A CONTRIBUTION TO THE MODELING OF NONSTATIONARY PROCESSES IN FUNCTIONALLY GRADED LAMINATES

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Abstract. The considerations are concerned with mathematical modeling of nonstationary thermomechanical processes in functionally graded laminates (FGL). The proposed modeling procedure is an extension of that based on the tolerance averaging technique [13].

Introduction

We summarize below some of studies which have been lately realized and published by a group of researches in Faculty of Mechanical Engineering and Computer Science, Czestochowa University of Technology. The object of analysis is the heat conduction and elastodynamics of two-phase multilayered solids having macroscopic properties continuously varying in the direction normal to the layering. The above solids will be referred to as the functionally graded laminates (FGL). They constitute a special case of structures made of functionally graded materials (FGM), cf. [9] and the list of references therein. The main aim of the studies is to answer how to describe thermomechanical processes occurring in FGL by means of PDEs with smooth functional coefficients. Two main lines of modeling were proposed. The first one is based on a certain generalization of the approach to the modeling of periodic structures using the periodic simplicial division and leading to the system of finite difference equations [7, 8]. The second line of modeling takes into account some concepts and assumptions of the tolerance averaging technique [13]. This technique was applied in the modeling of elastodynamics of functionally graded laminated plates [1], functionally graded laminated shells [14], functionally graded laminates with interlaminar microcracks [5, 15]. This procedure makes it possible to analyze also boundary layer phenomena in elastodynamics of functionally graded laminates [6]. Moreover, introducing the concept of slowly graded laminates with a weak transversal inhomogeneity we can decompose tolerance averaging equations into two asymptotic approximations. This approach was applied independently to elastodynamics [10-12] and heat conduction problems [2, 3].

In this contribution the considerations will be focused on the boundary layer phenomena in elastodynamics of functionally graded laminates and asymptotic approximations in elastodynamics and heat conduction of slowly graded laminates with a weak transversal inhomogeneity.

Denotations. By $0x_1x_2x_3$ we denote Carthesian orthogonal coordinate system in the physical space. Let $\Pi\times(0,L)$, $\Pi\subset \mathrm{R}^2$, be the region in this space occupied by the laminated solid in the reference configuration in which the x_3 - axis is normal to the lamina interfaces. We denote $\mathbf{e}\equiv(0,0,1)$, $\mathbf{x}\equiv(x_1,x_2)$, and t stands for the time coordinate. The partial differentiation with respect to arguments x_k, k = 1, 2, 3, is denoted by ∂_k and time differentiation by the overdot. We introduce gradient operators $\nabla=(\partial_1,\partial_2,\partial_3)$ and $\overline{\nabla}=(\partial_1,\partial_2,0)$. Throughout Sec. 2 the tensor notation is used with “dot” and “double dot” as the scalar and the double scalar products, respectively. In Sec. 3 we apply the index notation where subscripts $\alpha,\beta=1,2.$ Vectors and vector fields are denoted by small bold face letters, second-order tensors and tensor fields by capital bold face letters and higher-order tensors and tensor fields by block letters.

For an arbitrary integrable function f (f can also depend on **x** and time t) defined in $(0,L)$ the *averaging* of this function is denoted by

$$\langle f\rangle(x_3)=\frac{1}{l}\int\limits_{x_3-l/2}^{x_3+l/2} f(y)\,dy$$

To make this paper self consistent we recall in the subsequent section some of the basic concepts which were presented in [6].

1. Preliminaries

The object of considerations is a two component functionally graded laminate consists of large number of thin layers. The thickness of every layer is assumed to be the same and will be denoted by l. Let us assume that FGL is divided on m layers along its thickness L such that $L = ml$, m is a natural number and $m^{-1} << 1$. Thicknesses of lamina in the n-th layer, n = 1,...,m, are denoted by l_n', l_n''. A cross section of FGL solid and its layer are shown in Figure 1, where, f', f'' stand for physical characteristics of lamina materials (mass densities, tensors of elastic moduli or/and specific heats per unit area and a symmetric heat conduction tensors) in every pair of adjacent laminae, respectively. By $\nu'(\cdot)$, $\nu''(\cdot)$ we denote smooth function defined on $[0,L]$ representing distributions of mean volume fractions of lamina materials, $\nu'(x_3)+\nu''(x_3)=1,\quad x_3\in[0,L]$. Setting $\nu=\sqrt{\nu'\nu''}$ we refer $\nu(\cdot)$ to as the phase distribution function.

