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A STUDY OF A COUPLED SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

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Abstract. In this study, we discuss the existence and uniqueness of solutions to a coupled system of nonlinear fractional differential equations of the Caputo-Fabrizio type with boundary conditions. We prove the existence and uniqueness of the solution using Banach's fixed-point theorem and the Leray-Schauder alternative theorem. Finally, we support our study with two illustrative examples.

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1. Introduction

Fractional calculus, the extension of differentiation and integration to non-integer orders, has experienced rapid expansion over the past thirty years, largely due to its powerful ability to describe complex systems exhibiting memory and hereditary behavior. Traditional fractional operators, such as the Riemann-Liouville and Caputo derivatives, rely on singular power-law kernels (e.g., $(t-\tau)^{-\alpha}$). While mathematically elegant, these kernels often introduce unphysical singularities at the initial point and pose challenges for numerical computation.

In their 2015 paper, Caputo and Fabrizio defined a fractional derivative without singular kernel. Their aim was to preserve the memory-capturing capability of fractional operators while eliminating the unrealistic assumption of infinite memory at the origin a feature that contradicts the behavior of many physical, biological, and engineering systems. This innovation triggered a major shift in fractional modeling and has since been widely embraced across both theoretical and applied disciplines. In particular, the Caputo-Fabrizio (CF) derivative is favored in contexts where abrupt, singular memory effects are physically implausible, especially in engineering and life sciences.

The primary motivation for investigating the given boundary value problem (BVP) lies in its capacity to represent a broad spectrum of real-world systems that cannot be accurately described by classical integer-order models or even by traditional fractional models with singular kernels. The problem combines several key features: Fractional dynamics ($\gamma_1, \gamma_2 \in (1,2)$): capturing memory and behavior intermediate between velocity and acceleration. The (CF) kernel: ensuring physical plausibility through smooth, exponential memory decay. The nonlinear coupling: modeling intricate interactions between state variables. The nonlocal and cross-variable boundary conditions: encoding feedback mechanisms, regulatory constraints, or design specifications.

Together, these elements render the system both mathematically rich and physically meaningful. Specifically, this BVP describes a coupled physical system with memory, where two state variables $r(s)$ and $w(s)$ evolve under fractional-order dynamics governed by the (CF) derivative. In viscoelasticity, the variables $r(s)$ and $w(s)$ represent the relationship between a beam's transverse displacement and its internal stress (or bending moment), respectively, where s often denotes a spatial coordinate, the (CF) derivative models the materialís memory with exponential relaxation, the condition $r(1) = \lambda r(\sigma)$ implies that the displacement at the endpoint is regulated based on the reading from a sensor located at $s = \sigma$, and the condition $w(1) = \beta w(\nu)$ indicates that the stress at the endpoint depends on the displacement measured at $s = \nu$, possibly due to an actuator responding to sensor feedback.
For further physical interpretations and theoretical background, see references [1–6]. We present the studied issue in the following form:

$$\begin{cases} \mathscr{D}^{(\gamma_1)} r(s) = h_1(s, r(s), w(s)), s \in I := [0,1] \\ \mathscr{D}^{(\gamma_2)} w(s) = h_2(s, r(s), w(s)), s \in I := [0,1] \\ r(0) = w(0) = 0 \\ r(1) = \lambda r(\sigma), w(1) = \beta w(\nu), 0 < \sigma, \nu < 1, \lambda, \beta > 0 \end{cases} \tag{1}$$

with γ_i being real numbers such that $1 < \gamma_i < 2$ for $i = 1,2$, $h_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,$i = 1,2$ is continuous functions, and $\mathscr{D}^{(\gamma_i)}$ is the new fractional derivative Caputo-Fabrizio of order $\gamma_i \in (1,2)$. This research is organized as follows. In the second part, we present some definitions and lemmas used in the rest of the research. In the third part, the existence and uniqueness of the solution to the dual system (1) was proven by applying Banach's fixed point theory to prove uniqueness by Leray-Schauder alternative theory to prove existence. In part four, we illustrate the main findings with two examples.

