SIMPLICIAL MODELLING OF DYNAMIC PROBLEMS IN A MICRO-PERIODIC COMPOSITE MATERIAL

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Abstract. The aim of this paper is to propose a certain new approach to the formulation of both discrete and continuum models for the analysis of dynamic problems in elastic composite solids with a periodic microstructure. The proposed approach is based on a periodic simplicial division of the unit cell and on the assumption of a uniform strain in every simplex. The main feature of the obtained discrete model is the finite-difference form of the governing equations. By applying smoothing operation the continuum models are derived directly from the discrete ones.

1. Introduction

In the existing literature we can find many approaches and methods leading to the different approximated mathematical models. The best known are those based on the asymptotic homogenization theory [3, 6, 10]. However, using this theory we are not able to account for the microstructural length-scale effect on the global dynamic behavior of the body. So, for analysis of these phenomena a number of alternative methods and theories have been proposed, e.g., effective stiffness theories [1], mixture theories [2], interacting continuum theories [5], general asymptotic methods approaches [4], and many others. A review of early papers on this subject can be found in [7]. Recently a number of dynamic problems have been investigated using the tolerance averaging method [11].

The aim of this paper is to show that after introducing a smoothing operations to the finite difference equations of a discrete model proposed in [9] it is possible to obtain a hierarchy of continuum models describing the macroscopic behaviour of a micro-periodic solid on different levels of accuracy. The simplest from the aforementioned models leads to the equations of a homogeneous equivalent medium which is not dispersive and can be obtained in the framework of the homogenization theory [3, 6, 10].

To make this paper self-consistent in the subsequent section we summarise the main concepts introduced in [9].

2. Preliminaries

Let the composite solid under consideration occupies a region Ω in E^3, has perfectly bounded linear-elastic constituents and a periodic structure determined by a vector basis $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ in E^3. We denote by Δ a polyhedron in E^3 such that for every $\mathbf{x} \in \partial\Delta$ and some $\mathbf{d}_\alpha$, $\alpha = 1, 2, 3$, we have either $\mathbf{x} + \mathbf{d}_\alpha \in \partial\Delta$ or $\mathbf{x} - \mathbf{d}_\alpha \in \partial\Delta$ (but not both). Let us also assume that the diameter l of Δ is sufficiently small when compared to the smallest characteristic length dimension of region Ω. In this case polyhedron Δ will be referred to as the unit cell.

Let Λ be the Bravais lattice in E^3

$$\Lambda := \left\{ \mathbf{z} \in E^3 : \mathbf{z} = \eta_1 \mathbf{d}_1 + \eta_2 \mathbf{d}_2 + \eta_3 \mathbf{d}_3,\ \eta_\alpha = 0, \pm 1, \pm 2, ...,\ \alpha = 1,2,3 \right\}$$

and let us denote

$$\Delta(\mathbf{z}) := \mathbf{z} + \Delta\,,\ \ \Lambda_0 := \left\{ \mathbf{z} \in \Lambda : \Delta(\mathbf{z}) \subset \Omega \right\},\ \ \overline{\Omega}_0 := \left\{ \mathbf{x} \in \overline{\Delta}(\mathbf{z}) : \mathbf{z} \in \Lambda_0 \right\}$$

