JAM CM
Journal of Applied Mathematics and Computational Mechanics 2026, 25(3), 5-19
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2026.3.01 e-ISSN 2353-0588

A NOTE ON AN EXPANSION METHOD FOR CALOGERO-TYPE SERIES INVOLVING ZEROS OF BESSEL FUNCTIONS

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Received: 22 March 2026; Accepted: 1 July 2026

Abstract. This paper presents a systematic reduction of selected infinite series involving the zeros of Bessel functions to fundamental Calogero-type expressions. The analysis is based on a general partial fraction decomposition formula for rational functions of the form $b^N/(a-b)^M$, which enables the transformation of seemingly complicated identities into structured sums with a transparent algebraic form. Applying this approach to formulas recently derived by Langowski and Nowak, we show that most of their results admit representations in terms of basic Calogero-type series. The method not only simplifies the structure of these identities but also reveals that several of them coincide with earlier results obtained by Urbanowicz. In addition, new identities are derived as direct consequences of the proposed decomposition scheme. The results provide a unified algebraic framework for handling broad classes of series constructed from Bessel zeros and contribute to a clearer understanding of their interrelations.

MSC 2010: *33C10, 40A05*
Keywords: *mathematics, computational mechanics, Bessel functions, Calogero series, Rayleigh series, partial fraction decomposition, Bessel functions roots*

1. Introduction

Bessel functions constitute a fundamental component of the mathematical description of numerous problems in physics and engineering. In many applications, the resulting solutions involve infinite series constructed from the zeros of these functions. One of the earliest and most fundamental problems involving such a series $\sum_{k=1}^{\infty} {j_{\nu,k}}^{-2l}$ (where l is a positive integer and ν is a function order), was studied in detail in the nineteenth century by Lord Rayleigh [1]. Recursive formulas for these series were later derived independently by Meiman [2] and Kishore [3].

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Related infinite series based on the zeros of derivatives of ordinary Bessel functions were investigated toward the end of the twentieth century by Muldoon and Reza [4]. More involved series, containing differences of squared zeros in the denominator $\sum_{k=1,k\neq n}^{\infty}\left({j_{\nu,n}}^2-{j_{\nu,k}}^2\right)^{-l}$, were introduced and analyzed in the late 1970s by Calogero [5-7]. Series of this type (for a fixed Bessel function order ν) have practical applications, as demonstrated in the work of Urbanowicz et al., where they were used in the context of inverse Laplace transform solutions of a dynamic viscosity function [8] used to model unsteady pipe flows. Further developments were presented by Pedersen [9], who analyzed non-relativistic quantum systems using the Thomas-Reiche-Kuhn sum rule. By employing an identity for static polarizability, he derived series involving the zeros of Bessel functions of two consecutive orders. He also referred to earlier work by Afanasiev [10], who studied eigenfunctions and eigenvalues of the free Schrödinger equation in a cylindrical domain and obtained Calogero-type identities in a related, though simpler, setting.

In the paper "*On new identities involving zeros of Bessel functions*", Langowski and Nowak [11] presented several previously unknown general formulas for series constructed from the zeros of Bessel functions and their derivatives. Their approach relies on the fundamental fact that the Fourier-Bessel and Fourier-Dini systems form an orthogonal basis in the Hilbert space $L^2\left((0,1),dx\right)$. In the present note, we employ a general partial fraction decomposition (PFD) formula of the form

$$\frac{b^N}{(a-b)^M}=\sum_{n=M-N}^{M}\binom{N}{M-n}a^{N-M+n}(-1)^{M-n}\frac{1}{(a-b)^n} \tag{1}$$

valid for integers $1\leq N<M$, where $n=M-N,M-N+1,\ldots,M$ and $a,b>0$. The function $\binom{x}{y}=\frac{x!}{y!(x-y)!}$ defines binomial coefficients that can be written with the help of the factorial function. This identity Eq. (1) will be used to decompose selected expressions from the Langowski and Nowak paper [11] into general solutions of the Calogero type. We also observe that formulas ($F2$), ($F4$), ($F9$), and ($F10$) from their paper [11] were previously derived in the work of Urbanowicz [12]; this will be confirmed in the present note. Other results from Langowski and Nowak paper, namely ($F6$), ($F8$), ($F12$), and ($F15$), concern Bessel zeros of negative order $\nu<0$, which have limited practical relevance in engineering applications and therefore fall outside the scope of our study.

2. Reduction of Langowski-Nowak identities to Calogero-type series

In this section, we will analyze the majority of the complicated series presented in the Langowski and Nowak paper [11], the discussion follows the original numbering of formulas adopted in that work, beginning with the first formula. It should be noted that in all formulas, when the exponent is even, the relations $(a^2-b^2)^{-n}=(b^2-a^2)^{-n}$ hold. In contrast, when the exponent is odd, we have $(a^2-b^2)^{-n}=-(b^2-a^2)^{-n}$.

a) Solution ($F1$). Using the universal formula for PFD (1) and the solutions (A5) and (A4) from the Urbanowicz paper [12], we obtain:

$$F1 = \sum_{k=1}^{\infty} \frac{{j_{\nu,k}}^2}{\left({j_{\nu+1,n}}^2 - {j_{\nu,k}}^2\right)^2} = {j_{\nu+1,n}}^2 \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,n}}^2 - {j_{\nu,k}}^2\right)^2} - \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,n}}^2 - {j_{\nu,k}}^2\right)} =$$
$${j_{\nu+1,n}}^2 \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu+1,n}}^2\right)^2} + \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu+1,n}}^2\right)} = {j_{\nu+1,n}}^2 \frac{1}{4{j_{\nu+1,n}}^2} + 0 = \frac{1}{4} \quad (2)$$

b) Solution ($F2$). Using Table A1 with results derived in the Urbanowicz paper [12], we find solution (A8), in which if we substitute ν instead of $\nu - 1$:

$$F2 = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,n}}^2 - {j_{\nu+1,k}}^2\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,k}}^2 - {j_{\nu,n}}^2\right)^2} = \frac{1}{4{j_{\nu,n}}^2} - \frac{\nu+1}{{j_{\nu,n}}^4} \quad (3)$$

c) Solution ($F3$). This solution can be rewritten similarly to $F1$ using PFD identity (1):

$$F3 = \sum_{k=1}^{\infty} \frac{{j_{\nu+1,k}}^2}{\left({j_{\nu,n}}^2 - {j_{\nu+1,k}}^2\right)^2} = {j_{\nu,n}}^2 \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,n}}^2 - {j_{\nu+1,k}}^2\right)^2} - \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,n}}^2 - {j_{\nu+1,k}}^2\right)} =$$
$${j_{\nu,n}}^2 \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,k}}^2 - {j_{\nu,n}}^2\right)^2} + \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,k}}^2 - {j_{\nu,n}}^2\right)} \quad (4)$$

