JAM CM Journal of Applied Mathematics and Computational Mechanics 2025, 24(4), 75-84
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2025.4.05 e-ISSN 2353-0588

A FEW REMARKS ABOUT YOUNG MEASURES WITH DENSITIES

Piotr Puchała

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
piotr.puchala@pcz.pl

Received: 12 July 2025; Accepted: 17 November 2025

Abstract. We introduce the notion of a density of a Young measure and investigate its first properties. The notion is illustrated with examples of nonhomogeneous Young measures with densities, also providing a link between Young measures and set-valued analysis *via* the Bressan-Colombo-Fryszkowski theorem on the existence of continuous selections of multifunctions with decomposable values.

MSC 2010: *46N10, 74B99, 26E25*
Keywords: *Young measure, density, multifunction, continuous selection*

1. Introduction

In the Introduction, we will sketch basic motivations for investigating Young measures. We will provide bibliographic suggestions in the following sections together while introducing necessary elements of the theory.

The deformation of crystals in certain elastic materials reveals periodic distortion patterns, known as microstructures, which are small in scale but visible under a microscope. The energy functionals of materials exhibiting a microstrucutre under deformation do not attend their infima. Mathematically this phenomenon is described by highly oscillatory in nature, minimizing sequences of such functionals.

The density of energy (i.e. the integrands in the energy functional) of these materials is not quasiconvex with respect to the gradient of the deformation. This is the reason why the energy functionals of such materials are not weakly lower semicontinuous and thus they do not attain their infima. This is the cause of highly oscillatory nature of its minimizing sequences, which is physically realized by the microstructure, which in turn can be observed through a microscope by an engineer. We cannot speak about classical infimum in this case. This difficulty can be addressed with in two ways. The first way is to calculate the so called quasiconvex envelope of the integrand and consider energy functionals with this form of energy density. However, obtaining an explicit form of quasiconvex envelope of the density of energy is usually very difficult, also some information about spatial behaviour of the minimizing sequence is lost in the process. The second idea is to enlarge the space of admissible integrands from the function space (usually a Sobolev one) into the space of weakly* measurable measure-valued mappings (we provide this and all other necessary definitions and results in the next Section). The measures of interest are called Young measures after Laurence Chisholm Young, who was the first one to prove their existence and investigate their basic properties in the one-dimensional case (1937).

Young measures have turned out to be the proper tool in analyzing the spatial oscillations of minimizing sequences of energy functionals of some elastic materials. Since they conserve some information about spatial oscillatory properties of minimizing sequences, Young measures can be regarded as a means summarizing these properties.

Although it is known that one can associate a Young measure with any bounded Borel function, calculation of an explicit form of a Young measure associated with a particular sequence of functions is usually difficult. This is connected with the difficulty of calculating weak* limits of sequences of functions, especially when the weak* limit is not a function defined on a subset of $\mathbb{R}^n$, but a set function (in our case – a Young measure) defined on a σ-algebra of subsets. However, it turns out that the problem becomes simpler in some cases. Namely, associating a Young measure with each element of the minimizing sequence yields a surprising result: in many cases the sequence of Young measures is constant (while the respective sequence of functions is neither constant nor convergent in norm), so it is trivially convergent to a Young measure that is obviously a generalized weak* limit of the considered function sequence. In this case, we deal with a homogeneous Young measure that is, a weak* measurable measure-valued mapping which is constant. Observe that despite the fact that any element of a highly oscillatory minimizing sequence is obviously not a constant function, the Young measure associated with it is a constant mapping. Thus the generalized limits of minimizing sequences of energy functionals of some elastic materials (in particular – shape memory materials) can serve as a source of real life examples of homogeneous Young measures.

Nonhomogeneous Young measures, i.e. the weakly* measurable probability measure-valued mappings that are not constant, appear in some variational problems in nonlinear elasticity. There are, however, relatively free many specific examples of nonhomogeneous Young measures. The aim of this article is to define the notion of a density of a Young measure and to use it in presenting a ”more concrete” example of a nonhomogeneous Young measure. This is done with the use of continuous selections of specific set-valued mapping. These selections are the densities of the Young measure of interest.