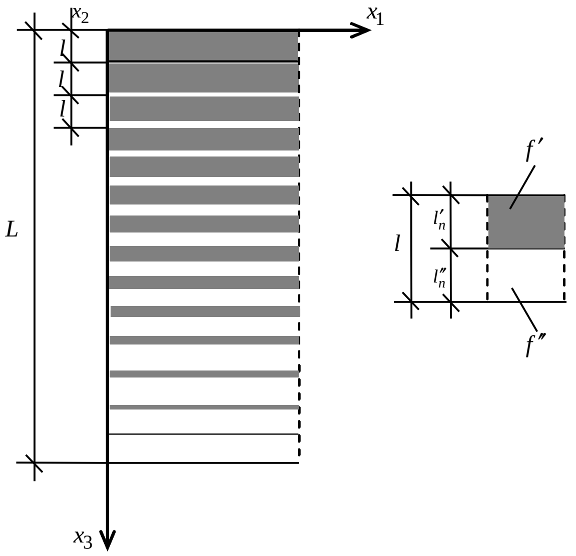

Fig. 1. A cross-section of the FGL solid and a fragment of its *n*-th layer

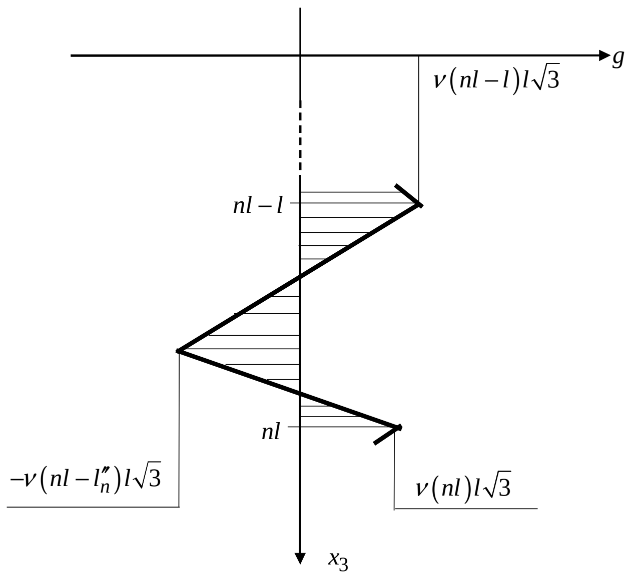

Fig. 2. A diagram of the fluctuation shape function in the *n*-th layer

Now, we recall two important notions occurring in tolerance averaging modelling technique. Function $F \in C^1([0,L])$ of argument (*F* can also depend on x and *t* as parameters) will be called slowly varying (related to length *l*, *l* << *L*, and a tolerance ε, 0 < ε << 1) if functions $l\partial_3 F$ and $O(\varepsilon F)$ are of the same order for $\varepsilon \to 0$. If this condition holds also for all derivatives of *F* then we shall write $F \in SV_\varepsilon(l)$, where ε is called a tolerance parameter. For a detailed discussion of this concept the reader is referred to [13].

Let $g:[0,L] \to \mathrm{R}$ be a continuous function the diagram of which in an arbitrary interval $[(n-1)l, nl]$, *n* = 1,...,*m*, is shown in Figure 2. This function will be referred to as the fluctuation shape function and represents a certain generalization of the saw-like function, well known in modelling of periodic laminates [13].

The basic assumption of the modelling procedure states that in every FGL mean volume fractions are slowly varying, i.e. they satisfy conditions $\nu'(\cdot) \in SV_\varepsilon(l)$, $\nu''(\cdot) \in SV_\varepsilon(l)$. This procedure will be also based on the formal assumption that for every slowly varying function $F \in SV_\varepsilon(l)$ terms $O(\varepsilon F)$ can be neglected as small when compared to *F*. This assumption will be referred to as the tolerance approximation.

