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ON STRUCTURE OF TICHÝ-INSPIRED LOGICAL SPACETIME

Samuel Novotný, William Steingartner, Valerie Novitzká

Faculty of Electrical Engineering and Informatics, Technical University of Košice
Košice, Slovakia
samuel.novotny@tuke.sk, william.steingartner@tuke.sk, valerie.novitzka@tuke.sk

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Abstract. This paper presents the construction and structural analysis of the logical spacetime – the single unique and real underlying structure of the extensional model of Transparent Intensional Logic. The proposed solution aims to help bridge the gap between the theoretical nature of this logical system and its practical applications while addressing two key challenges. Tichý's objections to the very existence of such a model as well as the ambiguity in Kripke's influential approach, which, rather than defining a single model, specifies various categories of models based on the relational properties of their underlying structures.

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1. Introduction

Since space and time are far from trivial concepts, it is only natural that humans have sought to understand them from a variety of perspectives. This has led to the development of models that capture different aspects of these notions. From a logical point of view, we can then talk about logical space, the set of possible worlds, which was originally defined in close connection with alethic modal logic (ML) [1], studying the concepts of necessity and possibility. Similarly, we can talk about the time, which is closely related to temporal logic (TL) [2], focusing on the formal representation of temporal relations. Please note that, from now on, all references to modality and modal logic are to be understood implicitly in the alethic sense.

Although the origins of modal and temporal logic go back to Aristotle, a significant breakthrough in the formalization of notions inherent to the semantics of these logical systems, such as possible world and time, came only in the 20th century in the works of Prior [3] and Kripke [4]. Until then, these notions were used only informally in philosophy and logic, without a rigorous formal semantics. This success in logic developed in parallel with a revolution in physical and mathematical models of space and time, most prominently represented by Minkowski's seminal work [5]. Although there are analogies between logical and physical understanding of these notions as presented in works of Goldblatt [6] and Hirsch and McLean [7], this study focuses exclusively on the logical aspects of modality and temporality.

Based on the above, Section 2 presents the analytical part of this work, clarifying the basic notions of modal and temporal logical systems. Section 3 concludes this analysis by presenting Tichý's interconnection of the logical aspects of modality and temporality in Transparent Intensional Logic (TIL), which finally allows us to formally define the logical spacetime – an extensional model of TIL. The core of this work, namely the structural analysis of the logical spacetime, is presented in Section 4. This section first presents the basic properties of this structure, then examines its relation to the redefined standard modal and temporal semantic model. Finally, Section 5 summarizes all the partial results of this work and discusses the advantages of the presented solution over existing standards.

2. Modality and temporality in logic

In this section, we focus on achieving the first of the outlined goals, namely the individual analysis of the logical aspects of modality and temporality to formally introduce their fundamental notions and models.

2.1. Modality

Modal logical systems explore and formalize sentences of natural language containing various modalities. Such modalities can, in most cases, ultimately be semantically interpreted in the same way as the modalities of necessity, possibility, or impossibility in alethic modal logic. Since the semantics of these statements is defined based on the notion of a possible world, in the following part, we take a closer look at its historical background, which led to the definition of the Kripke frame and Kripke model – the semantic model of modal logical systems.

2.1.1. Possible worlds in logic

The notion of a possible world is quite prominent in contemporary logic [8]; however, as early as the 17th century, Leibniz formulated many of his writings that often represented a combination of logical, metaphysical, and even theological investigations, based on this notion. Therefore, Leibniz's distinction between necessarily true and contingently or potentially true statements can be considered as one of the first applications of the idea of possible worlds in logic. According to Andreansky's work [9], there are multiple interpretations of Leibniz's definition of the notion of a possible world, which are, however, the results of independent critical analyses of his work; therefore, we do not focus on them.

