Journal of Applied Mathematics and Computational Mechanics 2026, 25(2), 125-138
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2026.2.09 e-ISSN 2353-0588

GENERALIZED ABSOLUTELY CONTINUOUS FUNCTIONS AND COMPOSITION OPERATOR

Małgorzata Wróbel[*]

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
malgorzata.wrobel@pcz.pl

Received: 11 March 2026; Accepted: 15 June 2026

Abstract. In the paper we give some properties of generalized, in the sense of Love, p-absolutely continuous functions defined on a compact interval I on the real line. Among others, we prove that the space of such functions equipped with Wiener's norm forms a Banach algebra. Moreover, we state the condition for the function space $X(I)$ under which the generator of any corresponding composition operator mapping $X(I)$ into $C(I)$ is continuous. As a consequence, we get that the generator of any Nemytskij operator acting between the spaces of generalized p-absolutely continuous functions defined on I is continuous.

MSC 2010: *47H30, 26A46*
Keywords: *Nemytskij composition operator, uniformly bounded operator, continuous function, absolutely continuous function, generalized p-absolutely continuous function*

1. Introduction

Let $I=[a,b]$ be an interval of a real line $\mathbb{R}$ ($a,b\in\mathbb{R}$, $a<b$), and let $X(I)$ and $Y(I)$ be two classes of functions $f:I\to\mathbb{R}$. For a given function $h:I\times\mathbb{R}\to\mathbb{R}$, the mapping $H:X(I)\to Y(I)$ defined by

$$H(f)(x):=h(x,f(x)),\quad f\in X(I),\ (x\in I),$$

is called a *composition (Nemytskij or superposition) operator of the generator h*.

This operator is frequently treated as a tool in many fields of science. Properties such as boundedness, continuity, compactness, and differentiability are used in PDEs theory in the context of mechanical models regarding reaction-diffusion systems, nonlinear optics, or nonlinear elasticity [1]. They are important for modeling interfaces, fractures, and discontinuities, where such operators represent local constitutive mappings. Moreover, they appear in signal processing and machine learning, where they are used in neural fields models [2]; in chemical kinetics, biology, and population dynamics. For more applications, see [3], for instance.

---

[*] Corresponding author

In the present paper we will examine the composition operators acting between two Banach spaces of generalized (in the sense of Love) absolutely continuous functions defined on I in terms of the regularity of the generating function h. The well-known theorem of Krasnosielskii shows that every generator of the composition operator acting between the spaces of continuous functions $C(I)$ has to be continuous. However, surprisingly enough [4], in the case where we replace the space $C(I)$ with the space $C^1(I)$ of continuously differentiable functions, this function h is not only not of class C^1, but it does not even have to be continuous. This shows that the degree of regularity of the function h depends on the nature of the function spaces both in its domain and range.

The main result of this paper says (Theorem 2) that the generator of any Nemytskij operator acting between the Banach spaces of generalized p-absolutely continuous functions defined on I is continuous. This is a consequence of Theorem 1, where we state the condition for the function space $X(I)$ under which the generator of each composition operator mapping $X(I)$ into $C(I)$ is continuous. For the sake of completeness, we also consider the composition operators acting between the spaces of generalized p-absolutely continuous functions that are uniformly bounded. In this case, we get (Theorem 4) that the generating function is affine in the function variable (traditionally, in the second variable). This relation was first observed by Matkowski [5], and later developed in many papers, such as [6–10] (see also the monograph [11]).

Moreover, in Section 2, we list some properties of generalized absolutely continuous functions. Among others, we prove that the space of such functions equipped with WBV_p-norm forms a Banach algebra.

2. Preliminaries

Let $I=[a,b]$ $(a,b\in\mathbb{R},\ a<b)$ stand for the interval of a real axis and let $\mathscr{S}([a,b])$ denote the family of all finite collections $S_n=\{[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]\}$, $n\in\mathbb{N}$, of pairwise nonoverlapping subintervals $[a_i,b_i]\subset[a,b]$, $i=1,\dots,n$, i.e., $(a_i,b_i)\cap(a_j,b_j)=\emptyset$, $i,j\in\{1,\dots,n\}$, $i\neq j$.

Definition 1 ([11], Definition 3.12). Given a positive number $p\geq 1$, we call the function $f:[a,b]\to\mathbb{R}$ *p-absolutely continuous on* $[a,b]$, and write $f\in AC_p([a,b])$, if it has the following property: for any $\varepsilon>0$ there exists a $\delta>0$ such that for all collections $S_n=\{[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]\}\in\mathscr{S}([a,b])$, $n\in\mathbb{N}$, the condition

$$\left(\sum_{i=1}^{n}|b_i-a_i|^p\right)^{\frac{1}{p}}\leq\delta \tag{1}$$

implies that

$$\left(\sum_{i=1}^{n}|f(b_i)-f(a_i)|^p\right)^{\frac{1}{p}}\leq\varepsilon.$$

For $p = 1$, we get the classical absolute continuous function, i.e., $AC_1([a,b]) = AC([a,b])$.