By substituting $\nu - 1$ for ν in Eq. (4) and using the solutions (A7) and (A8) from [12], we obtain:

$$F3 = {j_{\nu-1,n}}^2 \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu-1,n}}^2\right)^2} + \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu-1,n}}^2\right)} = {j_{\nu-1,n}}^2 \left(\frac{1}{4{j_{\nu-1,n}}^2} - \frac{\nu}{{j_{\nu-1,n}}^4}\right) +$$
$$\frac{\nu}{{j_{\nu-1,n}}^2} = \frac{1}{4} \quad (5)$$

d) Solution ($F4$). Using solution (A5) from the Urbanowicz paper [12] the result is quickly received:

$$F4 = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+1,n}}^2 - {j_{\nu,k}}^2\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu+1,n}}^2\right)^2} = \frac{1}{4{j_{\nu+1,n}}^2} \quad (6)$$

e) Solution ($F5$). Here, it is necessary to use the formulas developed in the accompanying paper [13], which will be recalled in this paper below:

$$F(z) = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - z^2\right)^1} = \frac{\nu}{2z^2} - \frac{1}{2z} \frac{J_\nu'(z)}{J_\nu(z)} \quad (7)$$

$$\frac{F(z)'}{2z} = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - z^2\right)^2} = \frac{1}{4z^2} \left(\frac{J_\nu'(z)}{J_\nu(z)}\right)^2 + \frac{1}{2z^3} \left(\frac{J_\nu'(z)}{J_\nu(z)}\right)^1 + \frac{z^2 - \nu(\nu+2)}{4z^4} \quad (8)$$

If we introduce now for $z = {j'_{\nu,n}}^2$ any zero of function $J_\nu'(z)$ in formulas (7) and (8) we receive the following solutions:

$$\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu,k}}^2-{j'_{\nu,n}}^2\right)^1}=\frac{\nu}{2{j'_{\nu,n}}^2}\ \text{and}\ \sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu,k}}^2-{j'_{\nu,n}}^2\right)^2}=\frac{{j'_{\nu,n}}^2-\nu(\nu+2)}{4{j'_{\nu,n}}^4} \tag{9}$$

Knowing the above solutions, the formula $F5$ can be decomposed with the help of PFD formula (1) into basic formulas in the following way:

$$F5=\sum_{k=1}^{\infty}\frac{{j_{\nu,k}}^2}{\left({j'_{\nu,n}}^2-{j_{\nu,k}}^2\right)^2}={j'_{\nu,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu,n}}^2-{j_{\nu,k}}^2\right)^2}-\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu,n}}^2-{j_{\nu,k}}^2\right)}=$$
$${j'_{\nu,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu,k}}^2-{j'_{\nu,n}}^2\right)^2}+\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu,k}}^2-{j'_{\nu,n}}^2\right)}={j'_{\nu,n}}^2\frac{{j'_{\nu,n}}^2-\nu(\nu+2)}{4{j'_{\nu,n}}^4}+\frac{\nu}{2{j'_{\nu,n}}^2}=$$
$$\frac{{j'_{\nu,n}}^2-\nu(\nu+2)}{4{j'_{\nu,n}}^2}+\frac{\nu}{2{j'_{\nu,n}}^2}=\frac{1}{4}-\frac{\nu(\nu+2)}{4{j'_{\nu,n}}^2}+\frac{\nu}{2{j'_{\nu,n}}^2}=\frac{1}{4}-\frac{\nu^2}{4{j'_{\nu,n}}^2}-\frac{\nu}{2{j'_{\nu,n}}^2}+\frac{\nu}{2{j'_{\nu,n}}^2}=\frac{1}{4}-\frac{\nu^2}{4{j'_{\nu,n}}^2} \tag{10}$$

f) Solution ($F7$). Using the Weierstrass product representation for $J'_\nu(z)$ and taking a logarithmic derivative, and then using the Bessel differential equation, Fattah [14] received such a series representation:

$$\sum_{k=1}^{\infty}\frac{1}{\left(j'_{\nu,k}\right)^2-z^2}=\frac{\nu-1}{2z^2}-\frac{1}{2z}\frac{J''_\nu(z)}{J'_\nu(z)} \tag{11}$$

By means of the well-known recurrence relations, we can express:

$$\frac{J''_\nu(z)}{J'_\nu(z)}=\left(\frac{\nu^2}{z^2}-1\right)\frac{J_\nu(z)}{J'_\nu(z)}-\frac{1}{z} \tag{12}$$

After simplification and rearrangement, the following expression is obtained:

$$\sum_{k=1}^{\infty}\frac{1}{\left(j'_{\nu,k}\right)^2-z^2}=\frac{\nu}{2z^2}-\left(\frac{\nu^2-z^2}{2z^3}\right)\frac{J_\nu(z)}{J'_\nu(z)} \tag{13}$$

By differentiating this formula and multiplying the result by $1/(2z)$, we obtain the series representation:

$$\sum_{k=1}^{\infty}\frac{1}{\left(\left(j'_{\nu,k}\right)^2-z^2\right)^2}=\frac{z^2-\nu(\nu+2)}{4z^4}-\frac{\nu^2}{2z^5}\left(\frac{J_\nu(z)}{J'_\nu(z)}\right)+\frac{\left(\nu^2-z^2\right)^2}{4z^6}\left(\frac{J_\nu(z)}{J'_\nu(z)}\right)^2 \tag{14}$$

For $z^2=j_{p,n}^2$, formulas (13) and (14) give the following series representation:

$$\sum_{k=1}^{\infty}\frac{1}{{j'_{\nu,k}}^2-{j_{\nu,n}}^2}=\frac{\nu}{2{j_{\nu,n}}^2}\ \text{and}\ \sum_{k=1}^{\infty}\frac{1}{{j_{\nu,n}}^2-{j'_{\nu,k}}^2}=-\frac{\nu}{2{j_{\nu,n}}^2} \tag{15}$$

$$\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu,k}}^2-{j_{\nu,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu,n}}^2-{j'_{\nu,k}}^2\right)^2}=\frac{{j_{\nu,n}}^2-\nu(\nu+2)}{4{j_{\nu,n}}^4} \tag{16}$$

