {A0.0,...A0.4, A3.5, A0.6,...,A0.13}.

Now, we add the following axioms to the axiom system **A01**
A3.14: $\forall x \neq 0\, \exists y \neq x\, P(x,y,y,-x)$
A3.15: $P(x,y,y,-x) \wedge P(x',y',y',-x') \Rightarrow P(x,y,y',x')$
A3.16: $\forall x,y \neq 0\, \exists z\, [P(y,z,z,y) \wedge \forall z'\, (P(x,z,x,z') \Rightarrow z = z')]$
A3.17: $P(x, z-y, x, y-z) \wedge P(y, z-x,y,x-z) \wedge z \neq 0 \Rightarrow P(z,x-y,z,y-x)$
A3.18: $P(x,y,-x,y) \wedge P(x,z,-x,z) \Rightarrow P(x,y+z,-x,y+z)$
A3.19: $P(x,y,y,x) \wedge y \neq -x \Rightarrow P(x+y, x-y, x+y, y-x)$.
A3.20: $\forall x,y\, [x \neq 0 \Rightarrow \exists z\, (P(x,y,x,z) \wedge \forall z'\, (P(x,y+z,x,z') \Rightarrow y+z = z'))]$
We denote by **A3** = **A01**∪{A3.14,...A3.20}. The paper [3] contains the axiom system **A3**. The structure M_F^n, described in the model IV, is the model of axiom system **A3** [see 4].

Now, we discuss the problem of independency of axioms for axiom system **A3**. The axioms A3.14, A3.15, A3.16 are identical as A2.14, A2.15, A2.16. The models of independence these axioms have been presenting early. We will precede the information of remaining axioms by proofs of properties of the relation P.

3. Some properties of the relation P and the relation collinearity L

We shall now consider the relation of similarity P and the additional relation of collinearity L.

Let us define a relation L(0,x,y).

DEFINITION 5

$L(0,x,y) \Leftrightarrow (P(x,y,x,z) \Rightarrow y = z))$

Now we prove some properties of relations P and L which we shall use in the sequel.

LEMMA 6 $P(x,y,x',y') \wedge x' \neq 0 \Rightarrow P(x',y',x,y)$.

Proof: By A0.1 and Lemma 2

$P(x,y,x',y') \Rightarrow x \neq 0 \Rightarrow P(x,y,x,y)$.

By A0.6 $P(x,y,x',y') \wedge P(x,y,x,y) \wedge x' \neq 0 \Rightarrow P(x',y',x,y)$. □

LEMMA 7 $P(x,y,x',y') \wedge y \neq 0 \Rightarrow P(y,x,y',x')$.

Proof: The assumptions and the axiom A0.11 imply:

$\exists\, u : (P(x,x,x',u) \wedge P(y,x,y',u))$.

We have to prove that: x' = u. By A0.8, A0.9, A0.10:

$P(x,x,x',u) \Rightarrow P(x,0,x', x'-u) \Rightarrow x' = u$. □

By A0.3, Lemma 6 and Lemma 7 we prove

LEMMA 8 $P(x,y,x',y') \wedge y' \neq 0 \Rightarrow P(y',x',y,x)$.

LEMMA 9 $P(x,y,x',y') \Leftrightarrow P(x, -y, -x',y')$.

The proof of ⇒) : By A0.4 and A0.9

$P(x,y,x',y') \Rightarrow P(x, -y\ x', -y') \Rightarrow P(x, -y, -x',y')$.

The proof of converse implication is analogous. □

LEMMA 10 $P(x,y,x',y') \Rightarrow P(-x,y, -x',y')$.

Proof: So, if $P(x,y,x',y')$ then by A0.1, Lemma 2 and A0.4 we get $P(x,y, -x, -y)$ hence, by A0.6 it follows that

$P(x,y, -x, -y) \wedge P(x,y,x',y') \Rightarrow P(-x, -y,x',y')$.

By Lemma 9 we get the thesis. □

LEMMA 11 $\forall x\, [L(0,x,0) \wedge L(0,x,x) \wedge L(0,x\ -x)]$.

Proof: The case x = 0 is obvious so, if x ≠ 0 then by A0.8 we obtain L(0,x,0) and by A0.8, A0.9, A0.10 we prove the other properties. □

In simple way by A0.9 we prove:

LEMMA 12 $L(0,x,y) \Rightarrow L(0,x, -y)$.