2. Preliminary and lemmas

In this section, we present some of the basic definitions, properties and lemmas of our research.

Definition 1 [1] The Caputo-Fabrizio fractional derivative (CFFD) of α order for a function $N \in H^1(e,d), d > e$ and $\alpha \in]0,1[$ is given as

$$\mathscr{D}^{(\alpha)}N(s) = \frac{\mathscr{Z}(\alpha)}{1-\alpha}\int_e^s N'(\eta)\exp\left[-\frac{\alpha(s-\eta)}{1-\alpha}\right]d\eta, \tag{2}$$

where $\mathscr{Z}(\alpha)$ is a normalization function with the main properties of $\mathscr{Z}(0) = \mathscr{Z}(1) = 1$. □

Lemma 1 *Let $n \geqslant 1$ the Caputo-Fabrizio fractional integral of order $0 < \alpha < 1$ for a function $N \in H^1(e,d), d > e$ is defined by*

$$I_e^{n+\alpha}N(s) = \frac{1}{\mathscr{Z}(\alpha).n!}\int_e^s (s-\eta)^{n-1}[\alpha(s-\eta)+n(1-\alpha)]N(\eta)d\eta,$$

Lemma 2 *[7] Let $\gamma \in (n, n+1), n = [\gamma] \geqslant 0$. Assume that*

1. *if $N(e) = 0$, then $\mathscr{D}^{(\gamma)}\left(I_e^{\gamma}N(s)\right) = N(s)$.*
2. *$I_e^{\gamma}\left(\mathscr{D}^{(\gamma)}N(s)\right) = N(s) + \sum_{i=0}^{n} e_i s^i, e_i \in \mathbb{R}, i = 0,1,\ldots,n.$* □

Lemma 3 *(Leray-Schauder alternative [8]) Let $B : K \to K$ be a completely continuous operator (i.e., a map that restricted to any bounded set in K is compact). Let $\mathscr{T}(B) = \{r \in K : r = \theta B(r)$, for some $0 < \theta < 1\}$. Then, either the set $\mathscr{T}(B)$ is unbounded or B has at least one fixed point.* □

3. Analytic study

Throughout the following, we assume that the function $\mathscr{Z}(\alpha) = 1$.

Lemma 4 *Let $F \in \mathscr{C}(I;\mathbb{R})$ be a given function and $1 < \gamma < 2$. Then, the unique solution of*

$$\begin{cases} \mathscr{D}^{(\gamma)}r(s) = F(s), s \in I := [0,1] \\ r(0) = 0, r(1) = \lambda r(\sigma), \end{cases} \tag{3}$$

is given by the integral equation

$$r(s) = \frac{1}{\sigma\lambda - 1} s \left[\int_0^1 [(1-\alpha\eta)]F(\eta)d\eta - \lambda \int_0^\sigma [\alpha(\sigma-\eta)+(1-\alpha)]F(\eta)d\eta \right]$$
$$+ \int_0^s [\alpha(s-\eta)+(1-\alpha)]F(\eta)d\eta \tag{4}$$

PROOF By applying Lemma 1, equation (3) can be transformed into an equivalent integral equation:

$$I_0^\gamma(\mathscr{D}^{(\gamma)}r(s)) = I_0^\gamma(F(s)),$$

we obtain,

$$r(s) = -c_0 - c_1 s + \int_0^s [\alpha(s-\eta)+(1-\alpha)]F(\eta)d\eta \tag{5}$$

where $c_0, c_1 \in \mathbb{R}$. Thus, from the initial condition $r(0) = 0$ and $r(1) = \lambda r(\sigma)$ we obtain $c_0 = 0$ and

$$-c_1 + \int_0^1 [(1-\alpha\eta)]F(\eta)d\eta = -c_1\sigma\lambda + \lambda \int_0^\sigma [\alpha(\sigma-\eta)+(1-\alpha)]F(\eta)d\eta,$$

which implies that

$$c_1 = -\frac{1}{\sigma\lambda - 1} \left[\int_0^1 [(1-\alpha\eta)]F(\eta)d\eta - \lambda \int_0^\sigma [\alpha(\sigma-\eta)+(1-\alpha)]F(\eta)d\eta \right]$$

We substitute c_0 and c_1 into the equation (5), completes solution (4).
Conversely, if r satisfies (4), then $r(0) = 0, r(1) = \lambda r(\sigma)$, and for each $s \in I := [0,1]$, we have

$$\mathscr{D}^{(\gamma)}r(s) = F(s)$$

Hence, r satisfies (3).