Definition 2 ([11], Definition 1.31). Given a real number $p \geq 1$, we say that $f : I \to \mathbb{R}$ is a *function of bounded p-variation (in Wiener's sense) in I*, and write $f \in WBV_p(I)$, if the *p-variation of f on I*, defined by

$$Var_p(f;I) := \sup\{var_p(f;P) : P \in \mathscr{P}(I)\},$$

is finite; here

$$var_p(f;P) := \sum_{i=1}^{m} |f(t_i) - f(t_{i-1})|^p,$$

and $\mathscr{P}(I)$ stands for the set of all partitions

$$P = \{t_0, t_1, \ldots, t_{m-1}, t_m\},\ a = t_0 < t_1 < \ldots < t_{m-1} < t_m = b$$

of the compact interval I.

In particular, for $p = 1$, $WBV_1(I)$ reduces to the usual class $BV_1(I)$ of functions of bounded variation in the sense of Jordan.

Remark 1. For any $p \geq 1$ and $s,t \in [0,\infty)$, the following inequality holds:

$$(t+s)^p \leq 2^{p-1}(t^p + s^p). \tag{2}$$

Indeed, by the convexity of a power function $\varphi(t) = t^p$, $p \geq 1$, we have

$$(t+s)^p = \left(\frac{2t+2s}{2}\right)^p \leq \frac{(2t)^p + (2s)^p}{2} = 2^{p-1}(t^p + s^p),$$

for all $s,t \in [0,\infty)$.

Let us recall some properties of the functional Var_p:

P1 Var_p is increasing with respect to the intervals, i.e.,

$$Var_p(f,[a,c]) \leq Var_p(f,[a,b]), \quad a,b,c \in I,\ a < c < b;$$

P2 Var_p is superadditive with respect to the intervals ([11], Proposition 2.10 (a)), i.e.,

$$Var_p(f,[a,c]) + Var_p(f,[c,b]) \leq Var_p(f,[a,b]), \quad a,b,c \in I,\ a < c < b;$$

P3 if $f,g \in WBV_p(I)$, then $f+g \in WBV_p(I)$ and

$$Var_p(f+g;[a,b]) \leq 2^{p-1}(Var_p(f;[a,b]) + Var_p(g;[a,b])). \tag{3}$$

Denoting by $C([a,b])$ the class of continuous functions $f:[a,b]\to\mathbb{R}$, we can give the relationships between introduced spaces with a chain of the following inclusions:

$$C([a,b])\cap WBV_p(I)\subset AC_q([a,b])\subset C([a,b])\cap WBV_q(I) \tag{4}$$

and the spaces $AC_p([a,b])$ are increasing with respect to p, i.e.,

$$AC_p([a,b])\subset AC_q([a,b])\subset C([a,b]), \tag{5}$$

for $1\le p<q<\infty$ (all inclusions are strict).

It is known ([11], Proposition 1.32) that the space $\left(WBV_p(I),\|\cdot\|_{WBV_p}\right)$, $p\ge 1$, with

$$\|f\|_{WBV_p} := |f(a)| + Var_p(f;I)^{\frac{1}{p}}, \quad f\in WBV_p(I), \tag{6}$$

forms a Banach space.

Let $p>1$. Denote by $CWBV_p(I)$ the set of all continuous functions $f:I\to\mathbb{R}$ satisfying

$$\inf\left\{\sum_{j=1}^{m} Var_p(f;[t_{j-1},t_j]) : \{t_0,t_1,\dots,t_{m-1},t_m\}\in\mathscr{P}(I)\right\}=0. \tag{7}$$

The crucial role in the paper plays the following Love's characterization of p-absolutely continuous functions using the class of $CWBV_p(I)$-functions.

Lemma 1 ([12], Theorem 1). *For $p>1$ the following equality holds:*

$$AC_p([a,b]) = CWBV_p([a,b]).$$

Remark 2. The above lemma is not true for $p=1$.

Indeed, by the additivity of functional Var_1 with respect to the intervals ([11], Proposition 1.13 (g)), condition (7) reduces to

$$\inf\{Var_p(f;[a,b]) : \{t_0,t_1,\dots,t_{m-1},t_m\}\in\mathscr{P}(I)\}=0,$$

hence, only constant functions would satisfy (7).

3. Some properties of AC_p spaces

In what follows, we will use some facts that we present as following lemmas.

Lemma 2. *Every affine function defined on a compact real interval $[a,b]$ is p-absolutely continuous for any $p\ge 1$.*

Proof. Fix arbitrarily $p\ge 1$ and take any affine function $f(x)=Ax+B$, $x\in[a,b]$, $A,B\in\mathbb{R}$. Let $\varepsilon>0$ and $S_n=\{[a_1,b_1],[a_2,b_2],\dots,[a_n,b_n]\}\in\mathscr{S}([a,b])$ be any collection satisfying (1) with $\delta = \dfrac{\varepsilon}{|A|}$. Then we have

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)|^p = \sum_{i=1}^{n} |A|^p |b_i - a_i|^p \leq \varepsilon^p,$$

which shows that $f \in AC_p([a,b])$. □

Lemma 3. *Let $p \geq 1$. Every continuous function defined on a compact real interval $[a,b]$ that is a gluing of a finite number of p-absolutely continuous functions is p-absolutely continuous.*