2. Elastodynamics

2.1. Tolerance averaging model equations

We are to present the tolerance averaging approach to the modeling of elastodynamic problems of a linear-elastic functionally graded laminated the scheme of which was illustrated in Figure 1. By ρ', ρ'' and Ł′, Ł″ we denote mass densities and tensors of elastic moduli in every pair of adjacent laminae, respectively.

The subsequent considerations will be restricted to problems in which displacements across the thickness of every lamina can be approximated (with a certain tolerance ε) by linear functions. Let us denote by $\mathbf{w}(\mathbf{x}, x_3, t)$, $\mathbf{x} = (x_1, x_2) \in \overline{\Pi}$, $x_3 \in [0,L]$ the displacement field at time *t*. Recalling the concept of the fluctuation shape function and that of the slowly-varying function, we conclude that the aforementioned restriction can be assumed in the form of the decomposition

$$\mathbf{w}(\mathbf{x}, x_3, t) = \mathbf{u}(\mathbf{x}, x_3, t) + g(z)\mathbf{v}(\mathbf{x}, x_3, t) \tag{1}$$

where **u**, **v** are slowly varying functions of argument x_3:

$$\mathbf{u}(\mathbf{x},\cdot,t)\in SV_{\varepsilon}(l), \quad \mathbf{v}(\mathbf{x},\cdot,t)\in SV_{\varepsilon}(l) \tag{2}$$

Using the tolerance approximation we also obtain

$$\mathbf{u}(\mathbf{x},x_3,t)=\langle \mathbf{w}\rangle(\mathbf{x},x_3,t)$$

for every $(\mathbf{x},x_3)\in \Pi\times[l/2, L-l/2]$ and every time t.

Governing equations for basic kinematic unknowns averaged displacement **u** and fluctuation amplitude **v** will be derived from the principle of stationary action. To this end the integrand in the action functional will be assumed in the form

$$\mathrm{L}=\frac{1}{2}\langle \rho\dot{\mathbf{w}}\cdot\dot{\mathbf{w}}\rangle-\frac{1}{2}\langle \nabla\mathbf{w}:\mathrm{C}:\nabla\mathbf{w}\rangle$$

where the displacement field **w** is restricted by conditions (1), (2). Using the tolerance approximation and recalling that $\mathbf{e}=(0,0,1)$ we shall approximate $\nabla\mathbf{w}$ by $\nabla\mathbf{u}+g'(x_3)\mathbf{e}\otimes\mathbf{v}+g(x_3)\overline{\nabla}\mathbf{v}$. Similarly we conclude that:

$$\begin{aligned}\langle\rho\rangle&=\nu'(x_3)\,\rho'+\nu''(x_3)\,\rho''\\ \langle\mathrm{C}\rangle&=\nu'(x_3)\,\mathrm{C}'+\nu''(x_3)\,\mathrm{C}''\end{aligned} \tag{3}$$

We shall also introduce denotations:

$$\begin{aligned}[\mathrm{C}]&\equiv 2\sqrt{3}\nu(x_3)(\mathrm{C}'-\mathrm{C}'')\cdot\mathbf{e}\\ [\mathrm{C}]^T&\equiv 2\sqrt{3}\,\nu(x_3)\,\mathbf{e}\cdot(\mathrm{C}'-\mathrm{C}'')\\ \{\mathbf{C}\}&\equiv 12\mathbf{e}\cdot\left(\mathrm{C}'\nu''(x_3)+\mathrm{C}''\nu'(x_3)\right)\cdot\mathbf{e}\end{aligned} \tag{4}$$

After rather lengthy manipulations the Euler-Lagrange equations for L lead to the following *equations of motion*:

$$\begin{aligned}&\langle\rho\rangle\ddot{\mathbf{u}}-\nabla\cdot\mathbf{S}=\mathbf{0}\\ &l^2\nu^2\langle\rho\rangle\ddot{\mathbf{v}}-l^2\nu^2\overline{\nabla}\cdot\left(\langle\mathrm{C}\rangle:\overline{\nabla}\mathbf{v}\right)+\mathbf{h}=\mathbf{0}\end{aligned} \tag{5}$$

and *constitutive equations*:

$$\begin{aligned}\mathbf{S}&=\langle\mathrm{C}\rangle:\nabla\mathbf{u}+[\mathrm{C}]\cdot\mathbf{v}\\ \mathbf{h}&=\{\mathbf{C}\}\cdot\mathbf{v}+[\mathrm{C}]^T:\nabla\mathbf{u}\end{aligned} \tag{6}$$

Equations (5), (6) for the basic kinematic unknowns **u** and **v**, with coefficients defined by (3) and (4), together with formulae (1), (2) represent an averaged mathematical model of the FGL solid under consideration.