The structure of the article is as follows. In the next section the basic notions of Young measures are presented. The third section introduces those notions of the multivalued analysis that are necessary to present a concrete example of nonhomogeneous Young measure. In the following section we introduce the notion of a density of Young measure with examples and first properties. In the Conclusions section, a summary of the article has been formulated.

2. Necessary facts about Young measures

In this section, we gather information about Young measures that will be of use in the sequel.

We will denote by $\mathrm{rca}(K)$ the space of regular, countably additive scalar measures on a Borel σ-algebra $\mathscr{B}(K)$ of subsets of a compact set $K \subset \mathbb{R}^l$. We equip this space with the norm $\|m\|_{\mathrm{rca}(K)} := |m|(K)$. Here $|\cdot|$ stands for the total variation of the measure m:

$$|m|(K) = \sup \sum_i |m(K_i)|,$$

where the supremum is taken over all partitions of the set K. Then the pair $(\mathrm{rca}(K), \|\cdot\|_{\mathrm{rca}(K)})$ is a Banach space. The Riesz representation theorem states that in this case, the space of all continuous linear functionals on $\big(C(K), \|\cdot\|_\infty\big)$ (shortly: the conjugate of $\big(C(K), \|\cdot\|_\infty\big)$) is isometrically isomorphic to $\mathrm{rca}(K)$.

If Z is a Banach space and Z^* – the conjugate of Z, then a real valued mapping $\langle \cdot, \cdot \rangle$ defined on $Z^* \times Z$, that is linear in each variable separately, is called a *dual pair*. We then say that a mapping $g\colon \Omega \to Z^*$ is *weakly*-measurable*, if for any $z \in Z$ the function $x \mapsto \langle g(x), z \rangle$ is measurable.

The elements of a space $L^\infty_{w^*}(\Omega; \mathrm{rca}(K))$ are the functions $\nu\colon \Omega \ni x \to \nu(x) \in \mathrm{rca}(K)$, that are weakly*-measurable and such that

$$\operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\} < +\infty,$$

where

$$\operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\} := \inf\big\{ \alpha \in \mathbb{R} \cup \{\infty\} : \|\nu(x)\|_{\mathrm{rca}(K)} \leq \alpha \text{ a.e. in } \Omega \big\}.$$

We endow this space with a norm

$$\|\nu\|_{L^\infty_{w^*}(\Omega, \mathrm{rca}(K))} := \operatorname{ess\,sup}\big\{ \|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega \big\}.$$

It is proved in [1] that the space $\big(L^1(\Omega, C(K))\big)^*$ is isometrically isomorphic to the space $L^\infty_{w^*}(\Omega; \mathrm{rca}(K))$.

The set $\mathscr{Y}(\Omega, K)$ of the *Young measures* consists of those functions from the space $\big(L^\infty_{w^*}(\Omega; \mathrm{rca}(K)), \|\nu\|_{L^\infty_{w^*}(\Omega, \mathrm{rca}(K))}\big)$, whose values are probability measures on K. The Theorem 3.6 in [1] allows one to conclude that for *any* measurable function $f\colon \Omega \to K$, there exists a Young measure ν^f associated with this function.

A particularly simple, yet important, example of a Young measure is a *homogeneous Young measure*. This is the case when the 'family' of probability measures $\nu = (\nu_x)_{x \in \Omega}$ is a single element one: $\nu_x = \nu$ for a.e. $x \in \Omega$. In this case, the set $\mathscr{Y}(\Omega, K)$ is a subset of the set of probability measures on K. The majority of the specific examples of the Young measures that can be found in the literature are homogeneous ones; see, for instance, the Section 5 in [2]. They play an important role in the characterization of the gradient Young measures – an important class of

Young measures appearing in variational problems of nonlinear elasticity; see, for example, the Section 7.5 in [3].