From the above Lemma, we can conclude the following Lemma.

Lemma 5 *Assume that $h_i : I \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions. Then (r,w) is a solution of (1) if and only if (r,w) is a solution of the coupled system of integral equations*

$$
\begin{aligned}
r(s) &= \frac{1}{\sigma\lambda-1}s\left[\int_0^1[(1-\alpha\eta)]h_1(\eta,r(\eta),w(\eta))d\eta\right.\\
&\left.-\lambda\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\right]\\
&+\int_0^s[\alpha(s-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\\
w(s) &= \frac{1}{\nu\beta-1}s\left[\int_0^1[(1-\alpha\eta)h_2(\eta,r(\eta),w(\eta))d\eta\right.\\
&\left.-\beta\int_0^\nu[\alpha(\nu-\eta)+(1-\alpha)]h_2(\eta,r(\eta),w(\eta))d\eta\right]\\
&+\int_0^s[\alpha(s-\eta)+(1-\alpha)]h_2(\eta,r(\eta),w(\eta))d\eta
\end{aligned}
\tag{6}
$$

4. Study the existence and uniqueness of the solution

4.1. Uniqueness

Proof of uniqueness by Banach's fixed point theorem

Let $\mathscr{C}(I;\mathbb{R})$ denote the space of all continuous functions. Let $E=\{r(s):r(s)\in\mathscr{C}(I;\mathbb{R})\}$ be a Banach space endowed with the norm $\|r\|_E=\sup\limits_{s\in[0,1]}|r(t)|$, and let $F=\{w(s):w(s)\in\mathscr{C}(I;\mathbb{R})\}$ be a Banach space endowed with the norm $\|w\|_F=\sup\limits_{s\in[0,1]}|w(t)|$. Thus, the product space $(E\times F,\|\cdot\|)_{E\times F}$ is also a Banach space with the norm $\|(r,w)\|_{E\times F}=\|r\|_E+\|w\|_F$ for $(r,w)\in E\times F$.

We define the operator as follows $B:E\times F\longrightarrow E\times F$ as

$$
B(r,w)(s)=\begin{pmatrix}B_1(r,w)(s)\\B_2(r,w)(s)\end{pmatrix}
\tag{7}
$$

where

$$B_1(r,w)(s) = \frac{1}{\sigma\lambda - 1} s \left[\int_0^1 [(1-\alpha\eta)] h_1(\eta, r(\eta), w(\eta)) d\eta \right.$$

$$\left. -\lambda \int_0^\sigma [\alpha(\sigma-\eta) + (1-\alpha)] h_1(\eta, r(\eta), w(\eta)) d\eta \right]$$

$$+ \int_0^s [\alpha(s-\eta) + (1-\alpha)] h_1(\eta, r(\eta), w(\eta)) d\eta \tag{8}$$

and

$$B_2(r,w)(s) = \frac{1}{\nu\beta - 1} s \left[\int_0^1 [(1-\alpha\eta) h_2(\eta, r(\eta), w(\eta)) d\eta \right.$$

$$\left. -\beta \int_0^\nu [\alpha(\nu-\eta) + (1-\alpha)] h_2(\eta, r(\eta), w(\eta)) d\eta \right]$$

$$+ \int_0^s [\alpha(s-\eta) + (1-\alpha)] h_2(\eta, r(\eta), w(\eta)) d\eta \tag{9}$$

To derive our main results, we impose the following assumptions.