2.2. Boundary layer equation

Let us decompose the fluctuation amplitude **v** in equations (5), (6) into the sum

$$\mathbf{v} = -\{\mathbf{C}\}^{-1} \cdot [\mathrm{C}]^{T} : \nabla \mathbf{u} + \mathbf{r} \tag{7}$$

where **r** is a new kinematical unknown slowly varying in x_3. Neglecting in equations (5) terms depending on the microstructure length l we obtain $\mathbf{r} \equiv \mathbf{0}$. That is why **r** is referred to as the intrinsic fluctuation amplitude. At the same time from (1) and (7) we obtain

$$\mathbf{w}(\mathbf{x}, x_3, t) = \mathbf{u}(\mathbf{x}, x_3, t) - g(x_3)\{\mathbf{C}\}^{-1} \cdot [\text{Ł}]^{T} : \nabla \mathbf{u}(\mathbf{x}, x_3, t) + g(x_3)\mathbf{r}(\mathbf{x}, x_3, t) \tag{8}$$

where $g\mathbf{r}$ represents the intrinsic fluctuation of displacement.

In order to formulate governing equations for functions **u** and **r** we shall use the notion of homogenized tensor of elastic moduli

$$\text{Ł}^{h} \equiv \langle \text{Ł} \rangle - [\text{Ł}] \cdot \{\mathbf{C}\}^{-1} \cdot [\text{Ł}]^{T}$$

We also introduce the following differential operators:

$$A\mathbf{u} \equiv \langle \rho \rangle \ddot{\mathbf{u}} - \nabla \cdot \left(\text{Ł}^{h} : \nabla \mathbf{u}\right),$$

$$D\mathbf{r} \equiv l^2 \nu^2 \left[\langle \rho \rangle \ddot{\mathbf{r}} - \overline{\nabla} \cdot \left(\langle \text{Ł} \rangle : \overline{\nabla} \mathbf{r} \right) \right] + \{\mathbf{C}\} \cdot \mathbf{r},$$

$$F\mathbf{u} \equiv l^2 \nu^2 \left[\langle \rho \rangle \{\mathbf{C}\}^{-1} \cdot [\text{Ł}]^{T} : \nabla \ddot{\mathbf{u}} - \overline{\nabla} \cdot \left(\langle \text{Ł} \rangle : \overline{\nabla} \cdot \left(\{\mathbf{C}\}^{-1} \cdot [\text{Ł}]^{T} : \nabla \mathbf{u} \right) \right) \right]$$

Combining equations (5) with formula (7) we obtain the coupled system of the model equations for **u** and **r**:

$$\begin{aligned} A\mathbf{u} &= [\text{Ł}] : \nabla \mathbf{r} \\ D\mathbf{r} &= F\mathbf{u} \end{aligned} \tag{9}$$

which is an alternative to model equations (5). It can be shown, [6], that equation

$$D\mathbf{r} = \mathbf{0} \tag{10}$$

describe certain near-initial and near-boundary phenomena strictly related to the initial and boundary conditions (on the part of boundary intersecting interfaces between laminae) imposed on **r**. That is why equation (10) will be referred to as the boundary-layer equation where the term "boundary" is related both to time and space.

2.3. Asymptotic approximations

Now we are to show that under certain conditions, the coupled macro-micro equations (9) can be decomposed into approximate model equations describing independently the macro- and micro-response of the laminated solid.