By Lemma 10 and A0.9 we deduce:

LEMMA 13 $L(0,x,y) \Rightarrow L(0,-x,-y)$.
By A0.9 and A0.10 we get:

LEMMA 14 $L(0,x,y) \Rightarrow L(0,x,x+y)$.
An interpretation of the next property is the following: if the triangles < 0,x,y > and < 0,x',y'> are similar and the points 0,x,y are collinear then the points 0,x',y' are collinear too.

LEMMA 15 $L(0,x,y) \wedge P(x,y,x',y') \Rightarrow L(0,x',y')$.
Proof: The cases $x' = 0$ and $y' = 0$ are obvious. Thus we assume that: $y' \neq 0$. We have to prove that $P(x',y',x',z) \Rightarrow y' = z$.
From the assumptions and $P(x',y',x',z)$ by Lemma 6 and Lemma 8 we have:

(1) $P(y',x',y,x)$, (2) $P(x',y',x'y)$, (3) $P(x',z,x',y')$

From (2) by A0.11 we get

(4) $\exists u : (P(y',z,y,u) \wedge P(x',z,x,u)$.

Applying A0.6 and from (2), (3) and (4) we obtain $P(x,y,x,u)$.
The assumption $L(0,x,y)$ and the condition $P(x,y,x,u)$ imply $y = u$ hence from (4) we have $P(y',z,y,y)$.
Now we can apply Lemma 14 because $y' \neq 0$ hence $y \neq 0$ and we get $P(y,y,y',z)$.
By A0.8, A0.9, A0.10 we prove that $y' = z$. □

LEMMA 16 $P(x,y,z,u) \wedge L(0,x,y) \Rightarrow P(x,y,x+z,y+u)$.
Proof: From the assumption $P(x,y,z,u)$ by A0.1 and A3.5 we get:

$\exists v,w\ (P(x,y,x,v) \wedge P(x,v,z,w) \wedge P(x,v,x+z,v+w)$

since $L(0,x,y)$ then from def. 1 it follows that: $y = v$. The case $z = 0$ is trivial.
When $z \neq 0$ by A0.6 and Lemma 15 we prove that $w = u$ hence $P(x,y,x+z,y+u)$. □

LEMMA 17 $L(0,x,x') \Rightarrow L(0,x',x)$.
Proof: The cases $x' = 0$ and $x = 0$ are obvious, thus we assume:

(1) $x \neq 0 \wedge x' \neq 0 \wedge P(x',x,x',z)$

We have to prove that $x = z$. By Lemma 7 and A0.4 (1) implies $P(x,x',-z,-x')$.
Applying the assumption $L(0,x,x')$ and Lemma 16 we obtain $P(x,x',x-z,0)$.
Then by Lemma 7 and A0.3 (1) implies $x = z$. □

LEMMA 18 $x \neq 0 \wedge L(0,x,x') \wedge L(0,x,x'') \Rightarrow L(0,x',x'')$.
Proof: Putting particular cases aside we assume that $x' \neq 0$ and $x'' \neq 0$.
We have to prove that $(P(x',x'',x',u) \Rightarrow x'' = u)$. By A0.11 we have:

$\exists z\ (P(x',x,x',z) \wedge P(x'',x',u,z))$

then from the assumptions, Lemma 17, Lemma 7 and def. 5 we obtain that $x = z$ and $P(x,x'',x,u)$. On the basis of def. 5 we get $x'' = u$. □
As a direct consequence of Lemma 17 and Lemma 18 we obtain:

LEMMA 19 $y \neq 0 \wedge L(0,x,y) \wedge L(0,y,z) \Rightarrow L(0,x,z)$

LEMMA 20 $L(0,x,y) \wedge L(0,x,y') \Rightarrow L(0,x, y+y')$

Proof: The case $x = 0$ is obvious. So we assume $x \neq 0$. From the assumptions by Lemma 12, Lemma 14, Lemma 18 we get

$L(0,x, x+y) \wedge L(0,x, x-y') \Rightarrow L(0, x+y, x-y') \Rightarrow L(0, x+y, y+y')$.