The Bessel differential equation for the function $J_\nu(z)$ is:

$$z^2 J_\nu''(z) + zJ_\nu'(z) + (z^2 - \nu^2)J_\nu(z) = 0 \tag{17}$$

In the special case when $z = \nu$, i.e., when the argument equals the order of the function, the following result from Eq. (17):

$$\frac{J_\nu''(z)}{J_\nu'(z)} = -\frac{1}{\nu} \tag{18}$$

By inserting the result derived above into formula (11), we arrive at the following new series solution:

$$\sum_{k=1}^{\infty} \frac{1}{\left(j_{\nu,k}'\right)^2 - \nu^2} = \frac{\nu-1}{2\nu^2} + \frac{1}{2\nu}\frac{1}{\nu} = \frac{1}{2\nu} \tag{19}$$

The series $F7$ under analysis can be decomposed into three components using partial fraction decomposition:

$$\frac{x}{(x-b)(a-x)^2} = \frac{b}{(a-b)^2} \cdot \frac{1}{x-b} + \frac{b}{(a-b)^2} \cdot \frac{1}{a-x} + \frac{a}{a-b} \cdot \frac{1}{(a-x)^2} \tag{20}$$

so

$$F7 = \sum_{k=1}^{\infty} \frac{{j_{\nu,k}'}^2}{\left({j_{\nu,k}'}^2 - \nu^2\right)\left({j_{\nu,n}}^2 - {j_{\nu,k}'}^2\right)^2} = \frac{\nu^2}{\left(j_{\nu,n}^2 - \nu^2\right)^2} \sum_{k=1}^{\infty} \left[\frac{1}{j_{\nu,k}'^2 - \nu^2} + \frac{1}{j_{\nu,n}^2 - j_{\nu,k}'^2}\right] + \frac{j_{\nu,n}^2}{j_{\nu,n}^2 - \nu^2} \sum_{k=1}^{\infty} \frac{1}{\left(j_{\nu,n}^2 - j_{\nu,k}'^2\right)^2} = \frac{\nu^2}{\left(j_{\nu,n}^2 - \nu^2\right)^2} \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}'^2 - \nu^2} + \frac{\nu^2}{\left(j_{\nu,n}^2 - \nu^2\right)^2} \sum_{k=1}^{\infty} \frac{1}{j_{\nu,n}^2 - j_{\nu,k}'^2} + \frac{j_{\nu,n}^2}{j_{\nu,n}^2 - \nu^2} \sum_{k=1}^{\infty} \frac{1}{\left(j_{\nu,n}^2 - j_{\nu,k}'^2\right)^2} \tag{21}$$

By employing the series obtained in (15), (16), and (19), we arrive at the following final result:

$$F7 = \frac{\nu^2}{\left(j_{\nu,n}^2 - \nu^2\right)^2} \frac{1}{2\nu} - \frac{\nu^2}{\left(j_{\nu,n}^2 - \nu^2\right)^2} \frac{\nu}{2{j_{\nu,n}}^2} + \frac{j_{\nu,n}^2}{j_{\nu,n}^2 - \nu^2} \frac{\left({j_{\nu,n}}^2 - \nu(\nu+2)\right)}{4{j_{\nu,n}}^4} = \frac{\nu}{2\left(j_{\nu,n}^2 - \nu^2\right)^2} - \frac{\nu^3}{2{j_{\nu,n}}^2\left(j_{\nu,n}^2 - \nu^2\right)^2} + \frac{\left({j_{\nu,n}}^2 - \nu(\nu+2)\right)}{4{j_{\nu,n}}^2\left(j_{\nu,n}^2 - \nu^2\right)} = \frac{1}{4j_{\nu,n}^2} \tag{22}$$

g) Solution ($F9$). This solution exactly matches the result (A11) derived in the Urbanowicz paper [12]:

$$F9 = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+2,n}}^2 - {j_{\nu,k}}^2\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j_{\nu+2,n}}^2\right)^2} = \frac{1}{16(\nu+1)^2} \tag{23}$$

h) Solution ($F10$). Under the substitution $\nu = p - 2$, solution ($F10$) coincides with solution (A14) derived by Urbanowicz [12]:

$$F10 = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,n}}^2 - {j_{\nu+2,k}}^2\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu+2,k}}^2 - {j_{\nu,n}}^2\right)^2} =$$
$$\sum_{k=1}^{\infty} \frac{1}{\left({j_{p,k}}^2 - {j_{p-2,n}}^2\right)^2} = \frac{1}{16(p-1)^2} - \frac{p}{{j_{p-2,n}}^4} = \frac{1}{16(\nu+1)^2} - \frac{\nu+2}{{j_{\nu,n}}^4} \tag{24}$$

i) Solution ($F11$). Similarly to solution ($F7$), we apply the partial fraction decomposition formula (20) to split the original Langowski and Nowak formula into simpler series:

$$F11 = \sum_{k=1}^{\infty} \frac{{j'_{\nu,k}}^2}{\left({j'_{\nu,k}}^2 - \nu^2\right)\left({j_{\nu+1,n}}^2 - {j'_{\nu,k}}^2\right)^2} = \frac{\nu^2}{\left(j_{\nu+1,n}^2 - \nu^2\right)^2} \sum_{k=1}^{\infty} \frac{1}{j'^2_{\nu,k} - \nu^2} +$$
$$\frac{\nu^2}{\left(j_{\nu+1,n}^2 - \nu^2\right)^2} \sum_{k=1}^{\infty} \frac{1}{j_{\nu+1,n}^2 - j'^2_{\nu,k}} + \frac{j_{\nu+1,n}^2}{j_{\nu+1,n}^2 - \nu^2} \sum_{k=1}^{\infty} \frac{1}{\left(j_{\nu+1,n}^2 - j'^2_{\nu,k}\right)^2} \tag{25}$$

From equation (19), the solution for the first series on the right-hand side is known, and the two remaining series solutions are determined below.

$$\sum_{k=1}^{\infty} \frac{1}{j_{\nu+1,n}^2 - j'^2_{\nu,k}} = -\sum_{k=1}^{\infty} \frac{1}{j'^2_{\nu,k} - j_{\nu+1,n}^2} = ? \text{ and}$$
$$\sum_{k=1}^{\infty} \frac{1}{\left(j_{\nu+1,n}^2 - j'^2_{\nu,k}\right)^2} = \sum_{k=1}^{\infty} \frac{1}{\left(j'^2_{\nu,k} - j_{\nu+1,n}^2\right)^2} = ? \tag{26}$$