where Ω_0 is a regular subregion of Ω. A simplicial division of E^3 will be called Δ-periodic if it implies the simplicial subdivision of every $\overline{\Delta}(\mathbf{z})$, $\mathbf{z} \in \Lambda$, into simplexes $T^k(\mathbf{z})$, $k = 1,...,m$, such that $T^k(\mathbf{z}) = T^k + \mathbf{z}$, $\mathbf{z} \in \Lambda$ where T^k, $k = 1,...,m$, are simplexes in Δ. Let $\left\{ \mathbf{p}_0^a \in \overline{\Delta} : a = 1,...,n+1 \right\}$, $n \geq 1$, be the smallest set of vertexes (nodal points) of T^k, $k = 1,...,m$, such that $\left\{ \mathbf{p}_0^a + \mathbf{z} : a = 1,...,n+1, \mathbf{z} \in \Lambda \right\}$ is the set of all nodal points in E^3 related to a certain Δ-periodic simplicial division of E^3. We shall also introduce a system of vectors $\mathbf{d}_A \in \Lambda$, $A = 0,1,...,N$, such that $\mathbf{d}_0 = \mathbf{0}$ and every vertex related to T^k, $k = 1,...,m$, can be uniquely represented by a sum $\mathbf{p}_0^a + \mathbf{d}_A$. It can be seen that $N = 7$ for spatial problem and $N = 3$ for plane problem. Setting $I := \left\{ (a, A) \in \{1,...,n+1\} \times \{0,1,...,N\} : \mathbf{p}_0^a + \mathbf{d}_A \in \overline{\Delta} \right\}$ and denoting $\mathbf{p}_A^a := \mathbf{p}_0^a + \mathbf{d}_A$ for every $(a, A) \in I$, we conclude that $\left\{ \mathbf{p}_A^a : (a, A) \in I \right\}$ is the set of all nodal points in $\overline{\Delta}$ which is related to the Δ-periodic simplicial division of E^3. Hence every simplex T^k can be represented by $T^k = \mathbf{p}_A^a \mathbf{p}_B^b \mathbf{p}_C^c \mathbf{p}_D^d$ where $(a, A),...,(d, D) \in I$. Setting $I_0 := \left\{ (a, A) \in I : A \neq 0 \right\}$ we see that $\mathbf{p}_A^a \in \partial\Delta$ if and only if $(a, A) \in I_0$. Here and hereafter it is assumed that a certain Δ-periodic simplicial division of E^3 is known.

For an arbitrary function $f(\cdot)$ defined on Λ_0 we shall define the finite differences

$$\begin{aligned} \Delta_A f(\mathbf{z}) &= f(\mathbf{z} + \mathbf{d}_A) - f(\mathbf{z}) \\ \overline{\Delta}_A f(\mathbf{z}) &= f(\mathbf{z}) - f(\mathbf{z} - \mathbf{d}_A) \end{aligned} \tag{1}$$

provided that $\mathbf{z}, \mathbf{z} + \mathbf{d}_A, \mathbf{z} - \mathbf{d}_A \in \Lambda_0$.

Throughout the paper it will be assumed that superscripts a, b, c, d run over $1,...,n+1$, $n \geq 1$, and subscripts A, B run over $0,1,...,N$, unless otherwise stated. We shall also introduce superscripts p, q which run over $1,...,n$. Summation convention with respect to all aforementioned indices holds.

Let $\mathbf{w}(\mathbf{x}, t)$, $\mathbf{x} \in \Omega$, stand for a displacement field at time t for the solid under consideration. Let us denote

$$\mathbf{u}_A^a(\mathbf{z}, t) := \mathbf{w}(\mathbf{p}_A^a(\mathbf{z}), t), \quad (a, A) \in I_0, \quad \mathbf{z} \in \Lambda_0$$

Subsequently we shall interpret simplexes T^k, $k = 1,...,m$, as finite elements of the unit cell Δ which are subjected to uniform strains. Hence $\mathbf{p}_A^a(\mathbf{z})$ are nodal points of these elements. Let us also „approximate” region Ω by Ω_0. In this case the displacement field $\mathbf{w}(\cdot, t)$ in every cell $\overline{\Delta}(\mathbf{z}), \mathbf{z} \in \Lambda_0$ will be uniquely determined by displacements $\mathbf{u}_A^a(\mathbf{z}, t)$ of the nodal points $\mathbf{p}_A^a(\mathbf{z})$, $(a, A) \in I$. Bearing in mind (1) we see that these displacements can be uniquely represented in the form