Another application of the idea behind the notion of possible world can be observed only in the 20th century, specifically with the emergence of intensional logics. An example of this is Carnap's work [10], which does not use the original notion of a possible world but instead introduces a new one called a state description, defined as a maximal consistent set of elementary statements. Nevertheless, assuming the use of sufficiently expressive language, this can be identified with the notion of a possible world, as argued by Peregrin in his work [11]. Based on the above, it can be asserted that Carnap's definition of a state description essentially represents the first modern logical formalization of the possible world.

In the following years, the notion of the possible world received significant attention from a large number of logicians, mathematicians, and philosophers. Therefore, following Peregrin's later work [12], we introduce two further perspectives on its explication in logic. The first one is Tichý's interpretation, which understands the possible world as an assignment of certain properties among individual elements of the universe of discourse. Here, the universe of discourse refers to the set of all objects that are considered within a given formal system. Each object can possess various contingent properties, and a possible world is completely determined by specifying which properties each object holds. This approach allows one to analyse possible worlds in a fine-grained, object-centered manner, rather than treating them as abstract entities (indivisible abstractions). It is not difficult to imagine that, based on such an assignment of properties, one could construct Carnap's state description corresponding to Tichý's interpretation of a possible world. Therefore, we can consider Tichý's interpretation simply as a different approach to constructing such a description. However, Stalnaker's interpretation of a possible world [13] is entirely different, as it defines a possible world implicitly based on its use in logic, as that to which truth is relative. This explication does not specify the possible world as a particular object but rather provides information about the (mental) concept of the term, leading us to an interesting conclusion.

The term *(mental) concept* is often used to explain Frege's distinction between the sense and meaning of an expression conceived in his work [14]. Later, Carnap developed this idea further by replacing these terms with the terms extension and intension and formally defining them based on the idea of possible worlds. Therefore, we could claim that Stalnaker's explication of the term possible world can be understood as its intension, and Carnap's or Tichý's explication as its extension, i.e., the concrete realization of what Stalnaker's explication expresses. It should be emphasized that the foregoing analysis of the notion of a possible world employed intensional logical analysis, which is based precisely on this notion.

2.1.2. Kripke's model

A common feature of modal logical systems is the use of modal operators whose proper interpretation presented a significant challenge for early logicians. Formally defining their semantics within a logical system proved nontrivial and required the development of new conceptual tools.

However, in the second half of the 20th century, the American logician Saul Kripke, in his work *Semantical Considerations on Modal Logic* [4], introduced a new approach to the formulation of the semantics of modal logics. Today, this is known as the Kripke model, with the following definition:

Definition 1 Let *Prop* be the set of all elementary propositions $p_1, p_2, p_3, \ldots$ The Kripke model is then an ordered triple $KM = (W, R, v)$, where:

- W is a non-empty set of possible worlds $w_1, w_2, w_3, \ldots$
- R is a binary accessibility relation on the set W where wRw' means that the possible world w' is accessible from the possible world w.
- v is a valuation function that assigns truth value from a boolean domain $\mathbb{B}$ to each elementary proposition from the set *Prop* in each possible world from the set W:

$$v : Prop \times W \to \mathbb{B}.$$

The underlying pair $KF = (W, R)$ of KM is a Kripke frame.

An important role is also played by the structure of the underlying Kripke frame, which allowed Kripke to identify the specific axioms valid in models implementing specific properties and thus construct the well-known hierarchy of modal logical systems. Despite its influence, the Kripke model did not remain without criticism. Tichý, in his critical work [15], questioned its adequacy, arguing that the accessibility relation lacked a proper definition and was introduced only through an informal discussion of its properties, which serves as a stepping stone for the research presented in this paper.

2.2. Temporality

A significant number of natural language sentences refer to time in some way, whether explicitly through linguistic expressions directly referencing it, or implicitly through the grammatical category of the verb. Time itself plays a very important role in their interpretation, which should make it clear that for logical systems dealing with the investigation and formalization of temporal aspects of natural language, the notion of time is an essential component of their semantic definition.

Firstly, we outline the parallel between temporality and modality, which was foundational to the development of Temporal Logic (TL) and allows it to be generally classified among modal logical systems.