(D_1) The functions $h_i : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,2$ are continuous, and there exist real constants $x_i > 0, i = 1,2$ such that

$$|h_1(s,r_1,w_1) - h_1(s,r_2,w_2)| \leq x_1(|r_1 - r_2| + |w_1 - w_2|)$$

$$|h_2(s,r_1,w_1) - h_2(s,r_2,w_2)| \leq x_2(|r_1 - r_2| + |w_1 - w_2|)$$

for $s \in I$ and $r_i, w_i \in \mathbb{R}, i = 1,2$.

(D_2) There exist real constants $d_i, e_i \geq 0, i = 1,2$ and $d_0 > 0, e_0 > 0$ such that

$$|h_1(s,r,w)| \leq d_0 + d_1|r| + d_2|w|, |h_2(s,r,w)| \leq e_0 + e_1|r| + e_2|w|$$

for $s \in I$ and $r_i, w_i \in \mathbb{R}, i = 1,2$.

To simplify, we use the following symbols:

$$\begin{cases} \mu_1 = \Big(\dfrac{2-\alpha}{2}\Big)\Big[\dfrac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big]M_1, \\ \mu_2 = \Big(\dfrac{2-\alpha}{2}\Big)\Big[\dfrac{1+|\beta|+|\nu\beta-1|}{|\nu\beta-1|}\Big]M_2, \\ \mu_3 = \Big(\dfrac{2-\alpha}{2}\Big)\Big[\dfrac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big]x_1, \\ \mu_4 = \Big(\dfrac{2-\alpha}{2}\Big)\Big[\dfrac{1+|\beta|+|\nu\beta-1|}{|\nu\beta-1|}\Big]x_2, \end{cases} \tag{10}$$

and

$$\begin{cases} \Lambda_0 = (2-\alpha)d_0\Big[\dfrac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big] + (2-\alpha)e_0\Big[\dfrac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big] \\ \Lambda_1 = (2-\alpha)d_1\Big[\dfrac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big] + (2-\alpha)e_1\Big[\dfrac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big] \\ \Lambda_2 = (2-\alpha)d_2\Big[\dfrac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big] + (2-\alpha)e_2\Big[\dfrac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big] \\ \Lambda^* = \max\{\Lambda_1, \Lambda_2\}. \end{cases} \tag{11}$$

Theorem 1 *Suppose that condition (D_1) is satisfied. Then the coupled system (1) possesses a unique solution on I, provided that*

$$(\mu_3 + \mu_4) < 1. \tag{12}$$

PROOF Assume that $\varphi > 0$ is a real number such that

$$\varphi \geq \frac{\mu_1 + \mu_2}{1 - (\mu_3 + \mu_4)}$$

First, we shall show that $B\mathscr{B}_\varphi \to \mathscr{B}_\varphi$, where B is given by (7) and

$$\mathscr{B}_\varphi = \{(r,w) \in E \times F : \|(r,w)\|_{E\times F} \leq \varphi\}$$

Set $\sup\limits_{s\in I}|h_1(s,0,0)| = M_1 < \infty$ and $\sup\limits_{s\in I}|h_2(s,0,0)| = M_2 < \infty$. For $(r,w) \in \mathscr{B}_\varphi, s \in I$,