$$\mathbf{u}_A^a(\mathbf{z}, t) = \mathbf{u}^a(\mathbf{z}, t) + \Delta_A \mathbf{u}^a(\mathbf{z}, t), \quad (a, A) \in I$$

where for $A = 0$ we obtain $\mathbf{u}_0^a(\mathbf{z}, t) = \mathbf{u}^a(\mathbf{z}, t)$. Let $\mathbf{u}$ be a certain averaged value of $\mathbf{u}^1,...,\mathbf{u}^{n+1}$, given by

$$\mathbf{u} = \nu_\alpha \mathbf{u}^\alpha$$

where $\nu_\alpha > 0$ and $\nu_1 + ... + \nu_{n+1} = 1$. The values ν_α will be specified in the subsequent section. Under the above denotations the strain and kinetic energy functions for the solid under consideration are respectively represented by:

$$\begin{aligned} U &= U\left(\Delta_A \mathbf{u}^a, \mathbf{u}^p - \nu_\alpha \mathbf{u}^\alpha\right) \\ K &= K\left(\Delta_A \dot{\mathbf{u}}^a, \dot{\mathbf{u}}^b\right) \end{aligned} \tag{2}$$

where $(a, A) \in I_0$, $b = 1,...,n+1$, $p = 1,...,n$. The coefficients of forms (2) can be uniquely determined for every periodic solid. Introducing the differences $\mathbf{u}^p - \mathbf{u}$ as arguments of the strain energy function we have taken into account the translational invariance of $U(\cdot)$. It can be shown [12], that for unknowns $\mathbf{u}^a(\mathbf{z}, t)$,

$a = 1,..,n+1$, $\mathbf{z} \in \Lambda_0$, in the absence of body forces, we obtain the system of ordinary differential equations which can be expressed in the following finite-difference form

$$\overline{\Delta}_A \mathbf{s}_A^a - \frac{\partial U}{\partial \mathbf{u}^a} = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{\mathbf{u}}^a} - \overline{\Delta}_A \mathbf{j}_A^a \right), \quad a = 1,...,n+1 \tag{3}$$

where:

$$\mathbf{s}_A^a = \frac{\partial U}{\partial \Delta_A \mathbf{u}^a}$$
$$\mathbf{j}_A^a = \frac{\partial K}{\partial \Delta_A \dot{\mathbf{u}}^a}, \ (a, A) \in I_0 \tag{4}$$

Equations (3), (4) are assumed to hold for every $\mathbf{z} \in \Lambda_0$ such that $\mathbf{z} \pm \mathbf{d}_A \in \Lambda_0$ for $A = 1,...,N$ and represent *a finite difference model* of the periodic composite under consideration. It has to be emphasised that this model has a physical sense only if diameters l_k of simplexes T^k, $k = 1,...,m$ are small as compared to the typical wavelength of the deformation pattern in the problem under consideration. Thus, for the high-frequency vibration problems the number m of simplexes T^k and hence also the number n of unknowns $\mathbf{u}^a(\mathbf{z},t)$ for every $\mathbf{z} \in \Lambda_0$ can be very large. Equations of the form (3), (4) constitute the foundations of the subsequent analysis leading to different continuum models of the micro-periodic solids under consideration.

3. Simplified finite difference models

For the given $\nu_1,...,\nu_{n+1}$ let us denote

$$\widetilde{\mathbf{u}}^a \coloneqq \mathbf{u}^a - \mathbf{u} = \mathbf{u}^a - \nu_b \mathbf{u}^b$$