From the well-known Bessel recurrence relations, we can derive that:

$$\frac{J_\nu(z)}{J'_\nu(z)} = \frac{z}{\nu}\left(1 + \frac{J_{\nu+1}(z)}{J'_\nu(z)}\right) \tag{27}$$

This identity relates the ratio in Eq. (13) to the desired form $J_{\nu+1}(z)/J'_\nu(z)$; after simplification and rearrangement, it can be written in the following form:

$$\sum_{k=1}^{\infty} \frac{1}{\left(j'_{\nu,k}\right)^2 - z^2} = \frac{1}{2\nu} - \left(\frac{\nu^2 - z^2}{2\nu z^2}\right) \frac{J_{\nu+1}(z)}{J'_\nu(z)} \tag{28}$$

Introducing $z^2 = j_{p+1,n}^2$ into the series representation (28) yields the desired series solution:

$$\sum_{k=1}^{\infty} \frac{1}{\left(j'_{\nu,k}\right)^2 - j_{\nu+1,n}^2} = \frac{1}{2\nu} \tag{29}$$

It is worth noting that the obtained series coincides exactly with the result derived when z was replaced by ν – see Eq. (19). Similarly, substituting (27) into equation (14) and rearranging yields the following result:

$$\sum_{k=1}^{\infty} \frac{1}{\left(\left(j'_{\nu,k}\right)^2 - z^2\right)^2} = \frac{z^4 - \nu^2 z^2 - 4\nu^3}{4\nu^2 z^4} + \frac{\left(\nu^2 - z^2\right)^2 - \nu^3}{2\nu^2 z^4}\left(\frac{J_{\nu+1}(z)}{J'_\nu(z)}\right) + \frac{\left(\nu^2 - z^2\right)^2}{4\nu^2 z^4}\left(\frac{J_{\nu+1}(z)}{J'_\nu(z)}\right)^2 \tag{30}$$

With its help, for the specific value $z^2 = j_{\nu+1,n}^2$, we obtain the following result:

$$\sum_{k=1}^{\infty}\frac{1}{\left((j'_{\nu,k})^2-j_{\nu+1,n}^2\right)^2}=\frac{j_{\nu+1,n}^4-\nu^2 j_{\nu+1,n}^2-4\nu^3}{4\nu^2 j_{\nu+1,n}^4}=\frac{1}{4\nu^2}-\frac{1}{4j_{\nu+1,n}^2}-\frac{\nu}{j_{\nu+1,n}^4} \tag{31}$$

Let us return with the presented solutions (29) and (31) to the initial formula for $F11$ Eq. (25):

$$F11=\frac{\nu^2}{\left(j_{\nu+1,n}^2-\nu^2\right)^2}\sum_{k=1}^{\infty}\frac{1}{j_{\nu,k}'^2-\nu^2}+\frac{\nu^2}{\left(j_{\nu+1,n}^2-\nu^2\right)^2}\sum_{k=1}^{\infty}\frac{1}{j_{\nu+1,n}^2-j_{\nu,k}'^2}+$$
$$\frac{j_{\nu+1,n}^2}{j_{\nu+1,n}^2-\nu^2}\sum_{k=1}^{\infty}\frac{1}{\left(j_{\nu+1,n}^2-j_{\nu,k}'^2\right)^2}=\frac{\nu^2}{\left(j_{\nu+1,n}^2-\nu^2\right)^2}\frac{1}{2\nu}-\frac{\nu^2}{\left(j_{\nu+1,n}^2-\nu^2\right)^2}\frac{1}{2\nu}+$$
$$\frac{j_{\nu+1,n}^2}{j_{\nu+1,n}^2-\nu^2}\left(\frac{1}{4\nu^2}-\frac{1}{4j_{\nu+1,n}^2}-\frac{\nu}{j_{\nu+1,n}^4}\right)=\frac{1}{4\nu^2}-\frac{\nu}{j_{\nu+1,n}^2\left(j_{\nu+1,n}^2-\nu^2\right)} \tag{32}$$

This solution, as can be seen, differs slightly from that obtained by Langowski and Nowak. However, numerical investigations carried out for selected values of ν and for the zeros of the ordinary Bessel function show that their solution yields results very close to those derived here. Apart from solution $F11$, Langowski and Nowak also presented another formula attributed to Buchholz. Let's denote this formula by $F11bis$:

$$F11bis=\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu,k}}^2-\nu^2\right)j_{\nu,k}'^2} \tag{33}$$

Using the modified different partial fraction expansion method proposed in paper [15]:

$$\sum_{k=1}^{\infty}(a-b)^{-M}b^{-N}=\sum_{l=1}^{N}\binom{N+M-l-1}{M-1}a^{-(N+M-l)}\sum_{k=1}^{\infty}b^{-l}+$$
$$\sum_{l=1}^{M}\binom{N+M-l-1}{N-1}a^{-(N+M-l)}\sum_{k=1}^{\infty}(a-b)^{-l} \tag{34}$$

and the previously derived solution (19) with the result of Muldoon and Raza (Eq. (21) in [4]), we obtain:

$$F11bis=\frac{1}{\nu^2}\left(\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu,k}}^2-\nu^2\right)}-\sum_{k=1}^{\infty}\frac{1}{j_{\nu,k}'^2}\right)=\frac{1}{\nu^2}\left(\frac{1}{2\nu}-\frac{\nu+2}{4\nu(\nu+1)}\right)=\frac{1}{2\nu^3}-\frac{\nu+2}{4\nu^3(\nu+1)}=$$
$$\frac{2(\nu+1)-(\nu+2)}{4\nu^3(\nu+1)}=\frac{2\nu+2-\nu-2}{4\nu^3(\nu+1)}=\frac{1}{4\nu^2(\nu+1)} \tag{35}$$

j) Solution ($F14$). To solve this problem, one needs to return to the solutions discussed above, together with the results of case $F5$ (Eqs. (7) and (8)), and replace the order ν with the higher order $\nu+1$.

$$\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu+1,k}}^2-z^2\right)^1}=\frac{\nu+1}{2z^2}-\frac{1}{2z}\frac{J'_{\nu+1}(z)}{J_{\nu+1}(z)} \tag{36}$$

$$\sum_{k=1}^{\infty}\frac{1}{\left({j_{\nu+1,k}}^2-z^2\right)^2}=\frac{1}{4z^2}\left(\frac{J'_{\nu+1}(z)}{J_{\nu+1}(z)}\right)^2+\frac{1}{2z^3}\left(\frac{J'_{\nu+1}(z)}{J_{\nu+1}(z)}\right)^1+\frac{z^2-(\nu+1)(\nu+3)}{4z^4} \tag{37}$$