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with the Lebesgue measure dx and let $M > 0$ be a Lebesgue measure of Ω. Define $d\lambda(x) := \frac{1}{M}dx$ and consider a compact set $K \subset \mathbb{R}^l$ with the Lebesgue measure $d\mu$ and a Borel measurable function $f\colon \Omega \to K$. Due to the fact that homogeneous Young measures are 'one-element families', we will write 'ν' instead of '$\nu = (\nu_x)_{x\in\Omega}$' for such measures and denote by 'ν^f' the homogeneous Young measure associated with the function f.

According to the Convention 3.1, the Theorem 3.6 in [1] and the Theorem 3.1 in [4], we will use the following definition of the homogeneous Young measure.

Definition 1 Let ν be a Young measure belonging to $\mathscr{Y}(\Omega, K)$, and let $f\colon \Omega \to K$ be a Borel function.

(i) We say that $\nu \in \mathscr{Y}(\Omega, K)$ is a *homogeneous Young measure* if it is constant on Ω.

(ii) Let ν^f be a Young measure associated with f. We say that $\nu^f \in \mathscr{Y}(\Omega, K)$ is a homogeneous Young measure if it is constant on Ω and is an image of the measure λ under f, i.e. $\nu^f = \lambda \circ f^{-1}$. □

In addition to the items cited above, the interested reader can find information about Young measures and their applications in [5–9] as well.

3. Necessary notions from set-valued analysis

Let Ω and K be any nonempty sets. A mapping $T\colon \Omega \to 2^K$ from Ω to the power set of K will be called a multifunction (or a set-valued mapping). We will write $T\colon \Omega \rightsquigarrow K$ instead of $T\colon \Omega \to 2^K$. Let Ω and K be topological spaces satisfying the first axiom of countability. We say that a multifunction T is lower semicontinuous at $\omega_0 \in \Omega$ if for any sequence (ω_n) in Ω convergent to ω_0 and for any $k_0 \in T(\omega_0)$ there exists a sequence (k_n) in K, convergent to k_0 and such that for any natural n there is $k_n \in T(\omega_n)$.

A selection of a multifunction T is a single-valued function $t\colon \Omega \to K$ such that for all $\omega \in \Omega$ there holds $t(\omega) \in T(\omega)$. By the Axiom of Choice, every multifunction has a selection, but one of the main problems of the set-valued analysis is looking for selections having certain regularity properties like measurability (for example the Kuratowski – Ryll-Nardzewski theorem) or continuity (for example the Michael theorem).

We say that a set $D \subset L^1(K)$ is *decomposable* if for all $u, v \in D$ and all $A \in \mathscr{B}(K)$, the function $\chi_A \cdot u + \chi_{K\setminus A} \cdot v$ is an element of D; as usual, the symbol χ_A stands for the characteristic function of the set A. As it is written in [10]: *The notion of decomposability was introduced by Rockafellar in 1968 in connections with integral* *functionals and since then decomposable sets became main tool in nonconvex analysis. They are in a sense a substitute of convexity and many properties of convex sets have counterparts for decomposable sets.*

The following theorem is proved in [10].

Theorem 1 (*Bressan-Colombo-Fryszkowski*) *Let S be a separable metric space, X – a Banach space and Y – a complete separable metric space equipped with a σ-algebra of Lebesgue measurable sets with a finite Radon measure on it. Let $T: S \rightsquigarrow L^1(Y,X)$ be a multifunction with nonempty, closed and decomposable values. Then T admits a continuous selection.* □

A reader interested in set-valued analysis and its applications may consult the aforementioned book by Fryszkowski, as well as references [11–14] and the works cited therein.

4. Density of a Young measure

Recall that if ξ is a measure on a set K, a function $w: K \to \mathbb{R}$ is integrable with respect to ξ and ρ is a measure on K, then if for any Borel subset A of K we have $\rho(A) = \int_A w(y)d\xi(y)$, the function w is called a density of the measure ρ. In this case we say that ρ is absolutely continuous with respect to ξ (shortly: ξ-continuous): $\xi(A) = 0 \Rightarrow \rho(A) = 0$.