we have

$$
\begin{aligned}
|B_1(r,w)(s)| \leq & \Big|\frac{1}{\sigma\lambda-1}s\Big[\int_0^1[(1-\alpha\eta)]h_1(\eta,r(\eta),w(\eta))d\eta \\
& -\lambda\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\Big]\Big| \\
& +\Big|\int_0^s[\alpha(s-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\Big| \\
\leq & \frac{1}{|\sigma\lambda-1|}\Big[\int_0^1[(1-\alpha\eta)](|h_1(\eta,r(\eta),w(\eta))-h_1(s,0,0)|+|h_1(s,0,0)|)d\eta \\
& +|\lambda|\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)](|h_1(\eta,r(\eta),w(\eta))-h_1(s,0,0)|+|h_1(s,0,0)|)d\eta\Big] \\
& +\int_0^s[\alpha(s-\eta)+(1-\alpha)](|h_1(\eta,r(\eta),w(\eta))-h_1(s,0,0)|+|h_1(s,0,0)|)d\eta \\
\leq & \frac{1}{|\sigma\lambda-1|}\Big[\int_0^1[(1-\alpha\eta)](x_1(|r(\eta)|+|w(\eta)|)+|h_1(s,0,0)|)d\eta \\
& +|\lambda|\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)](x_1(|r(\eta)|+|w(\eta)|)+|h_1(s,0,0)|)d\eta\Big] \\
& +\int_0^s[\alpha(s-\eta)+(1-\alpha)](x_1(|r(\eta)|+|w(\eta)|)+|h_1(s,0,0)|)d\eta \\
\leq & \frac{1}{|\sigma\lambda-1|}\Big[\int_0^1[(1-\alpha\eta)](x_1(\|r\|_E+\|w\|_F)+M_1)d\eta \\
& +|\lambda|\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)](x_1(\|r\|_E+\|w\|_F)+M_1)d\tau\Big] \\
& +\int_0^s[\alpha(s-\eta)+(1-\alpha)](x_1(\|r\|_E+\|w\|_F)+M_1)d\eta \\
\leq & \Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big](x_1\varphi+M_1)
\end{aligned}
$$

which implies that

$$
\|B_1(r,w)\|_E \leq \Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big](x_1\varphi+M_1) \tag{13}
$$

Similarly, we can find that

$$\|B_2(r,w)\|_F \leq \Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\beta|+|\nu\beta-1|}{|\nu\beta-1|}\Big](x_2\varphi+M_2) \tag{14}$$

Consequently, from (13) and (14), we obtain the following

$$\begin{aligned}\|B(r,w)\|_{E\times F} &\leq \Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big](x_1\varphi+M_1)\\ &\quad+\Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\beta|+|\nu\beta-1|}{|\nu\beta-1|}\Big](x_2\varphi+M_2)\\ &= (\mu_1+\mu_2)+(\mu_3+\mu_4)\varphi\\ &\leq \varphi\end{aligned}$$

Hence, $B\mathscr{B}_\varphi \subset \mathscr{B}_\varphi$. Now, for $(r_1,w_1),(r_2,w_2) \in E\times F$, and for any $s \in I$, we get

$$\begin{aligned}&|B_1(r_1,w_1)(s)-B_1(r_2,w_2)(s)|\\ &\leq \frac{1}{|\sigma\lambda-1|}\Big[\int_0^1[(1-\alpha\eta)](|h_1(\eta,r_1(\eta),w_1(\eta))-h_1(\eta,r_2(\eta),w_2(\eta))|)d\eta\\ &\qquad+|\lambda|\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)](|h_1(\eta,r_1(\eta),w_1(\eta))-h_1(\eta,r_2(\eta),w_2(\eta))|)d\eta\Big]\\ &\qquad+\int_0^s[\alpha(s-\eta)+(1-\alpha)](|h_1(\eta,r_1(\eta),w_1(\eta))-h_1(\eta,r_2(\eta),w_2(\eta))|)d\eta\\ &\leq \frac{x_1}{|\sigma\lambda-1|}\Big[\int_0^1[(1-\alpha\eta)](|r_1(\eta)-r_2(\eta)|+|w_1(\eta)-w_2(\eta)|)d\eta\\ &\qquad+|\lambda|\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)](|r_1(\eta)-r_2(\eta)|+|w_1(\eta)-w_2(\eta)|)d\eta\Big]\\ &\qquad+x_1\int_0^s[\alpha(s-\eta)+(1-\alpha)](|r_1(\eta)-r_2(\eta)|+|w_1(\eta)-w_2(\eta)|)d\eta\\ &\leq \Big(\frac{2-\alpha}{2}\Big)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{|\sigma\lambda-1|}\Big]x_1(\|r_1-r_2\|+\|w_1-w_2\|)\\ &\leq \mu_3(\|r_1-r_2\|+\|w_1-w_2\|)\end{aligned}$$

which implies that

$$\|B_1(r_1,w_1)-B_1(r_2,w_2)\|_E \leq \mu_3(\|r_1-r_2\|+\|w_1-w_2\|) \tag{15}$$

Similarly, we can find that

$$\|B_2(r_1,w_1)-B_2(r_2,w_2)\|_F \leq \mu_4(\|r_1-r_2\|+\|w_1-w_2\|) \tag{16}$$

It follows from (15) and (16) that

$$\|B(r_1,w_1)-B(r_2,w_2)\|_{E\times F} \leq (\mu_3+\mu_4)(\|r_1-r_2\|+\|w_1-w_2\|)$$