When $y \neq -x$ by Lemma 19 we get the thesis. If $y = -x$ then from assumption $L(0,x,y')$ by Lemma 12, Lemma 13 and Lemma 14 we obtain $L(0,x, y+y')$. □

LEMMA 21 $P(x,y,x',y') \wedge P(y,z,y',z') \wedge L(0,y,z) \Rightarrow P(x,z,x',z')$.

Proof: A0.1 and $P(y,z,y',z')$ imply $y \neq 0$ thus by A0.11 we get:

(1) $\exists$ u $(P(x,z,x',u) \wedge P(y,z,y',u)$.

If $y' = 0$ from (1) and the assumption $P(y,z,y',z')$ by A0.3 it follows that $u = z'$.

If $y' \neq 0$ from (1) and $P(y,z,y',z')$ by A0.6 it follows that $P(y',z',y',u)$, by Lemma 15 we get $L(0,y',z')$. We get $z' = u$ by virtue of def. 5 and this completes the proof. □

The Figure 3 presents the interpretation of the property Lemma 21 for the model II.

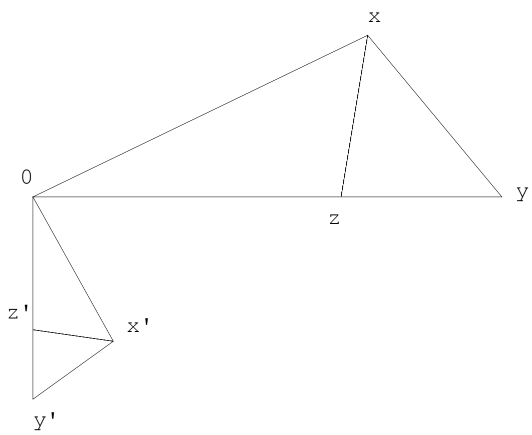

Fig. 3

LEMMA 22 $x+x = 0 \Rightarrow x = 0$

Proof: Suppose $x \neq 0$.

By Lemma 1 we get $P(x,x,x,x)$ since $x = -x$ hence $P(x,x,x,-x)$.

So $x \neq 0$ then by A2.14 (1) $\exists\, y \neq x$: $P(x,y,y,-x)$ thus by A2.15 $P(x,x,y,x)$. By A0.8, A0.9, A0.10 we prove that: $P(x,x,y,x) \Rightarrow P(x,0,y, y-x) \Rightarrow y = x$

what contradicts (1). □

LEMMA 23 $P(x,y,-x,y) \wedge y \neq 0 \wedge L(0,y,z) \Rightarrow P(x,z,-x,z)$.

Proof: From the assumptions by lemma 2 and A0.4 we have $P(y,z,-y,-z)$, hence by lemma 21 we obtain the thesis. □

LEMMA 24 $P(x,y,-x,y) \wedge L(0,x,z) \wedge z \neq 0 \Rightarrow P(z,y,-z,y)$

Proof: In the case $y = 0$ the thesis implies from lemma 2 and A0.8. In the case $y \neq 0$ the thesis implies from lemmas 23 and 7.□

LEMMA 25 $P(x, y, x, z) \Rightarrow P(x, z-y, -x, z-y)$.

Proof: If $y = 0$ from the assumption by A0.8 and A0.3 we obtain the thesis. The case $y = z$ is obvious so, we assume (1) $y \neq 0 \wedge y \neq z$.

From the assumption and (1) by A0.1, lemma 1, lemma 7, A0.13 we have

(2) $P(y, z, z, y)$.

By lemma 11 and lemma 14 we get

(3) $L(0, x. x+x)$.

If $z = -y$ then by lemma 11, lemma 14 and lemma 13 we get $L(0, y, -(y+y))$, so, by lemma 23 we obtain $P(x, -(y+y), x, y+y)$ and $P(x, z-y, -x, z-y)$.

If $y+z = -(x+x)$ then by A0.1 and lemma 22 we get $z \neq -y$ hence by (2) and A3.19 it follows that $P(y+z, y-z, y+z, z-y)$. By (3) and lemma 12, lemma 24, A0.9 we obtain the thesis. Thus we assume that

(4) $y+z \neq -(x+x) \wedge y \neq z \wedge z \neq -y$ (in this case $y \neq -x$).