Let us denote by $\|\cdot\|_n$ an arbitrary but fixed norm in the linear space of all *n*-th order tensors related to space E^3. Let us also define

$$\eta = \sup_{x_3 \in [0,L]} \frac{\| [Ł] \|_3}{\| \langle Ł \rangle \|_4}$$

as a transversal inhomogeneity parameter of the laminates under consideration. These laminates are said to have a weak transversal inhomogeneity provided that η satisfies condition $0 < \eta << 1$. This kind of inhomogeneity takes place for laminae reinforced by long high-strength fibres. In this case, the components of the elastic moduli tensor Ł which are related to the Ox_1x_2 - plane are strongly different in adjacent laminae; the remaining components attain only small jumps across the lamina interfaces. The above condition holds true for many laminated materials used in civil and mechanical engineering.

The subsequent analysis will be restricted to laminated solids with a weak transversal inhomogeneity, where η is treated as a certain small parameter.

Notice that the values of $[Ł]^T : \nabla \mathbf{r}$ and $F\mathbf{u}$ are of an order $O(\mathbf{r}\eta)$, $O(\mathbf{u}\eta)$, respectively. Moreover, $A\mathbf{u}$ and $D\mathbf{r}$ are of the same order as **u** and **r**, respectively. Let us assume that the solutions to Eqs. (9) can be represented in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_\Delta, \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}_\Delta \tag{11}$$

where: $\mathbf{u}_0 \in O(\eta^0)$, $\mathbf{r}_0 \in O(\eta^0)$, $\mathbf{u}_\Delta \in O(\eta)$, $\mathbf{r}_\Delta \in O(\eta)$. Bearing in mind (11) and applying the limit passage $\eta \to 0$ to Eqs. (9), we obtain the following system of equations for $\mathbf{u}_0$, $\mathbf{r}_0$:

$$\begin{aligned} A\mathbf{u}_0 &= \mathbf{0} \\ D\mathbf{r}_0 &= \mathbf{0} \end{aligned} \tag{12}$$

We shall assume that $\mathbf{u}_0, \mathbf{r}_0$ satisfy the boundary/initial conditions which coincide with those imposed on **u** and **r**, respectively. From (9), (11), (12) we conclude that $\mathbf{u}_\Delta, \mathbf{r}_\Delta$ have to satisfy the equations:

$$\begin{aligned} A\mathbf{u}_\Delta &= [Ł]^T : \nabla(\mathbf{r}_0 + \mathbf{r}_\Delta) \\ D\mathbf{r}_\Delta &= F(\mathbf{u}_0 + \mathbf{u}_\Delta) \end{aligned} \tag{13}$$

as well as the corresponding homogeneous boundary/initial conditions. It has to be emphasized that Eq. $(12)_1$ represents the model obtained by the homogenization technique. Equation $(12)_2$ describes the phenomena related to the fluctuations of boundary and initial displacements. Equations (12) will be referred to as the first order approximation model for slowly graded laminates with a weak transversal inhomogeneity. In the framework of this model the basic kinematic unknowns **u**, **r** are approximated by $\mathbf{u}_0, \mathbf{r}_0$, respectively. In this case formula (11) yields

$$\mathbf{u} = \mathbf{u}_0 + O(\eta), \quad \mathbf{r} = \mathbf{r}_0 + O(\eta)$$

i.e. we deal with an asymptotic approximation of an order $O(\eta)$.

Now we assume that $\mathbf{u}_\Delta, \mathbf{r}_\Delta$ can be written in the form

$$\mathbf{u}_\Delta = \mathbf{u}_1 + o(\eta), \quad \mathbf{r}_\Delta = \mathbf{r}_1 + o(\eta)$$

where $\mathbf{u}_1$, $\mathbf{r}_1$ are assumed to be linear functions of η. Applying limit passage $\eta \to 0$ to equations (13) we obtain the following system of equations for $\mathbf{u}_1$, $\mathbf{r}_1$:

$$\begin{aligned} A\mathbf{u}_1 &= [Ł]^T : \nabla\mathbf{r}_0 \\ D\mathbf{r}_1 &= F\mathbf{u}_0 \end{aligned} \tag{14}$$

The above equations are assumed to hold together with homogeneous boundary and initial conditions. These conditions are assumed to have the same form as pertinent homogeneous conditions for $\mathbf{u}_\Delta$, $\mathbf{r}_\Delta$, respectively. Equations (11) together with (13) will be referred to as the second order approximation model. In this case we deal with an asymptotic approximation of order $o(\eta)$ given by

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 + o(\eta), \quad \mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1 + o(\eta)$$

It can be seen that the right-hand sides of Eqs. (14) are known provided that the boundary/initial value problem for Eqs. (12) has been previously solved. Summarizing the obtained results we state that model equations (9) for **u** and **r** can be decomposed to the simplified asymptotic form given by equations (12) and

(14). It can be seen that the presented modelling line leads to the formulation of higher-order approximation models.