From the above inequality, we deduce that operator B is a contraction in light of the condition (12). Thus, according to Banach's fixed point theorem, we conclude that there exists a unique fixed point for operator B. Thus, the conditions of Banach's fixed point theorem are fully satisfied, and the proof of his uniqueness of the solution is complete. ■

4.2. Existence

Proof of Existence via the Leray-Schauder Alternative theorem.

Theorem 2 *Suppose that (D_2) holds. If $\Lambda^* < 1$, then, the coupled system (1) has at least one solution on I, where Λ^* is given in (11).* □

PROOF First, we show that the operator $B : E\times F \to E\times F$ is completely continuous. By the continuity of functions h_1 and h_2, the operator B is continuous. Let $K \in E\times F$ be bounded. Then, there exist constants $N_1 > 0, N_2 > 0$ such that $h_1(s,r(s),w(s)) \leq N_1$ and $h_2(s,r(s),w(s)) \leq N_2$. Then, for any $(r,w) \in K$, we get

$$\begin{aligned}
|B_1(r,w)(s)| \leq \frac{1}{|\sigma\lambda-1|}\Big[&\int_0^1 [(1-\alpha\eta)]|h_1(\eta,r(\eta),w(\eta))|d\eta \\
&+|\lambda|\int_0^\sigma [\alpha(\sigma-\eta)+(1-\alpha)]|h_1(\eta,r(\eta),w(\eta))|d\eta\Big] \\
&+\int_0^s [\alpha(s-\eta)+(1-\alpha)]|h_1(\eta,r(\eta),w(\eta))|d\eta \\
&\leq (2-\alpha)N_1\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big]
\end{aligned}$$

which implies that

$$\|B_1(r,w)(s)\|_E \leq (2-\alpha)N_1\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big] \tag{17}$$

Similarly, we get

$$\|B_2(r,w)(s)\|_F \leq (2-\alpha)N_2\Big[\frac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big] \tag{18}$$

From (17) and (18), it follows that B is uniformly bounded.
Then, we shall show that the operator B is equicontinuous. Suppose that $s_1, s_2 \in I$ with $s_1 < s_2$. Then we have

$$\begin{aligned}
|B_1(r,w)(s_2) - B_1(r,w)(s_1)| &= \Big|\frac{1}{\sigma\lambda-1}s_2\Big[\int_0^1 [(1-\alpha\eta)]h_1(\eta,r(\eta),w(\eta))d\tau \\
&-\lambda\int_0^\sigma [\alpha(\sigma-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\tau\Big] \\
&+\int_0^{s_2}[\alpha(s_2-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta \\
&-\frac{1}{\sigma\lambda-1}s_1\Big[\int_0^1[(1-\alpha\eta)h_1(\eta,r(\eta),w(\eta))d\eta \\
&-\lambda\int_0^\sigma[\alpha(\sigma-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\Big] \\
&-\int_0^{s_1}[\alpha(s_1-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\Big| \\
&\leq \frac{1}{|\sigma\lambda-1|}(s_2-s_1)\Big[\Big(\frac{2-\alpha}{2}\Big)\Big(1+|\lambda|\Big)N_1\Big] \\
&+\Big|\int_0^{s_2}[\alpha(s_2-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta \\
&+\int_{s_1}^0[\alpha(s_1-\eta)+(1-\alpha)]h_1(\eta,r(\eta),w(\eta))d\eta\Big| \\
&\leq \frac{N_1}{|\sigma\lambda-1|}(s_2-s_1)\Big[\Big(\frac{2-\alpha}{2}\Big)\Big(1+|\lambda|\Big)\Big] \\
&+N_1\Big(\frac{\alpha}{2}(s_2^2-s_1^2)-\alpha(s_2-s_1)+(s_2-s_1)\Big) \\
&\leq N_1(s_2-s_1)\Big[\frac{1}{|\sigma\lambda-1|}\Big[\Big(\frac{2-\alpha}{2}\Big)\Big(1+|\lambda|\Big)\Big]+\frac{\alpha}{2}(s_2-s_1)+1-\alpha\Big]
\end{aligned}$$