Proof. Fix arbitrarily $c \in (a,b)$ and take any continuous function $f : [a,b] \to \mathbb{R}$ such that $f\big|_{[a,c]} \in AC_p([a,c])$ and $f\big|_{[c,b]} \in AC_p([c,b])$. Given $\varepsilon > 0$, by Definition 1, there exist positive numbers δ_1 and δ_2 such that

$$\sum_{i=1}^{k} |f(c_i) - f(a_i)|^p \leq \left(\frac{\varepsilon}{2}\right)^p \tag{8}$$

for any $S_k^1 = \{[a_i, c_i] : i = 1, ..., k\} \in \mathscr{S}([a,c])$ satisfying $\sum_{i=1}^{k} |c_i - a_i|^p \leq \delta_1^p$ and

$$\sum_{i=1}^{l} |f(b_i) - f(c_i)|^p \leq \left(\frac{\varepsilon}{2}\right)^p \tag{9}$$

for any $S_l^2 = \{[c_i, b_i] : i = 1, ..., l\} \in \mathscr{S}([c,b])$ satisfying $\sum_{i=1}^{l} |b_i - c_i|^p \leq \delta_2^p$.

Define $\delta := \min\{\delta_1, \delta_2\}$. If $S_n = \{[a_1,b_1], [a_2,b_2], ..., [a_n,b_n]\} \in \mathscr{S}([a,b])$ is any collection satisfying (1), then c lies in at most one subinterval (a_i, b_i). Suppose first that $c \in (a_{i_0}, b_{i_0})$ for any $i_0 \in \{1, ..., n\}$ and put $K_1 := \{i \in \{1, ..., n\} : [a_i, b_i] \subset [a,c]\}$ and $K_2 := \{i \in \{1, ..., n\} : [a_i, b_i] \subset [c,b]\}$. Since, by (2),

$$|f(b_{i_0}) - f(a_{i_0})|^p \leq 2^{p-1} \left(|f(b_{i_0}) - f(c)|^p + |f(c) - f(a_{i_0})|^p\right),$$

therefore

$$\begin{aligned}\sum_{i=1}^{n} |f(b_i) - f(a_i)|^p \quad \leq \quad & 2^{p-1} \left(\sum_{i \in K_1} |f(b_i) - f(a_i)|^p + |f(c) - f(a_{i_0})|^p \right) \\ & + 2^{p-1} \left(|f(b_{i_0}) - f(c)|^p + \sum_{i \in K_2} |f(b_i) - f(a_i)|^p \right) \leq \varepsilon^p,\end{aligned}$$

by (8) and (9) with $S^1_{|K_1|+1} = \{[a_i, b_i] : i \in K_1\} \cup \{[a_{i_0}, c]\}$ and $S^2_{|K_2|+1} = \{[a_i, b_i] : i \in K_2\} \cup \{[c, b_{i_0}]\}$. Because the case where c does not belong to any of intervals (a_i, b_i) is obvious, the proof is completed. □

Lemma 4. *Given* $p \geq 1$, *the space* $AC_p([a,b])$ *equipped with the norm (6) is a Banach algebra and*

$$\|fg\|_{WBV_p} \leq 2\|f\|_{WBV_p}\|g\|_{WBV_p}, \tag{10}$$

for all $f,g \in AC_p([a,b])$.

Proof. In the case $p = 1$, we get the required claim by Proposition 1.10 and Proposition 3.24 of [11] (where (10) holds with constant 1).

Assume that $p > 1$. Applying Definition 1, we immediately get the linearity of $AC_p([a,b])$. To show that $AC_p([a,b])$ is closed in $WBV_p([a,b])$ in the sense of WBV_p-norm (6), take any sequence $f_n : [a,b] \to \mathbb{R}$, $n \in \mathbb{N}$, of p-absolutely continuous functions pointwise convergent to $f \in WBV_p([a,b])$ such that

$$\|f_n - f\|_{WBV_p} \to 0 \quad as \qquad n \to \infty. \tag{11}$$

Since the convergence in the sense of WBV_p-norm implies the convergence in the *sup*-norm ([11], Proposition 2.44(d)), i.e., in the uniform convergence, f is continuous. Let us fix $\varepsilon > 0$. By (11), there exists $n_0 \in \mathbb{N}$ such that

$$\|f_n - f\|_{WBV_p} < \frac{\varepsilon}{2^p}, \quad n \geq n_0. \tag{12}$$

Because $f_{n_0} \in AC_p([a,b])$, by Lemma 1 and the definition of the family $CWBV_p(I)$, we can find a partition $\{t_0, t_1, \ldots, t_{m-1}, t_m\} \in \mathscr{P}([a,b])$ such that

$$\sum_{j=1}^{m} Var_p(f_{n_0};[t_{j-1},t_j]) < \frac{\varepsilon}{2^p}. \tag{13}$$

On the other hand, taking into account Definition 2 with $f = f - f_{n_0}$, by virtue of (6) and (12), we obtain

$$Var_p(f - f_{n_0};[a,b]) < \frac{\varepsilon}{2^p},$$

and, consequently,

$$\sum_{j=1}^{m} Var_p(f - f_{n_0};[t_{j-1},t_j]) < \frac{\varepsilon}{2^p}. \tag{14}$$

by property **P2**. Representing f as $f = (f - f_{n_0}) + f_{n_0}$, by (3), we get

$$Var_p(f;[t_{j-1},t_j]) \leq 2^{p-1}\left(Var_p(f - f_{n_0};[t_{j-1},t_j]) + Var_p(f_{n_0};[t_{j-1},t_j])\right),$$

for all $j \in \{1,\ldots m\}$, and, finally,

$$\begin{aligned} &\sum_{j=1}^{m} Var_p(f;[t_{j-1},t_j]) \\ \leq\ & 2^{p-1}\sum_{j=1}^{m} Var_p(f_{n_0};[t_{j-1},t_j] + 2^{p-1}\sum_{j=1}^{m} Var_p((f - f_{n_0});[t_{j-1},t_j]) \leq \varepsilon, \end{aligned}$$

which gives that $f \in AC_p([a,b])$, by (13), (14), and Lemma 1.