Let Ω be a nonempty, bounded open subset of $\mathbb{R}^n$, K – compact subset of $\mathbb{R}^l$.

Definition 2 We say that a family $h = (h_x)_{x\in\Omega}$ is a density of a Young measure ν with respect to the measure ξ defined on $\mathscr{B}(K)$ if for any $x \in \Omega$, the function h_x is a density of the measure ν_x i.e. for any $A \in \mathscr{B}(K)$, there holds

$$\nu_x(A) = \int_A h_x(y)d\xi(y).$$

The next proposition follows directly from the above definition.

Proposition 1 *Let ν be a Young measure and let $h = (h_x)_{x\in\Omega}$ be a density of ν with respect to the measure ξ. Then ν is a homogeneous Young measure if and only if the family $h = (h_x)_{x\in\Omega}$ consists of one element only, up to a set of ξ-measure* 0. □

We now illustrate the introduced notion with some examples.

Example 1 Consider a function $f: (0,1) \to [0,1]$ given by the formula

$$f(x) := \begin{cases} 2x, & \text{if} x \in \left(0, \frac{1}{2}\right) \\ \frac{1}{2} & \text{if} x \in \left[\frac{1}{2}, 1\right). \end{cases}$$

Then the Young measure associated with f has no density. □

Example 2 Consider a function sequence (f_n) serving in the literature as a typical one generating Young measure: $f_n(x) := \sin(2n\pi x)$, $x \in (0,1)$. The Young measure associated with each element of this sequence is a homogeneous one and it is absolutely continuous with respect to the Lebesgue measure dy on $[-1,1]$: $\nu = \frac{1}{\pi\sqrt{1-y^2}}dy$. Therefore, its density $h = (h_x)_{x\in(0,1)}$ is of the form $h_x(y) = \frac{1}{\pi\sqrt{1-y^2}}$ for all $x \in (0,1)$. □

Example 3 Let $\Omega = (0,1)$, $K = [0,1]$ and denote by dy a Lebesgue measure on $\mathscr{B}(K)$. Consider a family $h = (h_x)_{x\in\Omega}$ of functions defined as follows: for each $x \in \Omega$ and $y \in K$

$$h_x(y) := \begin{cases} \frac{1}{x}y, & \text{if } y \in [0,x) \\ -\frac{1}{1-x}y + \frac{1}{1-x} & \text{if } y \in [x,1). \end{cases}$$

Then $h = (2h_x)_{x\in\Omega}$ is a density of a nonhomogeneous Young measure $\nu = (\nu_x)_{x\in\Omega}$, where for each $x \in \Omega$, we have $\nu_x := 2h_x dy$. □

Example 4 Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of positive Lebesgue measure and denote by $\mathrm{cl}\Omega$ its topological closure. Let $K \subset \mathbb{R}^l$ be nonempty and compact and μ – a Lebesgue measure on $\mathscr{B}(K)$. Let $T\colon \mathrm{cl}\Omega \rightsquigarrow L^1(K,\mathbb{R})$, be a lower semicontinuous multifunction having nonempty, closed and decomposable values. By the Bressan--Colombo-Fryszkowski theorem, there exists a continuous selection for T, that is a continuous function $t\colon \mathrm{cl}\Omega \to L^1(K,\mathbb{R})$ such that for each $x \in \mathrm{cl}\Omega$ there holds $t(x) \in T(x)$. Assume that the multifunction T is such that any of its selections is a nonzero function μ-a.e. Define:

(a) $\forall x \in \Omega\ h_x := \frac{1}{\mu(K)} \cdot \frac{|t(x)|}{\|t\|_{L^1}}$;

(b) $h := (h_x)_{x\in\Omega}$.

Then the family $\nu := (\nu_x)_{x\in\Omega}$, where $\nu_x = h_x d\mu$, is a nonhomogeneous Young measure with density h. □

We will now recall classical theorems concerning weak sequential convergence of functions and measures. These results, together with its corollary, will be needed in the sequel. Despite being simple, the corollary is stated and proved because its special form plays important role in Theorems 3 and 4.