Now, a formula is derived from recurrence Bessel solutions:

$$\frac{J'_{\nu+1}(z)}{J_{\nu+1}(z)} = \frac{z}{\nu}\frac{J'_\nu(z)}{J_{\nu+1}(z)} - \frac{\nu(\nu+1)-z^2}{\nu z} \tag{38}$$

can be substituted to Eq. (37), giving:

$$\sum_{k=1}^{\infty}\frac{1}{\left(j_{\nu+1,k}{}^2-z^2\right)^2} = \frac{1}{4z^2}\left(\frac{z}{\nu}\frac{J'_\nu(z)}{J_{\nu+1}(z)} - \frac{\nu(\nu+1)-z^2}{\nu z}\right)^2 + \frac{1}{2z^3}\left(\frac{z}{\nu}\frac{J'_\nu(z)}{J_{\nu+1}(z)} - \frac{\nu(\nu+1)-z^2}{\nu z}\right)^1 + \frac{z^2-(\nu+1)(\nu+3)}{4z^4} \tag{39}$$

Substituting now $z^2 = {j'_{p,n}}^2$, it is evident that the following remains:

$$\sum_{k=1}^{\infty}\frac{1}{\left(j_{\nu+1,k}{}^2-{j'_{p,n}}^2\right)^2} = \frac{1}{4{j'_{p,n}}^2}\left(-\frac{\nu(\nu+1)-{j'_{p,n}}^2}{\nu j'_{p,n}}\right)^2 + \frac{1}{2{j'_{p,n}}^3}\left(-\frac{\nu(\nu+1)-{j'_{p,n}}^2}{\nu j'_{p,n}}\right)^1 + \frac{{j'_{p,n}}^2-(\nu+1)(\nu+3)}{4{j'_{p,n}}^4} = \frac{\left[(j'_{p,n})^2-\nu(\nu+1)\right]^2+2\nu\left({j'_{p,n}}^2-\nu(\nu+1)\right)+\nu^2\left({j'_{p,n}}^2-(\nu+1)(\nu+3)\right)}{4\nu^2(j'_{p,n})^4} = \frac{1}{4\nu^2} - \frac{\nu}{{j'_{p,n}}^4} - \frac{4+{j'_{p,n}}^2}{4{j'_{p,n}}^4} = \frac{1}{4\nu^2} - \frac{\nu+1}{{j'_{p,n}}^4} - \frac{1}{4{j'_{p,n}}^2} \tag{40}$$

being exactly the same result as those presented by Langowski and Nowak.

k) Solution ($F16$). The series being analyzed is very similar to the previously solved series $F7$ and $F11$. Using partial fraction decomposition with the help of Eq. (20), we get:

$$F16 = \sum_{k=1}^{\infty}\frac{{j'_{\nu+1,k}}^2}{\left({j'_{\nu+1,k}}^2-(\nu+1)^2\right)\left(j_{\nu,n}{}^2-{j'_{\nu+1,k}}^2\right)^2} = \frac{(\nu+1)^2}{\left(j_{\nu,n}{}^2-(\nu+1)^2\right)^2}\sum_{k=1}^{\infty}\frac{1}{{j'_{\nu+1,k}}^2-(\nu+1)^2} + \frac{(\nu+1)^2}{\left(j_{\nu,n}{}^2-(\nu+1)^2\right)^2}\sum_{k=1}^{\infty}\frac{1}{j_{\nu,n}{}^2-{j'_{\nu+1,k}}^2} + \frac{j_{\nu,n}{}^2}{j_{\nu,n}{}^2-(\nu+1)^2}\sum_{k=1}^{\infty}\frac{1}{\left(j_{\nu,n}{}^2-{j'_{\nu+1,k}}^2\right)^2} \tag{41}$$

Substituting now $\nu+1$ instead of ν in Eq. (19) gives:

$$\sum_{k=1}^{\infty}\frac{1}{{j'_{\nu+1,k}}^2-(\nu+1)^2} = \frac{1}{2(\nu+1)} \tag{42}$$

The other two infinite sums appearing on the rhs of Eq. (41) are defined in this note for the first time. We will use again Eq. (13) and Eq. (14) in which we will introduce $\nu+1$ instead of ν

$$\sum_{k=1}^{\infty}\frac{1}{{j'_{\nu+1,k}}^2-z^2} = \frac{\nu+1}{2z^2} - \left(\frac{(\nu+1)^2-z^2}{2z^3}\right)\frac{J_{\nu+1}(z)}{J'_{\nu+1}(z)} \tag{43}$$

$$\sum_{k=1}^{\infty}\frac{1}{\left({j'_{\nu+1,k}}^2-z^2\right)^2} = \frac{z^2-(\nu+1)(\nu+3)}{4z^4} - \frac{(\nu+1)^2}{2z^5}\left(\frac{J_{\nu+1}(z)}{J'_{\nu+1}(z)}\right) + \frac{\left((\nu+1)^2-z^2\right)^2}{4z^6}\left(\frac{J_{\nu+1}(z)}{J'_{\nu+1}(z)}\right)^2 \tag{44}$$

As the next step we will introduce a relation following from standard Bessel identities:

$$\frac{J_{\nu+1}(z)}{J'_{\nu+1}(z)} = \frac{z}{\nu+1}\left(\frac{J_\nu(z)}{J'_{\nu+1}(z)} - 1\right) \tag{45}$$

then it is evident that for the characteristic value $z^2 = j_{\nu,n}^2$ we obtain:

$$\sum_{k=1}^{\infty} \frac{1}{{j'_{\nu+1,k}}^2 - {j_{\nu,n}}^2} = \frac{\nu+1}{2{j_{\nu,n}}^2} - \left(\frac{(\nu+1)^2 - {j_{\nu,n}}^2}{2{j_{\nu,n}}^2}\right)\left(-\frac{j_{\nu,n}}{\nu+1}\right) = \frac{2(\nu+1)^2 - {j_{\nu,n}}^2}{2{j_{\nu,n}}^2(\nu+1)} \tag{46}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\left({j'_{\nu+1,k}}^2 - {j_{\nu,n}}^2\right)^2} = \frac{{j_{\nu,n}}^2 - (\nu+1)(\nu+3)}{4{j_{\nu,n}}^4} + \frac{\nu+1}{2{j_{\nu,n}}^4} + \frac{\left((\nu+1)^2 - {j_{\nu,n}}^2\right)^2}{4{j_{\nu,n}}^4(\nu+1)^2} = \frac{1}{4(\nu+1)^2} - \frac{1}{4{j_{\nu,n}}^2} \tag{47}$$