From (4) and the assumption $P(x,y,x,z)$ by A0.10, lemma 7 it follows that $P(x+y, x, x+z, x)$. So, by lemma 1 and A0.13 we get

(5) $P(x+y, x+z, x+z, x+y)$.

From (4) and (5) by A3.19, lemma 7, A0.9 it follows that

(6) $P(z-y, x+x+y+z, y-z, x+x+y+z)$.

From (2) and (4) by A3.19 we obtain

(7) $P(y+z, y-z, y+z, z-y)$.

From (7) and (4) by A0.9 and lemma 7 we get

(8) $P(z-y, y+z, y-z, y+z)$.

From (8) and (6) by A3.18 and A0.9 we get

(9) $P(z-y, x+x, y-z, x+x)$.

The thesis implies from (9) and (3) by lemma 23 and lemma 7. □

Now we shall prove that the axiom A3.19 depends on the others.

THEOREM 26

$P(x, y, y, x) \wedge y \neq -x \Rightarrow P(x+y, x-y, x+y, y-x)$.

Proof: If $y = x$ then by lemma 2 and lemma 22 we get the thesis.

So we assume (i) $y \neq x$.

By A0.9, A0.10, lemma 7 and (i) we get

$P(x,y,y,x) \Rightarrow P(x, -y, y, -x) \Rightarrow P(x, x-y, y, y-x) \Rightarrow P(x-y, x, y-x, y)$.

By A0.1, lemma 1 and A0.10 we prove that $P(x, x+x, y, y+y)$.

By lemma 11, lemma 14 and lemma 21 we get $P(x-y, x+x, y-x, y+y)$ hence by A0.9 and A0.10 it follows that $P(x-y, -y-x, y-x, -x-y)$.

By A0.9 and lemma 7 we obtain the thesis. □

The axiom A3.17 follows from the condition of simple form:

A4.15 : $P(x, y, y, x) \wedge P(x, z, z, x) \Rightarrow P(y, z, z, y)$.

Therefore A3.17 can be replaced by A4.15.
The Figure 4 presents the interpretation of A4.15 in the model II.

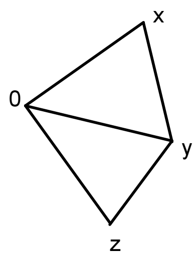

Fig. 4

THEOREM 27

$$P(x, z-y,x, y-z) \wedge P(y, z-x, y, x-z) \wedge z \neq 0 \Rightarrow P(z, x-y, z, y-x)$$

Proof: We denote the conditions from the antecedent of the implication by (1), (2), (3) respectively. Let us suppose that $z = x+y$.
From (1) by A0.9, A0.10, A0.8 we get $x+x = 0$ which by lemma 22 and A0.1 leads to contradiction, hence

(4) $z \neq x+y$.

From (1) and (2) by A0.4 and A0.10 we get
$P(x, z-y+x, -x, z-y-x)$ and $P(y, z-x+y, -y, z-x-y)$.
Applying to these formulae and (4) by lemma 8 and lemma 9 we obtain
$P(z-y-x, x, y-x-z, x)$ and $P(z-x-y, y, x-y-z, y)$.
By lemma 1 and A0.13 we have

$P(z-y-x, y-x-z, y-x-z, z-x-y)$ and $P(z-y-x, x-y-z, x-y-z, z-x-y)$

then by A4.15

$P(y-x-z, x-y-z, x-y-z, y-x-z)$

hence by theorem 26 it follows that

$P(-z-z, -x-x+y+y, z+z, y+y-x-x)$.

Then by lemma 21, lemma 10 and lemma 9 we obtain the thesis. □

Now, we add the following axioms to the axiom system **A01**

A4.18 : $P(x,y,-x,y) \wedge P(x,z,-x,z) \wedge P(y,z,-y,z) \Rightarrow z = 0$.
A4.19 : $\exists\, x,y,z\, (P(x,y,-x,y) \wedge P(x,z,-x,z) \wedge P(y,z,-y,z) \wedge z \neq 0)$.
The axiom A4.18 can be also written in equivalent form
A4.18’ $\neg \exists\, x,y,z\, (P(x,y,-x,y) \wedge P(x,z,-x,z) \wedge P(y,z,-y,z) \wedge z \neq 0)$
It is the dimension axiom, it restricts the dimensions of the space to 2. It is also the negation of the condition A4.19.