To show that $AC_p\left([a,b], \|.\|_{WBV_p}\right)$ is an algebra, fix arbitrarily $f, g \in AC_p([a,b])$. Since f, g are p-absolutely continuous on $[a,b]$, both are continuous on $[a,b]$ and, consequently,

$$|f(x)| \leq \|f\|_\infty, \qquad |g(x)| \leq \|g\|_\infty, \tag{15}$$

for all $x \in [a,b]$. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)|^p \leq \left(\frac{\varepsilon}{2\|g\|_\infty}\right)^p, \quad \sum_{i=1}^{n} |g(b_i) - g(a_i)|^p \leq \left(\frac{\varepsilon}{2\|f\|_\infty}\right)^p, \tag{16}$$

for any collection $S_n = \{[a_1,b_1],[a_2,b_2],...,[a_n,b_n]\} \in \mathscr{S}([a,b])$, $n \in \mathbb{N}$, satisfying (1). Since, by (15),

$$\begin{aligned} |(fg)(b_i) - (fg)(a_i)| &= |g(b_i)[f(b_i) - f(a_i)] + f(a_i)[g(b_i) - g(a_i)]| \\ &\leq \|g\|_\infty |f(b_i) - f(a_i)| + \|f\|_\infty |g(b_i) - g(a_i)|, \end{aligned}$$

for all $i = 1,...,n$, the Minkowski inequality yields the following estimates:

$$\begin{aligned} &\left(\sum_{i=1}^{n} |(fg)(b_i) - (fg)(a_i)|^p\right)^{\frac{1}{p}} \\ \leq\ &\left(\sum_{i=1}^{n} (\|g\|_\infty |f(b_i) - f(a_i)| + \|f\|_\infty |g(b_i) - g(a_i)|)^p\right)^{\frac{1}{p}} \\ \leq\ &\|g\|_\infty \left(\sum_{i=1}^{n} |f(b_i) - f(a_i)|^p\right)^{\frac{1}{p}} + \|f\|_\infty \left(\sum_{i=1}^{n} |g(b_i) - g(a_i)|^p\right)^{\frac{1}{p}}. \end{aligned}$$

Hence, $fg \in AC_p([a,b])$, by (16), and

$$Var_p(fg;[a,b])^{\frac{1}{p}} \leq \|g\|_\infty Var_p(f;[a,b])^{\frac{1}{p}} + \|f\|_\infty Var_p(g;[a,b])^{\frac{1}{p}}, \tag{17}$$

by Definition 2. Finally, let us prove (10). Applying (6), (15), and (17), we get the following estimates,

$$\begin{aligned} \|fg\|_{WBV_p} &= |f(a)g(a)| + Var_p(fg;[a,b])^{\frac{1}{p}} \\ &\leq \|f\|_\infty |g(a)| + \|g\|_\infty Var_p(f;[a,b])^{\frac{1}{p}} + \|f\|_\infty Var_p(g;[a,b])^{\frac{1}{p}} \\ &= \|f\|_\infty \left(|g(a)| + Var_p(g;[a,b])^{\frac{1}{p}}\right) + \|g\|_\infty Var_p(f;[a,b])^{\frac{1}{p}} \\ &\leq \|f\|_\infty \|g\|_{WBV_p} + \|g\|_\infty \|f\|_{WBV_p}, \end{aligned}$$

which, combined with $\|f\|_\infty \leq \|f\|_{WBV_p}$ ([11], Proposition 1.32 (d)), yield (10), and the proof is completed. □

4. The composition operator in AC_p spaces

Theorem 1. *Given a real compact interval I, let $X(I)$ be a function space containing all constant functions and satisfying the following condition:*
$(*)$ *for every $s_0 \in I$, $t_0 \in \mathbb{R}$, and for every sequence $(s_k,t_k) \in I \times \mathbb{R}$, $k \in \mathbb{N}$, convergent to (s_0,t_0), where $(s_k)_{k\in\mathbb{N}}$ is strictly monotonic and $(t_k)_{k\in\mathbb{N}}$ is monotonic, there exists a continuous function $f \in X(I)$ such that*

$$f(s_k) = t_k, \tag{18}$$

for all $k \in \mathbb{N}_0$.

If the composition operator H of the generator $h : I \times \mathbb{R} \to \mathbb{R}$ maps $X(I)$ into $C(I)$, then h is continuous.