It follows that

$$\nu_a \widetilde{\mathbf{u}}^a = \mathbf{0}$$

and hence the fields $\widetilde{\mathbf{u}}^a(\mathbf{z},t)$, $\mathbf{z} \in \Lambda_0$, are linear dependent. In order to satisfy the above condition we shall introduce new linear independent fields $\mathbf{v}^q = \mathbf{v}^q(\mathbf{z},t)$ $\mathbf{z} \in \Lambda_0$, such that

$$\widetilde{\mathbf{u}}^a = l h^{aq} \mathbf{v}^q$$

where l is a diameter of Δ and h^{aq} are elements of the $(n+1)\times n$ of an order n, satisfying conditions

$$\nu_a h^{aq} = 0$$

Hence

$$\mathbf{u}^a = \mathbf{u} + l h^{aq} \mathbf{v}^q \tag{5}$$

and we shall take u and $\mathbf{v}^q$ as the basic unknowns. It can be seen that the above formula represents a decomposition of the displacement field $\mathbf{u}^a$ into the averaged $\mathbf{u} = \nu_a \mathbf{u}^a$ and fluctuating $\widetilde{\mathbf{u}}^a$ parts.

Subsequently we shall restrict ourselves to problems in which the increments $\Delta_A \widetilde{\mathbf{u}}^a$ of fluctuations can be neglected as small with respect to the increments $\Delta_A \mathbf{u}$ of the averaged displacements. Thus, we shall apply to (2) an approximation

$$\Delta_A \mathbf{u}^a \cong \Delta_A \mathbf{u} \tag{6}$$

which holds for every $(a, A) \in I_0$. The above formula states that in an arbitrary but fixed periodicity cell $\overline{\Delta}(\mathbf{z})$, $\mathbf{z} \in \Lambda_0$, the displacement fluctuations can be treated as periodic: $\widetilde{\mathbf{u}}^a(\mathbf{z} + \mathbf{d}^A, t) \cong \widetilde{\mathbf{u}}^a(\mathbf{z}, t)$, $(a, A) \in I_0$.

Subsequently, for the sake of simplicity we shall also approximate the mass distribution in the periodic medium by a periodic system of concentrated masses m^a, $a = 1,...,n+1$, assigned to the nodal points. Setting $m = m^1 + ... + m^{n+1}$ we shall assume that $\nu_a = m^a / m$. Hence the kinetic energy function will take the form

$$K = \frac{1}{2|\Delta|} \sum_{a=1}^{n+1} m^a \, \dot{\mathbf{u}}^a \cdot \dot{\mathbf{u}}^a$$

where $|\Delta|$ is a measure of the cell Δ. Taking into account formula (5) we obtain the kinetic energy function in the form

$$\widetilde{K} = \frac{1}{2} \rho \, \dot{\mathbf{u}} \cdot \dot{\mathbf{u}} + \frac{1}{2} l^2 M^{pq} \dot{\mathbf{v}}^p \cdot \dot{\mathbf{v}}^q \tag{7}$$

where

$$\rho = \frac{m}{|\Delta|}, \quad M^{pq} = \frac{1}{|\Delta|} \sum_{a=1}^{n+1} m^a h^{aq} h^{ap}$$

Taking into account (5) and (6) we obtain from (2) the strain energy function

$$\widetilde{U} = \frac{1}{2} a_{AB} \Delta_A \mathbf{u} \cdot \Delta_B \mathbf{u} + \frac{1}{2} b^{pq} \mathbf{v}^p \cdot \mathbf{v}^q + c_A^q \mathbf{v}^q \cdot \Delta_A \mathbf{u} \tag{8}$$

which is a density per the unit measure of Δ. Because of $\Delta_A \mathbf{u} \in O(l)$ we have $a_{AB} \in O(l^{-2})$, $c_A^q \in O(l^{-1})$ and $b^{pq} \in O(1)$, i.e., all terms in (8) are the same order. Using (7) and (8) we shall transform equations (3), (4) to the form:

$$\begin{gathered} \overline{\Delta}_A \mathbf{s}_A + c_A^q \overline{\Delta}_A \mathbf{v}^q = \rho \ddot{\mathbf{u}} \\ l^2 M^{pq} \ddot{\mathbf{v}}^q + b^{pq} \mathbf{v}^q + c_A^p \Delta_A \mathbf{u} = 0 \end{gathered} \tag{9}$$