which implies that $\|B_1(r,w)-B_1(r,w)\| \to 0$ independent of $(r,w) \in K$ as $s_2 - s_1 \to 0$.

Also, we get

$$|B_2(r,w)(s_2)-B_2(r,w)(s_1)| \leq N_2(s_2-s_1)\Big[\frac{1}{|\nu\beta-1|}\Big[\Big(\frac{2-\alpha}{2}\Big)\Big(1+|\beta|\Big)\Big]+\frac{\alpha}{2}(s_2-s_1)+1-\alpha\Big]$$

which implies that $\|B_2(r,w)-B_2(r,w)\| \to 0$ independent of $(r,w) \in K$ as $s_2-s_1 \to 0$. Therefore, the operator B is equicontinuous.
Consequently, by the Arzelà-Ascoli theorem, we deduce that the operator B is completely continuous. Finally, we shall show that the set

$$\mathscr{T} = \{(r,w) \in E \times F : (r,w) = \theta B(r,w), 0 < \theta < 1\}$$

is bounded. Let $(r,w) \in \mathscr{T}$. Then $(r,w) = \theta B(r,w)$. For any $s \in I$, we have

$$r(s) = \theta B_1(r,w)(s), w(s) = \theta B_2(r,w)(s)$$

Then, we have

$$|r(s)| = |\theta B_1(r,w)(s)| \leq |B_1(r,w)(s)|$$

$$\leq (2-\alpha)(d_0+d_1|r|+d_2|w|)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big]$$

and

$$|w(s)| = |\theta B_2(r,w)(s)| \leq |B_2(r,w)(s)|$$

$$\leq (2-\alpha)(e_0+e_1|r|+e_2|w|)\Big[\frac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big]$$

Hence, we get

$$\|r(s)\|_E \leq (2-\alpha)\Big[\frac{1+|\lambda|+|\sigma\lambda-1|}{2|\sigma\lambda-1|}\Big](d_0+d_1\|r\|_E+d_2\|w\|_F),$$

$$\|w(s)\|_F \leq (2-\alpha)\Big[\frac{1+|\beta|+|\nu\beta-1|}{2|\nu\beta-1|}\Big](e_0+e_1\|r\|_E+e_2\|w\|_F),$$

which implies that

$$\|r\|_E+\|w\|_F \leq \Lambda_0+\max\{\Lambda_1,\Lambda_2\}\|r+w\|_{E\times F} \leq \Lambda_0+\Lambda^*\|r+w\|_{E\times F},$$

where $\Lambda_0, \Lambda_1, \Lambda_2$ and Λ^* are given in (11). Consequently, we get

$$\|(r,w)\|_{E\times F} \leq \frac{\Lambda_0}{1-\Lambda^*}$$

This proves that the set $\mathscr{T}$ is bounded. Hence, by Lemma 3, the operator B has at least one fixed point. Thus, the coupled system (1) has at least one solution on I. The proof is completed. ■

Examples

Example 1 Consider the following coupled system of Caputo-Fabrizio fractional differential equations:

$$\begin{cases} \mathscr{D}^{(\frac{3}{2})}r(s) = \dfrac{e^{-3s}}{60+s} + \dfrac{e^{2s}}{50+s^2}\left(\sin w(s) + |r(s)|\right), \quad s \in I := [0,1] \\ \mathscr{D}^{(\frac{19}{10})}w(s) = \sin s - 3 + \dfrac{1}{s+200}\left(\dfrac{|w(s)|}{|w(s)-1|} + \sin r(s)\right), \quad s \in I := [0,1] \\ r(0) = w(0) = 0 \\ r(1) = r(\frac{1}{2}), w(1) = w(\frac{1}{3}), \end{cases} \tag{19}$$