Using the formulas established above, we now return to formula $F16$ (41):

$$F16 = \frac{(\nu+1)}{2\left({j_{\nu,n}}^2 - (\nu+1)^2\right)^2} - \frac{(\nu+1)^2}{\left({j_{\nu,n}}^2 - (\nu+1)^2\right)^2}\frac{2(\nu+1)^2 - j_{\nu,n}^2}{2j_{\nu,n}^2(\nu+1)} + \frac{{j_{\nu,n}}^2}{{j_{\nu,n}}^2 - (\nu+1)^2}\left(\frac{1}{4(\nu+1)^2} - \frac{1}{4j_{\nu,n}^2}\right) = \frac{4(\nu+1)^3 + j_{\nu,n}^2(j_{\nu,n}^2 - (\nu+1)^2)}{4j_{\nu,n}^2(\nu+1)^2(j_{\nu,n}^2 - (\nu+1)^2)} = \frac{\nu+1}{j_{\nu,n}^2(j_{\nu,n}^2 - (\nu+1)^2)} + \frac{1}{4(\nu+1)^2} \tag{48}$$

l) Solution ($F17$). The solution to this problem will be obtained using the previously derived formula (8) and relation coming from common Bessel identities:

$$\left(\frac{J'_{\nu+1}(z)}{J_\nu(z)}\frac{z}{\nu+1} - \frac{z}{\nu+1} + \frac{\nu}{z}\right) = \frac{J'_\nu(z)}{J_\nu(z)} \tag{49}$$

With the help of Eq. (49), we obtain the formula:

$$\sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - z^2\right)^2} = \frac{1}{4z^2}\left(\frac{J'_{\nu+1}(z)}{J_\nu(z)}\frac{z}{\nu+1} + \frac{\nu}{z} - \frac{z}{\nu+1}\right)^2 + \frac{1}{2z^3}\left(\frac{J'_{\nu+1}(z)}{J_\nu(z)}\frac{z}{\nu+1} + \frac{\nu}{z} - \frac{z}{\nu+1}\right)^1 + \frac{z^2 - \nu(\nu+2)}{4z^4} \tag{50}$$

If we now substitute $z^2 = j_{\nu+1,n}^2$, we get series solution:

$$\sum_{k=1}^{\infty} \frac{1}{\left({j_{\nu,k}}^2 - {j'_{\nu+1,n}}^2\right)^2} = \frac{1}{4{j'_{\nu+1,n}}^2}\left(\frac{\nu}{j'_{\nu+1,n}} - \frac{j'_{\nu+1,n}}{\nu+1}\right)^2 + \frac{1}{2{j'_{\nu+1,n}}^3}\left(\frac{\nu}{j'_{\nu+1,n}} - \frac{j'_{\nu+1,n}}{\nu+1}\right)^1 + \frac{{j'_{\nu+1,n}}^2 - \nu(\nu+2)}{4{j'_{\nu+1,n}}^4} = \frac{1}{4(\nu+1)^2} - \frac{1}{4{j'_{\nu+1,n}}^2} \tag{51}$$

which is the desired formula $F17$ in the Langowski-Nowak publication.

3. Construction of higher-order series via partial fraction decomposition

Using the series from Urbanowicz [12], Table A1, together with formula (1), it is possible to obtain new analytical solutions for a series constructed from the zeros of Bessel functions. To remain consistent with the notation used in Urbanowicz [12], the order of the Bessel function will be denoted by the letter p throughout this Section. In Subsection 3.1, we present the solutions for:

$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p-1,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p+1,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p-2,n}{}^2\right)^3};$$
$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p+2,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p,k}{}^2-j_{p+3,n}{}^2\right)^3} \tag{52}$$

In Subsection 3.2, we present solutions for other formulas not derived in the paper of Langowski and Nowak [11]:

$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p-2,n}{}^2\right)^2};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p+2,n}{}^2\right)^2};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^2};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p+3,n}{}^2\right)^2} \tag{53}$$

and

$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p-1,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p+1,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p-2,n}{}^2\right)^3};$$
$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p+2,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^3};\ \sum_{k=1}^{\infty}\frac{j_{p,k}{}^2}{\left(j_{p,k}{}^2-j_{p+3,n}{}^2\right)^3} \tag{54}$$

3.1. Decomposition into three series of Calogero-type

For functions (52), where $N = 2$ and $M = 3$, Eq. (1) gives:

$$\frac{b^2}{(a-b)^3}=\frac{1}{a-b}-\frac{2a}{(a-b)^2}+\frac{a^2}{(a-b)^3}=-\frac{1}{b-a}-\frac{2a}{(b-a)^2}-\frac{a^2}{(b-a)^3} \tag{55}$$

By substituting $b = j_{p,k}^2$ and, successively, $a = j_{p-3,n}^2, j_{p-2,n}^2, j_{p-1,n}^2, j_{p+1,n}^2, j_{p+2,n}^2, j_{p+3,n}^2$, we obtain the following new solutions:

$$\sum_{k=1}^{\infty}\frac{j_{p,k}{}^4}{\left(j_{p-3,n}{}^2-j_{p,k}{}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^1}-2j_{p-3,n}{}^2\sum_{k=1}^{\infty}\frac{1}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^2}-$$
$$j_{p-3,n}{}^4\sum_{k=1}^{\infty}\frac{1}{\left(j_{p,k}{}^2-j_{p-3,n}{}^2\right)^3}=-\left(\frac{p}{j_{p-3,n}^2}-\frac{p-2}{4(p-1)(p-2)-j_{p-3,n}^2}\right)-$$