Proof. We start with showing that h is continuous with respect to the second variable. To this end, fix $(x_0,y_0) \in I \times \mathbb{R}$ and take an arbitrary sequence $(y_n)_{n\in\mathbb{N}}$ of real numbers convergent to y_0. Choose a monotonic subsequence $(y_{n_k})_{k\in\mathbb{N}}$ and define functions $P_k : I \to \mathbb{R}$, $k \in \mathbb{N}_0$, by

$$P_k(t) := y_{n_k}, \quad t \in I, \quad k \in \mathbb{N}, \quad P_0(t) := y_0. \tag{19}$$

Of course, by the assumption, P_k, $k \in \mathbb{N}_0$, as constant functions, belong to $X(I)$.

Fix an $\varepsilon > 0$. Since all the functions $H(P_k)$, $k \in \mathbb{N}_0$, are continuous, there exist $\delta_k > 0$ such that the following implication holds:

$$(|x - x_0| < \delta_k \wedge x \in I) \Longrightarrow |H(P_k)(x) - H(P_k)(x_0)| < \varepsilon, \quad k \in \mathbb{N}_0. \tag{20}$$

Choose a strictly monotonic sequence $(x_k)_{k\in\mathbb{N}}$ such that

$$|x_k - x_0| < \delta_k, \quad x_k \in I, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} x_k = x_0.$$

In virtue of condition $(*)$ with $s_k = x_k$ and $t_k = y_{n_k}, k \in \mathbb{N}_0$, we get the existence of continuous function $f \in X(I)$ such that

$$f(x_k) = y_{n_k}, \ k \in \mathbb{N}, \ f(x_0) = y_0. \tag{21}$$

Thus, applying the triangle inequality, (19) and (21), we get

$$\begin{aligned} |h(x_0,y_{n_k}) - h(x_0,y_0)| &\le |h(x_k,y_{n_k}) - h(x_0,y_{n_k})| + |h(x_k,y_{n_k}) - h(x_0,y_0)| \\ &= |h(x_k,P_k(x_k)) - h(x_0,P_k(x_k))| \\ &\quad + |h(x_k,f(x_k)) - h(x_0,f(x_0))| \\ &= |h(x_k,P_k(x_k)) - h(x_0,P_k(x_0))| \\ &\quad + |h(x_k,f(x_k)) - h(x_0,f(x_0))|, \end{aligned}$$

whence, by the definition of a composition operator,

$$|h(x_0,y_{n_k}) - h(x_0,y_0)| \le |H(P_k)(x_k) - H(P_k)(x_0)| + |H(f)(x_k) - H(f)(x_0)|,$$

and, consequently, by (20),

$$|h(x_0, y_{n_k}) - h(x_0, y_0)| \le \varepsilon + |H(f)(x_k) - H(f)(x_0)|.$$

Now, the continuity of $H(f)$ at x_0 implies that h is continuous with respect to the second variable.

To show that h is continuous, fix $(x_0, y_0) \in I \times \mathbb{R}$ and take an arbitrary pair of sequences $x_k \in I$, $y_k \in \mathbb{R}$, $k \in \mathbb{N}$, convergent to x_0, y_0, respectively, such that $(x_k)_{k\in\mathbb{N}}$ is non-constant. Choose a strictly monotonic sequence $(x_{n_k})_{k\in\mathbb{N}}$ and monotonic sequence $(y_{n_k})_{k\in\mathbb{N}}$, convergent to x_0, y_0, respectively, and define functions $P_k : I \to \mathbb{R}$, $k \in \mathbb{N}_0$, by (18). Again, from condition $(*)$ with $s_k = x_{n_k}$ and $t_k = y_{n_k}$, $k \in \mathbb{N}_0$, there exists $f \in X(I)$ satisfying (21). Thus, we have,

$$|h(x_{n_k}, y_{n_k}) - h(x_0, y_0)| = |h(x_{n_k}, f(x_{n_k})) - h(x_0, f(y_0))|,$$

whence, by the definition of a composition operator,

$$|h(x_{n_k}, y_{n_k}) - h(x_0, y_0)| = |H(f)(x_{n_k}) - H(f)(x_0)|.$$

Since $H(f)$ is continuous at x_0, the proof is completed. □

Corollary 1. *If the function space $X(I)$ contains the family of all continuous and monotonic functions defined on a real compact interval I, then the generator $h : I \times \mathbb{R} \to \mathbb{R}$ of any composition operator H mapping $X(I)$ into $C(I)$ is continuous.*

Remark 3. *By Corollary 1, we immediately get Theorem 1 of [13] and Theorem 1 of [14].*

Lemma 5. Let $I = [a,b] \subset \mathbb{R}$ *be a compact interval. The function space $AC_p(I)$ satisfies $(*)$ for all* $p > 1$.

Proof. Fix $p > 1$, $s_0 \in I$, $t_0 \in \mathbb{R}$, and take any sequence $(s_k, t_k) \in I \times \mathbb{R}$, $k \in \mathbb{N}$, convergent to (s_0, t_0), such that $(s_k)_{k\in\mathbb{N}}$ is strictly monotonic and $(t_k)_{k\in\mathbb{N}}$ is monotonic. We consider two cases: (i) $s_0 \ne b$ and $(s_k)_{k\in\mathbb{N}}$ is strictly decreasing; (ii) $s_0 \ne a$ and $(s_k)_{k\in\mathbb{N}}$ is strictly increasing.