Here, $\gamma_1 = \frac{3}{2}, \gamma_2 = \frac{19}{10}, \sigma = \frac{1}{2}, \nu = \frac{1}{3}, \lambda = \beta = 1,$

$$h_1(s,r,w) = \frac{e^{-3s}}{60+s} + \frac{e^{2s}}{50+s^2}\left(\sin w(s) + |r(s)|\right)$$

and

$$h_2(s,r,w) = \sin s - 3 + \frac{1}{s+200}\left(\frac{|w(s)|}{|w(s)-1|} + \sin r(s)\right)$$

$$|h_1(s,r_1,w_1) - h_1(s,r_2,w_2)| \leq \frac{1}{50}(|r_1 - r_2| + |w_1 - w_2|)$$

and

$$|h_2(s,r_1,w_1) - h_2(s,r_2,w_2)| \leq \frac{1}{200}(|r_1 - r_2| + |w_1 - w_2|)$$

from which we get $x_1 = \frac{1}{50}$ and $x_1 = \frac{1}{200}$. Using the given data, the condition (12) becomes

$$\mu_3 + \mu_4 = 0.025 + 0.004 = 0.029 < 1$$

Therefore, all the conditions of Theorem 1 are met, implying that the coupled system (19) admits a unique solution on I. □

Example 2 To enhance our study, we consider the following system of coupled fractional Caputo-Fabrizio differential equations:

$$\begin{cases} \mathscr{D}^{(\frac{3}{2})}r(s) = \dfrac{e^{-s}}{\sqrt{s+2500}} + \dfrac{1}{170}\sin r(s) + \dfrac{1}{7(40+s)}\sin w(s), \quad s \in I := [0,1] \\ \mathscr{D}^{(\frac{19}{10})}w(s) = \dfrac{1}{3\sqrt{400+s}}\cos s + \dfrac{e^{s}}{50} + \sin r(s) + \dfrac{1}{600}\dfrac{w(s)|r(s)|}{1+|r(s)|}, \quad s \in I := [0,1] \\ r(0) = w(0) = 0 \\ r(1) = r(\dfrac{1}{2}), w(1) = w(\dfrac{1}{3}), \end{cases} \tag{20}$$

Obviously, we get

$$|h_1(s,r,w)| \leq \frac{1}{50} + \frac{1}{170}\|r\|_E + \frac{1}{280}\|w\|_F,$$

$$|h_2(s,r,w)| \leq \frac{1}{60} + \frac{1}{50}\|r\|_E + \frac{1}{600}\|w\|_F,$$

Thus $d_0 = \frac{1}{50}$, $d_1 = \frac{1}{170}$, $d_2 = \frac{1}{280}$, $e_0 = \frac{1}{60}$, $e_1 = \frac{1}{50}$, $e_0 = \frac{1}{600}$. Using (11), we find that

$$\Lambda^* = \max\{\Lambda_1, \Lambda_2\} = \max\{0.00446429, 0.00033333\} = 0.00446429 < 1$$

Therefore, the assumptions of Theorem 2 are satisfied. Hence, the coupled system (20) has at least one solution on I. □

Conclusion

In this paper, we focus on the analysis of a coupled system of nonlinear fractional differential equations that incorporate Caputo-Fabrizio fractional derivatives with non-singular exponential kernels. This class of derivatives has gained significant attention for its ability to model many complex systems. Our primary objective is to examine the existence and uniqueness of solutions to the system under appropriate conditions. To achieve this, we employ two powerful analytical tools: the Leray--Schauder alternative theorem and Banach's fixed point theorem. The first is employed to prove the existence of at least one solution under the previously stated assumptions, while the second offers a framework for demonstrating the uniqueness of the solution by verifying the contraction property of the operator B introduced earlier in the paper. In addition, to demonstrate the applicability and effectiveness of the theoretical results, we provide two illustrative examples that meet the required conditions and confirm the theoretical results.

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