$$2{j_{p-3,n}}^2\left(\frac{1}{4}\frac{j_{p-3,n}^2-4(p-2)}{\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^2}-\frac{p}{j_{p-3,n}^4}\right)-{j_{p-3,n}}^4\left(\frac{p}{j_{p-3,n}^6}-\frac{(p-2)\left[3j_{p-3,n}^2+8-4p(p-1)\right]}{8\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^3}\right)=$$
$$\frac{p-2}{4(p-1)(p-2)-j_{p-3,n}^2}-\frac{1}{2}\frac{j_{p-3,n}^2\left(j_{p-3,n}^2-4(p-2)\right)}{\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^2}+\frac{j_{p-3,n}^4(p-2)\left[3j_{p-3,n}^2+8-4p(p-1)\right]}{8\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^3}=$$
$$\frac{(3p-2)j_{p-3,n}^6-4(p-1)(p-2)(p+4)j_{p-3,n}^4+128(p-1)^2(p-2)^3}{8\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^3} \tag{56}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^4}{\left({j_{p-2,n}}^2-{j_{p,k}}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^1}-2{j_{p-2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^2}-$$
$${j_{p-2,n}}^4\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^3}=-\left(\frac{p}{j_{p-2,n}^2}-\frac{1}{4(p-1)}\right)-2j_{p-2,n}^2\left(\frac{1}{16(p-1)^2}-\frac{p}{j_{p-2,n}^4}\right)-$$
$$j_{p-2,n}^4\left(\frac{p}{j_{p-2,n}^6}-\frac{1}{64(p-1)^3}+\frac{1}{32(p-1)j_{p-2,n}^2}\right)=\frac{j_{p-2,n}^4-2(p+3)(p-1)j_{p-2,n}^2+16(p-1)^2}{64(p-1)^3} \tag{57}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^4}{\left({j_{p-1,n}}^2-{j_{p,k}}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^1}-2{j_{p-1,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^2}-$$
$${j_{p-1,n}}^4\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^3}=-\left(\frac{p}{j_{p-1,n}^2}\right)-2j_{p-1,n}^2\left(\frac{1}{j_{p-1,n}^2}\left(\frac{1}{4}-\frac{p}{j_{p-1,n}^2}\right)\right)-$$
$$j_{p-1,n}^4\left(-\frac{1}{j_{p-1,n}^4}\left(\frac{1}{4}-p\left(\frac{1}{8}+\frac{1}{j_{p-1,n}^2}\right)\right)\right)=-\frac{p+2}{8} \tag{58}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^4}{\left({j_{p+1,n}}^2-{j_{p,k}}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^1}-2{j_{p+1,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^2}-$$
$${j_{p+1,n}}^4\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^3}=-(0)-2j_{p+1,n}^2\left(\frac{1}{4j_{p+1,n}^2}\right)-j_{p+1,n}^4\left(-\frac{1}{j_{p+1,n}^4}\left(\frac{1}{4}+\frac{p}{8}\right)\right)=\frac{p-2}{8} \tag{59}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^4}{\left({j_{p+2,n}}^2-{j_{p,k}}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^1}-2{j_{p+2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^2}-$$
$${j_{p+2,n}}^4\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^3}=-\left(\frac{1}{4(1+p)}\right)-2j_{p+2,n}^2\left(\frac{1}{4^2(1+p)^2}\right)-$$
$$j_{p+2,n}^4\left(\frac{1}{4^3(1+p)^3}-\frac{1}{32(p+1)j_{p+2,n}^2}\right)=\frac{-j_{p+2,n}^4+2(p-3)(p+1)j_{p+2,n}^2-16(p+1)^2}{64(p+1)^3} \tag{60}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^4}{\left({j_{p+3,n}}^2-{j_{p,k}}^2\right)^3}=-\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^1}-2{j_{p+3,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^2}-$$
$${j_{p+3,n}}^4\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^3}=-\left(\frac{p+2}{4(p+1)(p+2)-j_{p+3,n}^2}\right)-$$
$$2j_{p+3,n}^2\left(\frac{1}{4}\frac{j_{p+3,n}^2+4(p+2)}{\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^2}\right)-j_{p+3,n}^4\left(\frac{(p+2)\left[3j_{p+3,n}^2+8-4p(p+1)\right]}{8\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^3}\right)=$$
$$\frac{-(3p+2)j_{p+3,n}^6+4(p+1)(p+2)(p-4)j_{p+3,n}^4-128(p+1)^2(p+2)^3}{8\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^3} \quad (61)$$

3.2. Decomposition into two series of Calogero-type

For functions (53), where $N=1$ and $M=2$, Eq. (1) gives:

$$\frac{b}{(a-b)^2}=-\frac{1}{a-b}+\frac{a}{(a-b)^2}=\frac{1}{b-a}+\frac{a}{(b-a)^2} \quad (62)$$

using it we can find the following solutions:

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p-2,n}}^2-{j_{p,k}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p-2,n}}^2}+{j_{p-2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^2}=$$
$$\left(\frac{p}{{j_{p-2,n}}^2}-\frac{1}{4(p-1)}\right)+{j_{p-2,n}}^2\left(\frac{1}{16(p-1)^2}-\frac{p}{j_{p-2,n}^4}\right)=\frac{j_{p-2,n}^2-4(p-1)}{16(p-1)^2} \quad (63)$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p+2,n}}^2}+{j_{p+2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^2}=$$
$$\left(\frac{1}{4(1+p)}\right)+{j_{p+2,n}}^2\left(\frac{1}{4^2(1+p)^2}\right)=\frac{j_{p+2,n}^2+4(p+1)}{16(p+1)^2} \quad (64)$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p-3,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p-3,n}}^2}+{j_{p-3,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-3,n}}^2\right)^2}=$$
$$\left(\frac{p}{j_{p-3,n}^2}-\frac{p-2}{4(p-1)(p-2)-j_{p-3,n}^2}\right)+{j_{p-3,n}}^2\left(\frac{1}{4}\frac{j_{p-3,n}^2-4(p-2)}{\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^2}-\frac{p}{j_{p-3,n}^4}\right)=$$
$$\frac{j_{p-3,n}^4-16(p-1)(p-2)^2}{4\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^2} \quad (65)$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p+3,n}}^2}+{j_{p+3,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^2}=$$
$$\left(\frac{p+2}{4(p+1)(p+2)-{j_{p+3,n}}^2}\right)+{j_{p+3,n}}^2\left(\frac{1}{4}\frac{{j_{p+3,n}}^2+4(p+2)}{\left[4(p+1)(p+2)-{j_{p+3,n}}^2\right]^2}\right)=\frac{{j_{p+3,n}}^4+16(p+1)(p+2)^2}{4\left[4(p+1)(p+2)-{j_{p+3,n}}^2\right]^2} \quad (66)$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p-4,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p-4,n}}^2}+{j_{p-4,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-4,n}}^2\right)^2}=$$
$$\left(\frac{p}{{j_{p-4,n}}^2}-\frac{{j_{p-4,n}}^2-4(p-2)(p-3)}{8(p-2)\left[{j_{p-4,n}}^2-2(p-1)(p-3)\right]}\right)+$$
$${j_{p-4,n}}^2\left(\frac{{j_{p-4,n}}^2-16(p-2)(p-3)^2}{4\left[8(p-1)(p-2)(p-3)-4(p-2){j_{p-4,n}}^2\right]^2}-\frac{p}{{j_{p-4,n}}^2}\right)=$$
$$\frac{{j_{p-4,n}}^6-8(p-2){j_{p-4,n}}^4+32(p-3)(p-2)(p-1){j_{p-4,n}}^2-64(p-1)(p-2)^2(p-3)^2}{64(p-2)^2\left({j_{p-4,n}}^2-2(p-1)(p-3)\right)^2} \tag{67}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+4,n}}^2\right)^2}=\sum_{k=1}^{\infty}\frac{1}{{j_{p,k}}^2-{j_{p+4,n}}^2}+{j_{p+4,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+4,n}}^2\right)^2}=$$
$$\left(\frac{j_{p+4,n}^2-4(p+2)(p+3)}{8(p+2)\left[j_{p+4,n}^2-2(p+1)(p+3)\right]}\right)+{j_{p+4,n}}^2\left(\frac{j_{p+4,n}^4+16(p+2)(p+3)^2}{4\left[8(p+1)(p+2)(p+3)-4(p+2)j_{p+4,n}^2\right]^2}\right)=$$
$$\frac{j_{p+4,n}^6+8(p+2)j_{p+4,n}^4-32(p+1)(p+2)(p+3)j_{p+4,n}^2+64(p+1)(p+2)^2(p+3)^2}{64(p+2)^2\left[j_{p+4,n}^2-2(p+1)(p+3)\right]^2} \tag{68}$$