(i) To prove the existence $f \in AC_p(I)$ satisfying (18), for all $k \in \mathbb{N}_0$, define a sequence of functions $f_k : [a,b] \to \mathbb{R}$, $k \in \mathbb{N}$, by

$$f_k(s) := \begin{cases} t_0 & \text{for} \quad s \in (-\infty, s_0] \cap I \\ \dfrac{t_k - t_0}{s_k - s_0}(s - s_0) + t_0 & \text{for} \quad s \in (s_0, s_k] \\ \dfrac{t_i - t_{i-1}}{s_i - s_{i-1}}(s - s_{i-1}) + t_{i-1} & \text{for} \quad s \in (s_i, s_{i-1}],\ i \in \{2, \ldots, k\} \\ t_1 & \text{for} \quad s \in (s_1, \infty) \cap I \end{cases}.$$

Arguing as in the proof of Lemma 2 of [15], with $x_k = s_k$ and $y_k = t_k$, $k \in \mathbb{N}_0$, we deduce that a function $f:[a,b]\to\mathbb{R}$ defined by

$$f(s) := \lim_{n\to\infty} f_n(s), \quad s\in I,$$

satisfies (18), is continuous and monotonic (i.e., it is decreasing if $(t_k)_{k\in\mathbb{N}}$ is decreasing, and it is increasing if $(t_k)_{k\in\mathbb{N}}$ is increasing). Thus, given $\varepsilon>0$, we can choose s_{k_0} such that

$$|f(s_{k_0})-f(s_0)|^p < \frac{\varepsilon}{2}.$$

On the other hand, we have

$$Var_p(f;[s_0,s_{k_o}]) = |f(s_{k_0})-f(s_0)|^p, \tag{22}$$

by monotonicity of f on $[s_0,s_{k_0}]$ and Lemma 2 of [14]. Since $f\Big|_{[s_{k_0},b]}$ is continuous and piecewise affine, by Lemmas 2-3, f is p-absolutely continuous on $[s_{k_0},b]$. Thus, by Lemma 1, we can find a partition $P=\{\zeta_0,\zeta_1,\dots,\zeta_{m-1},\zeta_m\}\in\mathscr{P}([s_{k_0},b])$, where $s_{k_0}=\zeta_0<\zeta_1<\ldots<\zeta_{m-1}<\zeta_m=b$, satisfying

$$\sum_{j=1}^{m} Var_p(f;[\zeta_{j-1},\zeta_j]) < \frac{\varepsilon}{2}. \tag{23}$$

Moreover, since f is constant on $[a,s_0]$,

$$Var_p(f;[a,s_0]) = 0. \tag{24}$$

Finally, by (22)-(24), for a partition $\{a,s_0,s_{k_o},\zeta_1,\dots,\zeta_{m-1},\zeta_m\}\in\mathscr{P}([a,b])$, we get

$$Var_p(f;[a,s_0]) + Var_p(f;[s_0,s_{k_o}]) + \sum_{j=1}^{m} Var_p(f;[\zeta_{j-1},\zeta_j]) < \varepsilon,$$

which, by Lemma 1, establishes the claim in case (i).

(ii) Modifying the sequence $f_k:[a,b]\to\mathbb{R}$, $k\in\mathbb{N}$, as follows

$$f_k(s) := \begin{cases} t_1 & \text{for} \quad s\in(-\infty,s_1)\cap I \\[2ex] \dfrac{t_k-t_0}{s_k-s_0}(s-s_0)+t_0 & \text{for} \quad s\in[s_k,s_0) \\[2ex] \dfrac{t_i-t_{i-1}}{s_i-s_{i-1}}(s-s_{i-1})+t_{i-1} & \text{for} \quad s\in[s_{i-1},s_i)\,,\ i\in\{2,\dots,k\} \\[2ex] t_0 & \text{for} \quad s\in[s_0,\infty)\cap I \end{cases},$$

and arguing as in (i), one obtains a continuous and monotone limiting function $f:[a,b]\to\mathbb{R}$ satisfying (18). To establish that $f\in AC_p(I)$, fix $\varepsilon>0$. By the continuity and monotonicity of f, there exists s_{k_0} such that

$$Var_p(f;[s_{k_o},s_0]) = |f(s_0) - f(s_{k_0})|^p < \frac{\varepsilon}{2}. \tag{25}$$

Since $f\Big|_{[a,s_{k_0}]}$ is continuous and piecewise affine, Lemmas 1-3 imply the existence of a partition $\{\eta_0,\eta_1,\ldots,\eta_{l-1},\eta_l\} \in \mathscr{P}([a,s_{k_0}])$ satisfying

$$\sum_{j=1}^{l} Var_p(f;[\eta_{j-1},\eta_j]) < \frac{\varepsilon}{2}. \tag{26}$$

Hence, by (25)-(26), for a partition $\{\eta_0,\eta_1,\ldots,\eta_{l-1},\eta_l,s_0,b\} \in \mathscr{P}([a,b])$, we obtain

$$\left(\sum_{j=1}^{l} Var_p(f;[\eta_{j-1},\eta_j])\right) + Var_p(f;[s_{k_0},s_0]) + Var_p(f;[s_0,b]) < \varepsilon,$$

as f is constant on $[s_0,b]$. This completes the proof, by Lemma 1. □

Lemma 6 ([6], Lemma 1). If a composition operator H of the generator h maps the set $AC(I)$ into the set $C(I)$, then h is continuous.

The main result reads as follows.