3. Heat conduction

3.1. Tolerance averaging equations

We are to show that the concept of modeling of functionally graded laminates with a weak transversal inhomogeneity can be applied also to the analysis of heat conduction problem. We will consider a functionally graded laminated rigid heat conductor the material geometry of which was described in Sec. 1, cf. Figure 1. By c', c'' and K'_{ij}, K''_{ij} we denote a specific heat (per unit area) and a symmetric heat conduction tensor in laminae in the n layer with thicknesses l'_n, l''_n, respectively. Every lamina is assumed to be homogeneous with $x_3 =$ const as the material symmetry plane. Hence K'_{ij}, K''_{ij} are constant and $K'_{\alpha 3} = K'_{3\alpha} = 0$, $K''_{\alpha 3} = K''_{3\alpha} = 0$, $\alpha = 1, 2$.

Let $\theta = \theta(\mathbf{x}, x_3, t)$, $\mathbf{x} = (x_1, x_2) \in \overline{\Pi}$, $x_3 \in [0, L]$, stand for a temperature field at time $t \geq 0$. Function $\theta(\cdot)$ is assumed to be continuous and satisfy in every lamina the Fourier heat conduction equation

$$c(x_3)\dot{\theta}(\mathbf{x}, x_3, t) - \partial_i \left[K_{ij}(x_3)\partial_j \theta(\mathbf{x}, x_3, t) \right] = 0 \tag{15}$$

together with the heat flux continuity conditions on the interfaces between adjacent laminae.

The line of modeling will be similar to that presented in Sec. 2 for the elastodynamic problems. We are to consider the class of temperature fields θ in the form

$$\theta(\mathbf{x}, x_3, t) = \vartheta(\mathbf{x}, x_3, t) + g(x_3)\psi(\mathbf{x}, x_3, t) \tag{16}$$

where functions ϑ and ψ are slowly varying in argument $x_3 \in [0, L]$. They constitute the basic unknowns of the modeling and will be called the averaged temperature and the temperature fluctuation amplitude, respectively.

It can be shown [13], that the governing equations for ϑ, ψ have the form:

$$\begin{aligned} &\langle c \rangle \dot{\vartheta} - \langle K_{\alpha\beta} \rangle \partial_\alpha \partial_\beta \vartheta - \langle K_{33} \rangle \partial_3 \partial_3 \vartheta - [K_{33}] \partial_3 \psi = 0 \\ &l^2 \nu^2 \dot{\psi} - l^2 \nu^2 \langle K_{\alpha\beta} \rangle \partial_a \partial_\beta \psi + \{K_{33}\} \psi + [K_{33}] \partial_3 \vartheta = 0 \end{aligned} \tag{17}$$

where:

$$
\begin{aligned}
&\langle c \rangle = \nu'(x_3)c' + \nu''(x_3)c'' \\
&\langle K_{\alpha\beta} \rangle = \nu'(x_3)K'_{\alpha\beta} + \nu''(x_3)K''_{\alpha\beta} \\
&[K_{33}] \equiv 2\sqrt{3}\nu(x_3)(K'_{33} - K''_{33}) \\
&\{K_{33}\} \equiv 12\left(K'_{33}\nu''(x_3) + K''_{33}\nu'(x_3)\right)
\end{aligned}
$$

Equations (17) represent a tolerance model of the heat conduction in a functionally graded laminate under considerations. They constitute a basis for the subsequent analysis.