where

$$\mathbf{s}_A = a_{AB} \Delta_B \mathbf{u}$$

The above equations hold for every $\mathbf{z} \in \Lambda_0$ and time t and represent *a simplified finite difference model* of the periodic composite medium under consideration.
Let us observe that $l^2 M^{pq} \ddot{\mathbf{v}}^q \in O(l^2)$ and the values of all other terms in (9) are independent of l. Hence, for a sufficiently small l we can apply the limit passage $l \to 0$. In this case the first term in the second from equations (9) will be neglected and we arrive at the equations

$$b^{pq} \mathbf{v}^q = -c_A^p \Delta_A \mathbf{u}$$

Since b^{pq} represent the non-singular $n \times n$ then denoting by B^{pq} elements of the inverse matrix and setting

$$a_{AB}^0 := a_{AB} - c_A^q B^{qp} c_A^p$$

the first from equations (9) yields

$$a_{AB}^0 \overline{\Delta}_A \Delta_B \mathbf{u} = \rho \ddot{\mathbf{u}} \tag{10}$$

Thus we have arrived at the single equation for the averaged displacement field $\mathbf{u}(\mathbf{z}, t)$, $\mathbf{z} \in \Lambda_0$. The above equation together with formulae

$$\mathbf{v}^q = -B^{qp} c_A^p \Delta_A \mathbf{u} \tag{11}$$

represent what will be called the asymptotic discrete finite element model of the periodic composite under consideration. Let us observe that for stationary problems the second from equations (9) coincide with equations (11).

Discrete models governed by equations (9) and (10) will be treated in subsequent analysis as a basis for the formulation of continuum models. The main advantage of the aforementioned equations is that they involve finite differences with respect to only one unknown field u, in contrast to equations (3), (4). This fact will imply the relatively simple form of pertinent continuum model equations which will be derived in the subsequent section. It has to be remembered that equations (9), (10) can be applied exclusively to the analysis of the long wave problems.

4. Continuum models

Let $\mathbf{u}(\cdot,t)$, $\mathbf{s}_A(\cdot,t)$ be arbitrary sufficiently smooth fields defined on Ω, which after restricting their domain Ω to Λ_0 reduce to fields $\mathbf{u}(\mathbf{z},t)$, $\mathbf{s}_A(\mathbf{z},t)$, $\mathbf{z}\in\Lambda_0$, occurring in (9). In order to obtain a continuum model of the periodic solid under consideration, we shall assume that for every $\mathbf{z}\in\Lambda_0$ and every y such that $|\mathbf{y}|\le l$ and $\mathbf{z}+\mathbf{y}\in\Omega$, the aforementioned smooth fields can be approximated by means of the formulae

$$\mathbf{w}(\mathbf{z}+\mathbf{y},t)\cong\mathbf{w}(\mathbf{z},t)+\mathbf{y}\cdot\nabla\mathbf{w}(\mathbf{z},t)+\frac{1}{2}(\mathbf{y}\otimes\mathbf{y}):(\nabla\otimes\nabla)\mathbf{w}(\mathbf{z},t)$$

where $\mathbf{w}$ stands for $\mathbf{u}$ and $\mathbf{s}_A$. From the above approximation we also obtain

$$\mathbf{w}(\mathbf{z},t)\cong\mathbf{w}(\mathbf{z}-\mathbf{y},t)+\mathbf{y}\cdot\nabla\mathbf{w}(\mathbf{z},t)-\frac{1}{2}(\mathbf{y}\otimes\mathbf{y}):(\nabla\otimes\nabla)\mathbf{w}(\mathbf{z},t)$$