Theorem 2. Let $1 \le p \le q < \infty$ *and* $I = [a,b] \subset \mathbb{R}$ *be a compact interval. If the composition operator* H *of the generator* $h : I \times \mathbb{R} \to \mathbb{R}$ *maps* $AC_p(I)$ *into* $AC_q(I)$, *then* h *is continuous.*

Proof. Fix $p,q \in [1,\infty)$ and $p \le q$. Let us notice, that if a composition operator H maps $AC_p(I)$ into $AC_q(I)$, then, by (5), H maps $AC_p(I)$ into $C(I)$. Thus, the continuity of h follows directly from Lemma 6 in the case where $p = 1$, and from Theorem 1 and Lemma 5 in the case where $p > 1$, completing the proof. □

For the sake of completeness, we will now consider the composition operators that are uniformly bounded.

Definition 3 ([5], Definition 1). Let $\mathscr{Y}$ and $\mathscr{Z}$ be metric (or normed) spaces. A mapping $H : \mathscr{Y} \to \mathscr{Z}$ is said to be *uniformly bounded*, if for any $t > 0$ there is a nonnegative real number $\gamma(t)$ such that for any set $B \subset \mathscr{Y}$ we have

$$\operatorname{diam} B \le t \Rightarrow \operatorname{diam} H(B) \le \gamma(t).$$

Definition 4 ([11], Definition 6.26). We say that a pair $(X(I),Y(I))$ of normed function spaces $(X(I),\|\cdot\|_X)$ and $(Y(I),\|\cdot\|_Y)$ has *the uniform weak Matkowski property*, if whenever the Nemytskij composition operator H maps the space $(X(I),\|\cdot\|_X)$ into the space $(Y(I),\|\cdot\|_Y)$ and is uniformly bounded, the corresponding generating *left regularization of* h, i.e., the function $h^- : I^- \times \mathbb{R} \to \mathbb{R}$ defined by

$$h^-(x,y) := \lim_{s \uparrow x} h(s,y), \quad x \in I^- = (a,b]; \quad y \in \mathbb{R},$$

must have the form

$$h^{-}(x,y)=\alpha(x)y+\beta(x), \quad x\in(a,b],\ y\in\mathbb{R}, \tag{27}$$

for some functions $\alpha,\beta\in Y(I)$. Similarly, we say that $(X(I),Y(I))$ has *the uniform Matkowski property,* if the generator h of the uniformly bounded Nemytskij superposition operator $H:X(I)\to Y(I)$ has the form (27) with $h^{-}=h$.

Let us quote the following theorem.

Theorem 3 ([11], Theorem 6.29). *Let* $X(I)$ *and* $Y(I)$ *be two function spaces over* $[a,b]$ *such that the space* $P_n([a,b])$ *of polynomials of degree not exceeding n, equipped with the norm of* $X(I)$*, is imbedded into* $X(I)$ *for each* $n\in N$ *and* $Y(I)$ *is imbedded into some space* $\Phi BV([a,b])$ *of functions of bounded Schramm variation with classical* $\|\cdot\|_{\Phi BV}$ *Schramm-norm. If a generator* $h:[a,b]\times\mathbb{R}\to\mathbb{R}$ *of a corresponding Nemytskij composition operator acting between* $X(I)$ *and* $Y(I)$ *is such that for any* $x\in[a,b]$ *a function* $h(x,\cdot):\mathbb{R}\to\mathbb{R}$ *is continuous with respect to the second variable, then* $(X(I),Y(I))$ *has the uniform weak Matkowski property.*

Let us recall that a function space $(X(I),\|\cdot\|_X)$ is *imbedded* into another function space $(Y(I),\|\cdot\|_Y)$ (symbolically $X\hookrightarrow Y$), if $X(I)\subseteq Y(I)$ and

$$\|f\|_Y\le c\|f\|_X, \quad f\in X(I),$$

for some *imbedding constant* $c>0$ independent of f.

Now, we can give a characterization formula for composition operators acting between the spaces of generalized p-absolutely continuous functions which are, additionally, uniformly bounded.

Theorem 4. Let $1\le p\le q<\infty$ *and* $I=[a,b]$ *be a real compact interval. If a composition operator H mapping* $AC_p(I)$ *into* $AC_q(I)$ *is uniformly bounded, then there exist* $\alpha,\ \beta\in AC_q(I)$ *such that*

$$H(f)(x)=\alpha(x)f(x)+\beta(x), \quad f\in AC_p(I), \quad (x\in I). \tag{28}$$

Conversely, if an operator $H:\mathbb{R}^I\to\mathbb{R}^I$ *is defined by (28) for some functions* $\alpha,\beta\in AC_q(I)$*, then the operator H maps* $AC_p(I)$ *into* $AC_q(I)$*, and is uniformly bounded composition operator.*

Proof. Fix $p,q\in[1,\infty)$, $p\le q$. Let us observe that $\left(WBV_q,\|.\|_{WBV_q}\right)$ is a special case of a Banach space $\Phi BV(I)$ of functions of bounded variation in the sense of Schramm where the Schramm sequence $\Phi=(\varphi_i)_{i=1}^{\infty}$ is defined by $\varphi_i(u)=u^q$, $i\in\mathbb{N}$. Thus, by (4), (5), and Lemma 4, $AC_q([a,b])$ is imbedded into some space $\Phi BV([a,b])$. Since $P_n([a,b])\subset AC(I)\hookrightarrow AC_p(I)$, by Theorem 3 and the continuity of the generator h, we get a required form (28) of an operator H.