The expression 'weak convergence of the sequence of measures' will be meant as the 'weak convergence of the sequence of measures as elements of the Banach space $\mathrm{rca}(K)$' (with the total variation norm). The term 'convergence' is always understood as 'sequential convergence'.

Let $(X,\mathscr{A},\rho)$ be a measure, space and consider a sequence (u_n) of scalar functions defined on X and integrable with respect to the measure ρ (that is, $\forall n \in \mathbb{N}\ u_n \in L^1_\rho(X)$) and a function $u \in L^1_\rho(X)$. Recall that (u_n) converges weakly sequentially to u if

$$\forall g \in L^\infty_\rho(X) \quad \lim_{n\to\infty} \int_X u_n g d\rho = \int_X u g d\rho.$$

The following theorem characterizes weak sequential L^1 convergence of functions and weak convergence of measures. We refer the reader to [5].

Theorem 2 *(a) (J. Dieudonné, 1957) Let X be a locally compact Hausdorff space and $(X, \mathscr{A}, \rho)$ – a measure space with ρ regular. A sequence $(u_n) \subset L^1_\rho(X)$ converges weakly to an $u \in L^1_\rho(X)$ if and only if $\forall A \in \mathscr{A}$ the limit*

$$\lim_{n\to\infty} \int_A u_n d\rho$$

exists and is finite.

(b) Let X be a locally compact Hausdorff space, and denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X. A sequence (ρ_n) of scalar measures on $\mathscr{B}(X)$ converges weakly to a scalar measure ρ on $\mathscr{B}(X)$ if and only if $\forall A \in \mathscr{B}(X)$ the limit

$$\lim_{n\to\infty} \rho_n(A)$$

exists and is finite. □

This Theorem has simple yet useful corollary for our purposes.

Corollary 1 *Let (ρ_n) be a sequence of measures having respective densities u_n, $n \in \mathbb{N}$. Then the sequence (u_n) is weakly convergent in $L^1(X)$ to a function h if and only if the sequence (ρ_n) is weakly convergent to a measure η.* □

PROOF ($\Rightarrow$) Since (u_n) is weakly convergent in $L^1(X)$, then for any measurable $A \subseteq X$ the limit

$$\lim_{n\to\infty} \int_A u_n d\rho \left(= \lim_{n\to\infty} \rho_n(A)\right)$$

exists and is finite. This is equivalent to the fact that the sequence (ρ_n) is weakly convergent to a measure η.

($\Leftarrow$) We proceed as above, but start the reasoning from the weak convergence of the sequence (ρ_n). ■

Let us state a special case of the above as a separated result since it refers directly to Young measures and is used in the following proofs.

Proposition 2 *Let (ρ_n) be a sequence of measures having respective densities u_n, $n \in \mathbb{N}$. Assume additionally, that $X \subset \mathbb{R}^l$ is compact and let (ρ_n) be a sequence of* *homogeneous Young measures having respective densities u_n. Then the sequence (u_n) is weakly convergent in $L^1(X)$ to a function h if and only if the sequence (ρ_n) is weakly convergent to a measure η.* ◻

Let the family $(h_x)_{x\in\Omega}$ be a density of a Young measure $(\nu_x)_{x\in\Omega}$. Then we have the following theorem.

Theorem 3 *Let (Ω, dist) be a metric space and assume that the mapping*

$$h\colon \Omega \ni x \to h(x) := h_x \in L^1(K)$$

is (Ω, dist) – weakly-sequentially-in-$L^1(K)$ continuous. Let (x_n) be a sequence in Ω convergent to $x_0 \in \Omega$. Then the sequence (h_{x_n}) converges weakly to h_{x_0} if and only if the sequence (ν_{x_n}) converges weakly to the measure ν_{x_0} having density h_{x_0}. ◻