Hence, under denotations (no summation over A!):

$$\mathbf{e}_A:=\mathbf{d}_A\,l^{-1}$$

$$\mathbf{E}_A:=\frac{1}{2}\mathbf{e}_A\otimes\mathbf{e}_A$$

we conclude that the following approximations:

$$\begin{aligned}\Delta_A\mathbf{u}(\mathbf{z},t)&\cong l\mathbf{e}_A\cdot\nabla\mathbf{u}(\mathbf{z},t)+l^2\mathbf{E}_A:(\nabla\otimes\nabla)\mathbf{u}(\mathbf{z},t)\\ \overline{\Delta}_A\mathbf{s}_A(\mathbf{z},t)&\cong l\mathbf{e}_A\cdot\nabla\mathbf{s}_A(\mathbf{z},t)-l^2\mathbf{E}_A:(\nabla\otimes\nabla)\mathbf{s}_A(\mathbf{z},t)\end{aligned}\tag{12}$$

hold for every $\mathbf{z}\in\Lambda_0$. Substituting the right-hand sides of the above formulae into (9) and denoting

$$\begin{aligned} &\mathbf{G} := a_{AB}\mathbf{E}_A \otimes \mathbf{E}_B\, l^2, \quad \mathbf{C} := a_{AB}\mathbf{e}_A \otimes \mathbf{e}_B\, l^2 \\ &\mathbf{H}^q := c_A^q \mathbf{E}_A\, l \qquad\quad , \quad \mathbf{h}^q := c_A^q \mathbf{e}_A\, l \end{aligned} \tag{13}$$

after simple manipulations we obtain

$$\begin{aligned} &-l^2(\nabla\otimes\nabla):\left[\mathbf{G}:(\nabla\otimes\nabla)\mathbf{u}\right]+\nabla\cdot(\mathbf{C}\cdot\nabla\mathbf{u})+ \\ &\quad -l\,\mathbf{H}^q:(\nabla\otimes\nabla)\mathbf{v}^q+\mathbf{h}^q\cdot\nabla\mathbf{v}^q=\rho\,\ddot{\mathbf{u}} \\ &l^2 M^{pq}\ddot{\mathbf{v}}^q+b^{pq}\mathbf{v}^q+\mathbf{h}^q\cdot\nabla\mathbf{u}+l\,\mathbf{H}^q:(\nabla\otimes\nabla)\mathbf{u}=0, \quad q=1,\dots,n \end{aligned} \tag{14}$$

Because $\mathbf{u}(\cdot,t), \mathbf{v}^q(\cdot,t)$ are functions defined for every time t on region Ω we have arrived at the system of n+1 differential equations (14) for n+1 unknown vector fields $\mathbf{u}$, $\mathbf{v}^q$. The aforementioned equations represent what will be called *the second order continuum model* of the periodic composite medium under consideration. It has to be emphasised that for an averaged displacement field $\mathbf{u}$ we have obtained the partial differential equation and for the displacement fluctuations $\mathbf{v}^q$ the system of n ordinary differential equations. It follows that the boundary conditions can be imposed only on the averaged displacement field; we deal here with a situation similar to that occurring in the tolerance averaging model equations [11].

Applying approximations (12) to equation (10) and denoting:

$$\begin{aligned} &\mathbf{G}_0 := a_{AB}^0\mathbf{E}_A \otimes \mathbf{E}_B\, l^2 \\ &\mathbf{C}_0 := a_{AB}^0\mathbf{e}_A \otimes \mathbf{e}_B\, l^2 \end{aligned} \tag{15}$$

we obtain

$$-l^2(\nabla\otimes\nabla):\left[\mathbf{G}_0:(\nabla\otimes\nabla)\mathbf{u}\right]+\nabla\cdot(\mathbf{C}_0\cdot\nabla\mathbf{u})=\rho\,\ddot{\mathbf{u}} \tag{16}$$

The above equation represent *the asymptotic second order continuum model* of the medium under consideration.

