A SIMPLE EXAMPLE OF A NONCLOSED SET OF REPRESENTING MEASURES

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Abstract. Let X be a locally convex Hausdorff space, K - its compact, convex and metrizable subset. We say, that a regular Borel probability measure μ on K represents point $x \in X$ if the equality $f(x) = \int_K f d\mu$ holds for every $f \in X^*$. We will show by a simple example, that the set of such measures supported on ext K need not be closed.

1. Definition Let A be a nonempty set. A point $a \in A$ is an extreme point of A, if it is not an internal point of any line interval whose endpoints are in A.

Compact convex set in locally convex space always has extreme points. The set of extreme points of a compact convex set is closed only in $\mathbf{R}^1$ and $\mathbf{R}^2$, it need not be closed even in $\mathbf{R}^3$.

2. Example Let A:=$\{(x, y, z) \in \mathbf{R}^3 : z = 0,\ x^2 + y^2 = 1\} \cup \{(1,0,1), (1,0,-1)\}$.

If K is closed convex hull of A, then K is compact and convex, but the set of its extreme points

$extK = \{(x, y, z) \in \mathbf{R}^3 : z = 0,\ x^2 + y^2 = 1\} \setminus \{(1,0,0)\} \cup \{(1,0,1), (1,0,-1)\}$ is not closed.

3. Definition Let X be a topological space and μ - a probability measure on X. We say that μ is regular, if

$$\mu(\mathrm{A}) = \sup\{\mu(C) : C \subseteq A, C \text{ - closed}\} = \inf\{\mu(U) : A \subseteq U, U \text{ - open}\}$$

4. Definition Let X be as above.

a) the support of a measure μ is a closed set $S \subset X$ such that:

(i) $\mu(S) = 1$;

(ii) if A is any closed set such that $\mu(A) = 1$, then $S \subseteq A$;

b) we say that a measure μ is supported by a set $A \subset X$ (not necessarily closed), if $\mu(A) = 1$.

A Borel measure μ cannot have more than one support. The terms „support" and „supported by" should not be confused.

5. Example If μ is Lebesgue measure on [0,1], then its support is [0,1], but it is supported by]0,1[.

The generalized sequence (μ_α) of measures is weakly*-convergent to the measure μ_0 if and oly if $\mu_\alpha(\varphi) = \int_X \varphi d\mu_\alpha \to \int_X \varphi d\mu_0 = \mu_0(\varphi)$ for every bounded continuous function φ on X. In general, weak*-topology is not metrizable, but under certain conditions it is possible to consider countable sequences of measures on a topological space instead of generalized ones.

6. Theorem Let X be a metrizable space and denote by $M(X)$ the space of regular probability Borel measures on X.

Then X is a Polish space if and only if $M(X)$ is a Polish space.

7. Definition Let X be a locally convex Hausdorff space and K - its compact convex subset. We say, that probability measure μ on K represents point $x \in X$, if for every $f \in X^*$ there holds an equality

$$f(x) = \int_K f(k)\mu(dk)$$

The set of measures representing points of a compact, convex and metrizable subset of a locally convex space, supported by extreme points of this subset is always nonempty by a Choquet integral representation theorem. It can be easily checked that it is also convex, however, it need not be closed, as the following simple example shows.

8. Example Consider the set K of Example 2. For $n \in N$ we define the sequence of measures (μ_n), $\mu_n := \frac{1}{2}\delta_{a_n} + \frac{1}{2}\delta_{b_n}$, where δ_{a_n} stands for the Dirac measure concentrated on the point a_n; points a_n and b_n have cylindrical coordinates $(1, \pi + \frac{\phi}{n}, 0)$ and $(1, \frac{\phi}{n}, 0)$, respectively. If $f \in (R^3)^*$, then $f(0) = 0$ and

$$\int_K f d\mu_n = \frac{1}{2} f(a_n) + \frac{1}{2} f(b_n) = \frac{1}{2} f(a_n) - \frac{1}{2} f(a_n) = 0$$

so each of the measures μ_n represents point (0,0,0).

As $\lim a_n = (1, \pi, 0) =: a_0$ and $\lim b_n = (1,0,0) =: b_0$, we get $\mu_n \to \mu_0 := \frac{1}{2}\delta_{a_0} + \frac{1}{2}\delta_{b_0}$ in the weak* topology. We also have

$\int_K f d\mu_0 = 0$, i.e. measure μ_0 represents point (0,0,0), but $0 \notin \text{ext } K$, so $\mu_0(\text{ext } K) < 1$ (what means that μ_0 is not supported by ext K).

Let C be a closed subset of a Polish space X and consider a sequence (μ_n) of regular probability Borel measures on X convergent weakly* to the measure μ_0 and such that the support of each μ_n is contained in C. Then for every bounded continuous function φ on X with support (i.e. the smallest closed set outside of which φ vanishes) contained in the complement of C we have $\mu_n(\varphi) = 0$ for each n, hence $\mu_0(\varphi) = 0$. From this and the definition of the support of measure we have that the closure of the set of representing measures supported by extreme points of compact, convex metrizable set consists of measures representing points of that set and supported on the closure of the set of its extreme points.

References

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