PROOF The first part of the theorem follows from the Proposition 2. Now let the sequence (ν_{x_n}) converges weakly to the measure η. By the Vitali-Hahn-Saks theorem, the measure η is μ-continuous, so by the Radon-Nikodym theorem it has a density r. Choose and fix set $A \in \mathscr{B}(K)$. The fact that $r = h_{x_0}$ up to the set of μ-measure 0 follows from arbitrariness of the choice of A and the inequality

$$\Big|\int_A r(y)d\mu - \int_A h_{x_0}(y)d\mu\Big| \leq \Big|\int_A r(y)d\mu - \eta(A)\Big| + \Big|\eta(A) - \nu_{x_n}(A)\Big| + \\ + \Big|\nu_{x_n}(A) - \int_A h_{x_n}(y)d\mu\Big| + \Big|\int_A h_{x_n}(y)d\mu - \int_A h_{x_0}(y)d\mu\Big|,$$

which in turn implies that $\eta = \nu_{x_0}$. ∎

The next result follows from the above theorem and weak sequential completness of $L^1(K)$ (Theorem 1.3.13 in [15]).

Corollary 2 *The density $h = (h_x)_{x\in\Omega}$ of a Young measure $\nu = (\nu_x)_{x\in\Omega}$ is a weakly sequentially closed set.* ◻

Denote by $\mathscr{M}$ – the set of all Borel measurable functions from Ω to K, by $\mathscr{M}_H \subset \mathscr{M}$ – the set of functions such that the Young measures associated with them are homogeneous and μ-continuous, and define

- $\mathscr{F} := \{\nu^h \in \mathscr{Y}(\Omega, K) : h \in \mathscr{M}_H\}$;
- $\mathscr{D} := \{u\colon K \to \mathbb{R} : u \text{ is the density of the measure from } \mathscr{F}\}$.

Recall that for convex subsets of a normed space, strong and weak closures coincide. We have the following theorem.

Theorem 4 *Assume that the set K is convex. Then the set $\mathscr{F}$ is closed in the norm topology of $\big(\mathrm{rca}(K), \|\cdot\|_{\mathrm{rca}(K)}\big)$ if and only if the set $\mathscr{D}$ is closed in the norm topology of the space $L^1(K)$.* ◻

PROOF We use Proposition 2. Since K is convex the sets $\mathscr{M}_H$, $\mathscr{F}$ and $\mathscr{D}$ are convex as well. Assume that the set $\mathscr{F}$ is weakly closed and let u_n be a sequence from $\mathscr{D}$ convergent weakly to u_0. Then the sequence (ν_n), where $\nu_n = \nu^{u_n} = u_n d\mu$, converges weakly to $\nu_0 \in \mathscr{F}$. Assumptions that the density w of ν_0 is not equal μ-almost everywhere to u_0 and that $u_0 \notin \mathscr{D}$ lead to a contradiction, proving that the set $\mathscr{D}$ is weakly closed in $L^1(K)$, hence strongly closed.

Conversely, assume that the set $\mathscr{D}$ is weakly closed and let (ρ_n) be a sequence of Young measures from the set $\mathscr{F}$, weakly convergent to the measure η. Denote their respective densities by u_n. By the Vitali-Hahn-Saks theorem and the Radon-Nikodym theorem, we infer respectively that the measure η is μ-continuous and has a density r. The fact that $r = h$ up to the set of μ-measure 0 is proved analogously, as in the Theorem 3, while the inequality

$$\left|1 - \int_K h d\mu\right| \leq \left|1 - \int_K u_n d\mu\right| + \left|\int_K u_n d\mu - \int_K h d\mu\right|$$

shows, that η is a probability measure on K.

The uniqueness of the weak limit gives the weak*-measurability of the mapping

$$\Omega \ni x \to \eta \in \mathrm{rca}(K),$$

proving that η is a Young measure. Its homogenity is a consequence of the Proposition 1. ■

Respective weak convergences of Young measures and their densities still hold without convexity assumptions on K.