It is possible to prove that if we add the axiom A4.18 to the axiom system **A3** then the axioms A3.18 and A3.20 become dependent on the others

THEOREM 28

$$[P(x,x',x,x'') \wedge P(y,x',y,x'') \wedge P(y',x',y',x'') \wedge P(x,y,x,y') \wedge x'' \neq x'] \Rightarrow y = y'.$$

Proof: From the assumptions by lemma 25, lemma 7, A0.9 we obtain

(1) P(x”-x’, x, x”-x’, -x) $\wedge$ P(x”-x’, y, x”-x’, -y) $\wedge$ P(x”-x’, -y’, x”-x’, y’) $\wedge$ P(x, y’-y, -x, y’-y).

From (1) by A3.18 we get

(2) P(x”-x’, y’-y, x’-x”, y’-y) .

From (1) and (2) by A4.18 it follows that y’ = y. □

As a direct consequence of theorem 28 we obtain

THEOREM 29

$$P(x,x',x,x'') \wedge P(y,x',y,x'') \wedge x' \neq x'' \Rightarrow L(0,x,y).$$

Figure 5 presents the interpretation of this theorem in the model II.

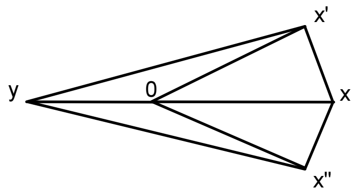

Fig. 5

THEOREM 30

$$P(x,x',x,x'') \wedge x' \neq x'' \Rightarrow L(0,x, x'+x'').$$

Proof: By lemma 11 the case x" = -x' is obvious so we assume:

(1) x" $\neq$ -x'.

By A0.8, lemma 1, lemma 7, and A0.13 we prove that: P(x',x",x",x') hence by A0.10 we get P(x', x'+x", x", x'+x") which by (1) and lemma 7 gives P(x'+x", x', x'+x", x").

Then the thesis follows from the theorem 29. □

Now, we present the proof of the dependence for A3.18.

THEOREM 31

$$P(x,y,-x,y) \wedge P(x,z,-x,z) \Rightarrow P(x, y+z, -x, y+z).$$

Proof: The cases y = 0 and z = 0 are obvious.

Let us assume y $\neq$ 0 and z $\neq$ 0.

By lemma 7, lemma 22 and theorem 29 we obtain L(0,y,z), thus the thesis follows from the assumption P(x,y,-x,y) by lemmas 20 and 21. □

LEMMA 31 $\quad P(x,y,-x,y) \Rightarrow y = 0 \vee \neg L(0,x,y)$

Proof: By lemma 16 we have P(x,y,-x,y) $\wedge$ L(0,x,y) $\Rightarrow$ P(x,y,0,y+y), hence by A0.3 and lemma 22 we get y+y = 0 and y = 0. □

LEMMA 32 $\quad P(x,y,y,-x) \Leftrightarrow P(x,y,-x,y) \wedge P(x,y,y,x).$

The proof of ⇒: By A2.15 we get P(x,y,y,x), hence by A0.1 and A0.8 $y \neq 0$. By A0.6 and lemma 6 we prove that:

$P(x,y,y,x) \wedge P(x,y,y,-x) \wedge y \neq 0 \Rightarrow P(y,x,y,-x) \Rightarrow P(x,y,-x,y)$.

By A0.1, A0.4, A0.8 and A0.9 we prove the converse implication. □

Now we present the proof of the dependence for A3.20, which by def. 5 is equivalent to:

THEOREM 33

$$x \neq 0 \Rightarrow \exists z\, P(x,y,x,z) \wedge L(0,x, y+z).$$

Proof: By A2.14 and A0.8 we prove that: $\exists y\, P(x,y,y,-x) \wedge y \neq 0$ then the lemma 31 yields $\neg L(0,x,y)$ hence $\exists z\, P(x,y,x,z) \wedge y \neq z$. By theorem 30 we get L(0,x, y+z). □

4. Final remarks

The paper contains the models of independence for the axioms A0.5, A2.14, A2.15, A2.16 and the proofs of dependence of the axioms A0.2, A0.7 and A3.19.