and for functions (54) where $N=1$ and $M=3$ Eq. (1) gives:

$$\frac{b}{(b-a)^3}=\frac{1}{(b-a)^2}+\frac{a}{(b-a)^3} \tag{69}$$

we then can derive such unknown earlier solutions:

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^3}=\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^2}+{j_{p-1,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-1,n}}^2\right)^3}=$$
$$\left(\frac{1}{4j_{p-1,n}^2}-\frac{p}{j_{p-1,n}^4}\right)+{j_{p-1,n}}^2\left(-\frac{1}{4j_{p-1,n}^4}+\frac{p}{8j_{p-1,n}^4}+\frac{p}{j_{p-1,n}^6}\right)=\frac{p}{8j_{p-1,n}^2} \tag{70}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^3}=\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^2}+{j_{p+1,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+1,n}}^2\right)^3}=$$
$$\left(\frac{1}{4j_{p+1,n}^2}\right)+{j_{p+1,n}}^2\left(-\frac{1}{j_{p+1,n}^4}\left(\frac{1}{4}+\frac{p}{8}\right)\right)=-\frac{p}{8j_{p+1,n}^2} \tag{71}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^3}=\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^2}+{j_{p-2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-2,n}}^2\right)^3}=$$
$$\left(\frac{1}{16(p-1)^2}-\frac{p}{j_{p-2,n}^4}\right)+{j_{p-2,n}}^2\left(\frac{p}{j_{p-2,n}^6}-\frac{1}{64(p-1)^3}+\frac{1}{32(p-1)j_{p-2,n}^2}\right)=\frac{2\left(p^2-1\right)-j_{p-2,n}^2}{64(p-1)^3} \tag{72}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^3}=\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^2}+{j_{p+2,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+2,n}}^2\right)^3}=$$
$$\left(\frac{1}{4^2(1+p)^2}\right)+{j_{p+2,n}}^2\left(\frac{1}{4^3(1+p)^3}-\frac{1}{32(p+1)j_{p+2,n}^2}\right)=\frac{j_{p+2,n}^2-2\left(p^2-1\right)}{64(p+1)^3} \tag{73}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p-3,n}}^2\right)^3} = \sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-3,n}}^2\right)^2} + {j_{p-3,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p-3,n}}^2\right)^3} =$$
$$\left(\frac{1}{4}\frac{j_{p-3,n}^2-4(p-2)}{\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^2}-\frac{p}{j_{p-3,n}^4}\right)+{j_{p-3,n}}^2\left(\frac{p}{j_{p-3,n}^6}-\frac{(p-2)\left[3j_{p-3,n}^2+8-4p(p-1)\right]}{8\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^3}\right) =$$
$$\frac{-(3p-4)j_{p-3,n}^4+4(p-1)(p-2)(p+2)j_{p-3,n}^2-32(p-1)(p-2)^2}{8\left[4(p-1)(p-2)-j_{p-3,n}^2\right]^3} \tag{74}$$

$$\sum_{k=1}^{\infty}\frac{{j_{p,k}}^2}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^3} = \sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^2} + {j_{p+3,n}}^2\sum_{k=1}^{\infty}\frac{1}{\left({j_{p,k}}^2-{j_{p+3,n}}^2\right)^3} =$$
$$\left(\frac{1}{4}\frac{j_{p+3,n}^2+4(p+2)}{\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^2}\right)+{j_{p+3,n}}^2\left(\frac{(p+2)\left[3j_{p+3,n}^2+8-4p(p+1)\right]}{8\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^3}\right) =$$
$$\frac{(3p+4)j_{p+3,n}^4-4(p+1)(p+2)(p-2)j_{p+3,n}^2+32(p+1)(p+2)^2}{8\left[4(p+1)(p+2)-j_{p+3,n}^2\right]^3} \tag{75}$$

4. Conclusions

It has been shown that complicated formulas (series) based on the zeros of Bessel functions, derived in the work of Langowski and Nowak [11], can be decomposed into fundamental Calogero-type formulas. The effectiveness of this decomposition is illustrated both by the expansion of the complicated series derived by Langowski and Nowak [11] and by the presentation of new formulas resulting from the expressions obtained in Urbanowicz [12].

Since the companion work [13] deals with the evaluation of derivatives of ratios of Bessel functions (both ordinary and modified, as well as ratios involving Bessel function derivatives), the remaining task for the future is to determine recursive formulas for the modified Calogero-type expressions. To date, recursive formulas are known only for Rayleigh-type series, derived by Meiman [2] and Kishore [3], and analogous series based on the zeros of derivatives of Bessel functions, developed by Muldoon and Raza [4].

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