Corollary 3 *Let $K \subset \mathbb{R}^l$ be compact. Let (ρ_n) be a sequence of μ-continuous homogeneous Young measures with respective densities u_n. Then the sequence (u_n) is weakly convergent in $L^1(K)$ to a function h if and only if the sequence (ρ_n) is weakly convergent in the Banach space $(\mathrm{rca}(K), \|\cdot\|_{\mathrm{rca}(K)})$ to a homogeneous Young measure η with density h.* □

Remark 1 It can be seen how the above result generalizes to more general cases of sequences of finite measures with densities. □

5. Conclusions

The concept of a density of a Young measure arises naturally from the notion of the 'common' density of a measure. This, in particular, means that applying this definition to a homogeneous case, we obtain this common notion. It is also 'natural' in the sense that its sequential closedness in the weak topology follows from the intrinsic property of the weak sequential closedness of the space of integrable functions.

Some of the most important theorems concerning Young measures theory are of course formulated in general term of a Young measure, not necessarily homogeneous one. Such are, for example, the existence theorem for Young measures and the characterization theorem for gradient Young measures (see Theorem 3.6 in [1] and Theorem 7.15 in [3] respectively). However, there is no abundance of specific examples of nonhomogeneous Young measures in the literature. As it is seen in the above section, the notion of density of a Young measure is useful in providing nontrivial examples of such measures. Another prospective feature is the fact that integrable selections of a multifunction (if they exist) form a decomposable set, so this can be a path for research. We thus have the link between the theory of Young measures and the set-valued analysis – the two dynamically developing areas of contemporary analysis, both having many applications also in applied fields.

References

[1] Roubíček, T. (2020). *Relaxation in Optimization Theory and Variational Calculus*. 2nd edition. Walter de Gruyter.

[2] Puchała, P. (2024). Certain convergence results for homogeneous Young measures with densities. *J. Appl. Math. Comput. Mech.*, *23*(4), 101-111.

[3] Rindler, P. (2018). *Calculus of Variations*. Springer International Publishing AG, part of Springer Nature.

[4] Puchała, P. (2017). A simple characterization of homogeneous Young measures and weak L^1 convergence of their densities. *Optimization*, *66*(2), 197-203.

[5] Florescu, L.C., & Godet-Thobie, Ch. (2012). *Young Measures and Compactness in Measure Spaces*, Walter de Gruyter GmbH & Co. KG.

[6] Kružík, M., & Roubíček, T. (2019). *Mathematical Methods in Continuum Mechanics of Solids*. Springer Nature.

[7] Azhmyakov, V. (2019). *A Relaxation-Based Approach to Optimal Control of Hybrid and Switched Systems. A practical guide for engineers*. Elsevier Inc.

[8] Pedregal, P. (2000). *Variational Methods in Nonlinear Elasticity*. Society for Industrial and Applied Mathematics.

[9] Puchała, P. (2021). Young measures – an abstract tool in investigation concrete problems. In: *Selected Topics in Contemporary Mathematical Modeling*. Częstochowa: Publishing Office of Czestochowa University of Technology, 91-105.

[10] Fryszkowski, A. (2004). *Fixed point theory for decomposable sets*. Topological Fixed Point Theory and its Applications, vol. 2. Dordrecht, Boston, London: Kluwer Academic Publishers.

[11] Hu, S., & Papageorgiou, N.S. (1997). *Handbook of Multivalued Analysis. Vol. I. Theory*, Mathematics and its Applications, Vol. 419. Dordrecht, Boston, London: Kluwer Academic Publishers.

[12] Arutyunov, A.A., & Obukhovskii, V. (2016). *Convex and Set-Valued Analysis. Selected Topics*. De Gruyter.

[13] Obukhovskii, V., & Gel'man, B. (2020). *Multivalued Maps and Differential Inclusions*. World Scientific.

[14] Repovš, D., & Semenov, P.V. (1998). *Continuous Selections of Multivalued Mappings*. Mathematics and its Applications, Vol. 455. Dordrecht, Boston, London: Kluwer Academic Publishers.

[15] Lin, P-K. (2004). *Köthe-Bochner Function Spaces*. New York: Springer Science+Business Media.