3.2. Asymptotic approximations

The main problem we are going to solve is how to separate model equations (17) for the averaged temperature and temperature fluctuations. The subsequent analysis will be based on the concept of the weak transversal inhomogeneity. To explain this concept let us reformulate the tolerance model equations. We shall introduce an alternative form of coefficients used before $k = (K'_{33} + K''_{33})/2$ and $\eta = (K''_{33} - K'_{33})/2.$ Under assumption $K'_{33} \cong K''_{33}$ what is taking place for laminates with a weak transversal inhomogeneity equations (17) can be written in the following form:

$$
\begin{aligned}
&\langle c \rangle \dot{\vartheta} - \langle K_{\alpha\beta} \rangle \partial_\alpha \partial_\beta \vartheta - k\partial_3\partial_3\vartheta = 4\sqrt{3}\eta\nu\partial_3\psi \\
&l^2 \langle c \rangle \dot{\psi} - l^2 \langle K_{\alpha\beta} \rangle \partial_\alpha \partial_\beta \psi + 12k\psi = -4\sqrt{3}\nu\partial_3\vartheta
\end{aligned}
\tag{18}
$$

It has to be emphasized that the parameter η for a weak transversal inhomogeneity satisfies condition $0 < \eta \ll 1$ and occurs only on the right-hand sides of equations (18). Thus, the mutual impact of the averaged temperature ϑ and the temperature fluctuations ψ depends directly on the value of parameter η.

Due to the presence of the small parameter η in the resulting tolerance model equations (18), we shall apply the asymptotic approach to the analysis of initial/boundary problems. We shall seek an asymptotic approximation of solution to equations (18) in the form of expansions $\vartheta = \vartheta_0 + \eta\vartheta_1 + O(\eta^2)$, $\psi = \psi_0 + \eta\psi_1 + O(\eta^2)$. Substituting these expansions into tolerance model equations (17) and neglecting terms depending on the small parameter η we obtain:

$$\begin{aligned} &\langle c\rangle \dot{\vartheta}_0 - \langle K_{\alpha\beta}\rangle \partial_\alpha \partial_\beta \vartheta_0 - k\partial_3\partial_3\vartheta_0 = 0 \\ &l^2\langle c\rangle \dot{\psi}_0 - l^2\langle K_{\alpha\beta}\rangle \partial_\alpha\partial_\beta \psi_0 + 12k\psi_0 = 0 \end{aligned} \tag{19}$$

The above equations are assumed to be considered together with the initial/boundary conditions coinciding with those imposed on ϑ and ψ. Equations (19) represent what will be called the first order asymptotic approximation of the tolerance model equations (18). The first approximation of solution to the equations (18) is $\vartheta = \vartheta_0$ and $\psi = \psi_0$. Similarly, substituting the proposed asymptotic expansions into tolerance model equations (18) and neglecting terms depending on the parameter η^2 we obtain:

$$\begin{aligned} &\langle c\rangle \dot{\vartheta}_1 - \langle K_{\alpha\beta}\rangle \partial_\alpha \partial_\beta \vartheta_1 - k\partial_3\partial_3\vartheta_1 = 4\sqrt{3}\eta v\partial_3\psi_0 \\ &l^2\langle c\rangle \dot{\psi}_1 - l^2\langle K_{\alpha\beta}\rangle \partial_\alpha\partial_\beta \psi_1 + 12k\psi_1 = -4\sqrt{3}v\partial_3\vartheta_0 \end{aligned} \tag{20}$$

Thus, the second order approximation is determined by formula $\vartheta = \vartheta_0 + \vartheta_1$ and $\psi = \psi_0 + \psi_1$, where ϑ_0, ϑ_1, ψ_0, ψ_1 are found as solutions to equations (19), (20) for the certain initial/boundary problem.

4. Final remarks

The main aim of the present contribution was to expose some basic ideas related to elastodynamics and heat conduction in functionally graded laminates (FGL). The general conclusion is that the tolerance averaging technique, so far applied to periodic structures also constitutes a proper tool of modeling for materials with deterministic but space varying structure. The results obtained above has been partly published and hence the contribution can be treated as general summary of results rather than a detailed discussion of the problem. It should be mentioned that the tolerance averaging technique can be also applied to more general form of solids with deterministic and slowly varying microstructure. The aforementioned problems are in the course of the present researches.

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