Now assume that instead of (12) we introduce the linear approximations:

$$\begin{aligned} &\Delta_A\mathbf{u}(\mathbf{z},t)\cong l\,\mathbf{e}_A\cdot\nabla\mathbf{u}(\mathbf{z},t) \\ &\overline{\Delta}_A\mathbf{s}_A(\mathbf{z},t)\cong l\,\mathbf{e}_A\cdot\nabla\mathbf{s}_A(\mathbf{z},t) \end{aligned} \tag{17}$$

In this case equations (9) reduce to the form:

$$\begin{aligned} &\nabla\cdot(\mathbf{C}\cdot\nabla\mathbf{u})+\mathbf{h}^q\cdot\nabla\mathbf{v}^q=\rho\,\ddot{\mathbf{u}} \\ &l^2 M^{pq}\ddot{\mathbf{v}}^q+b^{pq}\mathbf{v}^q+\mathbf{h}^q\cdot\nabla\mathbf{u}=0, \quad q=1,\dots,n \end{aligned} \tag{18}$$

where $\mathbf{C}$ and $\mathbf{h}^q$ are defined by formulae (13). The above equations represent *the first order continuum model* of the periodic composite medium. Similarly, from (10) we derive equation

$$\nabla \cdot \left(\mathbf{C}_0 \cdot \nabla \mathbf{u}\right) = \rho\, \ddot{\mathbf{u}} \tag{19}$$

representing *the asymptotic first order continuum model* of the medium under consideration.

Subsequently we shall apply the obtained model equations only to the analysis of the wave propagation in an unbounded medium; that is why in this paper we shall not discuss the physical meaning of boundary conditions related to equations (14), (16), (18) and (19). It can be shown [8], that the aforementioned equations together with pertinent natural boundary conditions can be also derived from the principle of stationary action.

Summarising the obtained results we shall state that the macroscopic dynamic behaviour of the elastic composites with a periodic microstructure can be analysed in the framework of different continuum models described by independent systems of equations (14), (16), (18) and (19). The above equations have constant coefficients which depend on the geometric and material structure of the unit cell, i.e. on vectors $\mathbf{e}_A$, $A = 1,...,N$, and coefficients of the quadratic forms (7), (8) related to a discrete model. Solutions to these equations have a physical sense only if the approximation formulae (12) or (17) are satisfied with a sufficient accuracy. Obviously, the derived continuum models describe the dynamic behaviour of the composite on different levels of accuracy. Thus, the problem arises what is the interrelation between these models and their accuracy when compared to the discrete model given by equations (3), (4). More detailed discussion of the above problem and applications of the proposed models will be given in the forthcoming paper.

Conclusions

Among new results obtained in this contribution the following ones seem to be most important.

1. The proposed discrete model makes it possible to obtain independent systems of equations for displacement fluctuations $\mathbf{v}^q(\mathbf{z},t)$, $q = 1,...,n$, in every cell $\Delta(\mathbf{z}), \mathbf{z} \in \Lambda_0$.
2. The proposed continuum models are governed by the partial differential equations only for the mean displacement field $\mathbf{u}(\cdot)$. Displacement fluctuation fields $\mathbf{v}^q(\cdot)$ are governed by ordinary differential equations involving only time derivatives of $\mathbf{v}^q(\cdot)$. It follows that in stationary problems fields $\mathbf{v}^q(\cdot)$ are governed by linear algebraic equations and can be eliminated from governing equations.

3. The displacement fluctuations $\mathbf{v}^q$ are governed by a system of linear algebraic equations also in dynamic problems provided that we apply the asymptotic approximation both to discrete and continuum model equations.
4. Apart from the asymptotic first order continuum model, all proposed models take into account the effect of the microstructure size on the dynamic behaviour of a composite solid, which plays an important role in the dispersive analysis of dynamic problems.
5. From a formal point of view the second order continuum model (14) correspond to that obtained in the framework of the tolerance averaging method [11].

It has to mentioned that in most engineering problems the number n of displacement fluctuations can be large and solution to these problems requires applications of the computational methods.

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