To prove the converse statement, suppose that there exist $\alpha,\beta \in AC_q(I)$ such that (28) is fulfilled. Hence, H is a composition operator with generator given by

$$h(x,y) := \alpha(x)y + \beta(x), \quad x \in I, \quad y \in \mathbb{R}.$$

Since $AC_p(I) \subset AC_q(I)$, $1 \leq p \leq q$, by (5), and the space $AC_q(I)$ is an algebra, by Lemma 4, therefore $H(f) \in AC_q(I)$, so that H maps $AC_p(I)$ into $AC_q(I)$.

Moreover, (6), (10) and (28) yield

$$\|H(f_1) - H(f_2)\|_{WBV_q} = \|\alpha(f_1 - f_2)\|_{WBV_q} \leq 2\|\alpha\|_{WBV_q} \|f_1 - f_2\|_{WBV_p},$$

for all $f_1, f_2 \in AC_p(I)$. Thus, setting $\gamma(t) = 2\|\alpha\|_{WBV_q} t$, $t > 0$, in Definition 3, we get the local boundedness of H, which completes the proof. □

Corollary 2. The pair $(AC_p(I), AC_q(I))$ of the normed spaces $\left(AC_p(I), \|\cdot\|_{WBV_p}\right)$ and $\left(AC_q(I), \|\cdot\|_{WBV_q}\right)$, for $1 \leq p \leq q < \infty$, has the uniform Matkowski property.

Remark 4. Theorem 4 completes the main result of [6], where the case $p = q = 1$ is considered.

5. Conclusion

The Nemytskij composition operators are treated as a tool in many fields of science and, consequently, it is necessary to consider them in many function spaces. Some examples show that regularity of a generating function depends largely on the properties of the function spaces both in its domain and range. In the present paper, we state the condition for the function space $X(I)$ under which the generator of each composition operator mapping $X(I)$ into $C(I)$ is continuous. As a corollary, we get that the generator of any Nemytskij operator acting between the Banach algebras $\left(AC_p(I), \|\cdot\|_{WBV_p}\right)$ of generalized, in the sense of Love, p-absolutely continuous functions defined on a real compact interval I is continuous. This completes the result of [6], where the case $p = 1$ is considered.

Acknowledgments

I am grateful to the anonymous reviewers for their extremely valuable comments and suggestions.

References

[1] Grützner, S., & Muntean, A. (2021). Identifying processes governing damage evolution in quasi--static elasticity, Part 1 – Analysis. *Adv. Math. Sci. Appl.*, *30*(2), 305-334.

[2] Christof, C., & Kowalczyk, J. (2024). On the Identification and Optimization of Nonsmooth Superposition Operators in Semilinear Elliptic PDEs. *ESAIM: Control Optim. Calc. Var.*, *30*, 16.

[3] Brunner, H. (2017). Volterra Integral Equations. Cambridge University Press.

[4] Lichawski, K., Matkowski, J., & Miś, J. (1989). Locally defined operators in the space of differentiable functions. *Bull. Polish Acad. Sci. Math.*, *37*, 315-325.

[5] Matkowski, J. (2011). Uniformly bounded composition operators between general Lipschitz functions normed spaces. *Topol. Methods Nonlinear Anal.*, *38*(2), 395-406.

[6] Głazowska, D., Matkowski, J., Merentes, N., & Sanchez, J.L. (2012). Uniformly bounded composition operators in the Banach space of absolutely continuous functions. *Nonlinear Anal. TMA*, *75*(13), 4995-5001.

[7] Głazowska, D., Guerrero, J.A., Matkowski, J., & Merentes, N. (2013). Uniformly bounded composition operators on a Banach space of ϕ-bounded Wiener-Young variation functions. *Bull. Korean Math. Soc.*, *50*(2), 675-685.

[8] Głazowska, D., & Matkowski, J. (2015). Uniformly bounded composition operators. *Bull. Aust. Math. Soc.*, *92*, 463-469.

[9] Aziz, W., Ereú, T., & Guerrero, J.A. (2016). Uniformly bounded composition operators on the space of bounded variation functions in the sense of Waterman. *Mediterr. J. Math.*, *13*, 4305-4320.

[10] Lupa, M., & Wróbel, M. (2017). Uniformly bounded Nemytskij operators acting between the Banach spaces of generalized Hölder functions. *J. Appl. Math. Comput. Mech.*, *16*(4), 37-45.

[11] Appell, J., Banaś, J., & Merentes, N. (2014). Bounded Variation and Around, Berlin, Boston: De Gruyter.

[12] Love, E.L. (1951). A generalization of absolute continuity. *J. London Math. Soc.*, *26*, 1-13.

[13] Wróbel, M. (2021). The form of locally defined operators in Waterman spaces. *Math. Slovaca*, *71*(6), 1529-1544.

[14] Wróbel, M. (2023). Schramm spaces and composition operators. *J. Appl. Math. Comput. Mech.*, *22*(2), 87-98.

[15] Wróbel, M. (2010). Representation theorem for local operators in the space of continuous and monotone functions. *J. Math. Anal. Appl.*, *372*, 45-54.