In the paper we have discussed the problem of dependence of the axioms A3.18 and A3.20.

The axioms A4.18 and A4.19 are the dimension axioms; they are independent.

The axiom A2.17 follows from A4.18 (see theorem 28). Therefore A2.17 can be replaced by A4.18.

Taking into consideration these results we notice that it is possible to obtain the axiom system of dimension-free geometry (for dimensions > 1).

We consider the structure (E, +, 0, P), where $+: E^2 \to E$, $0 \in E$, $P \subset E^4$, satisfying the following axioms:

A4.0: (E, +, 0) is the abelian group with at least two elements,

A4.1: $P(x,y,z,u) \Rightarrow x \neq 0$,

A4.2: $P(x,y,0,y') \Rightarrow y' = 0$,

A4.3: $P(x,y,x',y') \Rightarrow P(x,y,-x',-y')$,

A4.4: $\forall x,x',y\, [x \neq 0 \Rightarrow \exists y'\, (P(x,y,x,y') \wedge P(x,y',z,u) \wedge P(x,y',x+z,y'+u))]$

A4.5: $P(x,y,x',y') \wedge P(x,y,x'',y'') \wedge x' \neq 0 \Rightarrow P(x',y',x'',y'')$,

A4.6: $P(x,0,y,y') \Rightarrow y' = 0$,

A4.7: $P(x,x',y,y') \Rightarrow P(x,-x',y,-y')$,

A4.8: $P(x,x',y,y') \Rightarrow P(x, x+x',y, y+y')$,

A4.9: $\forall x,x',x'',y,y'\, [x' \neq 0 \wedge P(x,x',y,y') \Rightarrow \exists y''\, (P(x,x'',y,y'') \wedge P(x',x'',y',y''))]$,

A4.10: $\forall x,x'',y\, [x \neq 0 \Rightarrow \exists y'\, (P(x,y,x',y') \wedge P(x,x',y,y')]$,

A4.11: $P(x,y,x',y') \wedge P(y,y',y',y) \Rightarrow P(x,x',x',x)$,

We denote by **A4.0** = {A4.0,...,A4.11}.

Now, we add the following axioms to the axiom system **A4.0**

A4.12: $\forall x \neq 0\, \exists y \neq x\, P(x,y,y,-x)$

A4.13: $P(x,y,y,-x) \wedge P(x',y',y',-x') \Rightarrow P(x,y,y',x')$

A4.14: $\forall x,y \neq 0\ \exists z\ [P(y,z,z,y) \wedge \forall z'\ (P(x,z,x,z') \Rightarrow z = z')]$

A4.15: $P(x, y, y, x) \wedge P(x, z, z, x) \Rightarrow P(y, z, z, y)$

A4.16: $P(x,y,-x,y) \wedge P(x,z,-x,z) \Rightarrow P(x,y+z,-x,y+z)$

A4.17: $\forall x,y\ [x\neq 0 \Rightarrow \exists z\ (P(x,y,x,z) \wedge \forall z'\ (P(x,y+z,x,z') \Rightarrow y+z = z'))]$

We denote by **A4.1** = **A4.0**$\cup${A4.12,...A4.17}.

Let **A4.2** = **A4.1**$\cup${A4.18}, **A4.3** = **A4.1**$\cup${A4.19}.

The axiom system **A4.2** is the axiomatic of the plane geometry. The axiom system **A4.3** is the axiomatic of dimension-free geometry (of dimensions ≥ 3).

It is also interesting that the system of primitive notions considered here can lead to the axiomatics of one dimensional geometry and universal axiomatics that does not neglect any case of axiomatics of dimension-free geometry. Some problems will be discussed in a separate paper.

References

[1] Szmielew W., From affine to Euclidean geometry, An axiomatic approach, PWN-Polish Scientific Publishers, Warszawa 1983.

[2] Waligóra B., On some system of axioms of central similarity geometry, ZN Geometria XV, Poznań 1986.

[3] Waligóra B., One some dimension-free central similarity system, ZN Geometria XVII, Poznań 1988.

[4] Waligóra B., Pewna aksjomatyka geometrii centralnej podobieństw, Doctoral dissertation, UMK, Toruń 1986.