2.2.1. The parallel between temporality and modality

The New Zealand logician Arthur Prior, often regarded as the founder of Temporal Logic (TL), laid its formal foundations and explored the parallel between temporality and modality in his work *Time and Modality* [3]. However, according to Copeland's work [16], Prior initially formulated ideas related to the modal understanding of time at the turn of the first half of the 20th century, when he proposed introducing temporal modalities, which he called quasi-modalities.

Before we proceed with the parallel between temporality and modality, it is important to note the chronological sequence of Prior's and Kripke's works. At the time Prior formulated the aforementioned parallel, Kripke's semantic model of modal logic did not exist yet; there was only the idea of a set of possible worlds without any defined relation on this set.

The fundamental idea from which Prior derived the parallel was the existence of propositions whose truth value is relative to time. Their semantics is thus based on the notion of time (a set of time instants), which represents a notion analogous to the set of possible worlds and provides a clear argument for classifying TL among non-classical logical systems.

Based on the above parallel, one can assume the existence of a semantic model of time that ultimately precedes Kripke's semantic model. Therefore, in the following section, we examine various approaches to the semantic formalization of time, based on which the semantics of TL can be defined.

2.2.2. Time model in logic

Arthur Prior addressed the issue of temporal relativity in the semantics of certain propositions by introducing a temporal context, specifically a particular time instance with respect to which their semantics were defined. He pointed out that there is a naturally defined ordering relation on the set of time instances that expresses the relationship between the past, present, and future, based on the intuitive understanding of time, as presented in his later work [17]. Based on this, a logical model of time can be defined as follows.

Definition 2 Let *Prop* be the set of all elementary propositions $p_1, p_2, p_3, \ldots$. A logical model of time is then an ordered triple $TM = (T, <, \pi)$, where:

- T is a non-empty set of time instances $t_1, t_2, t_3, \ldots$
- $<$ is a binary ordering relation on the set T, where $t < t'$ means that time instance t precedes time instance t'.
- π is a valuation function that assigns truth value from a boolean domain $\mathbb{B}$ to each elementary proposition from the set *Prop* in each time instance from the set T:

$$\pi : Prop \times T \to \mathbb{B}.$$

The underlying pair $TF = (T, <)$ of TM is called a time frame.

At the same time, when Prior pointed out the existence of an ordering relation on the set T, he also specified it more closely as a relation of linear ordering. Based on this, such a temporal framework can be referred to as a linear time model.

However, based on the work of Ploug and Øhrstrøm [18], which is based on private correspondence between Kripke and Prior, Kripke criticized the aforementioned approach to the formalization of time. He proposed abandoning the linearity of the ordering relation on the set of time instances in favor of a non-linear branching structure captured by a partial ordering relation supplemented by the property of backward linearity. He supported his approach with two arguments, namely non-determinism of the future and determinism of the past. Another important argument in support of Kripke's approach was formulated by Hintikka, who, drawing on Copeland's work [16], argued that the linearity of the ordering relation on the set of time instances ties this model to classical physics, while the branching future model of time presents a parallel with modern 20th-century physics.

The various approaches to the formalization of time have also led to the definition of different variants of TL. While the linear time model is part of the Linear Temporal Logic (LTL), which can be considered an extended variation of Prior's TL, the branching future model is a part of Computation Tree Logic (CTL*), introduced by Emerson and Halpern [19] as an extension of LTL.

2.3. Summary of modality and temporality

After introducing the fundamental modal and temporal principles of logic in the previous section, we can conclude that these are significantly analogous aspects, as demonstrated by the following parallels:

- the relativity of truth values of propositions with respect to the elements of their underlying sets (the set of possible worlds, the set of time instances),
- the internal structure of their underlying sets (the accessibility relation on the set of possible worlds, the ordering relation on the set of time instances),
- the formal representation of models of space and time (Kripke frame, time frame),
- the formal representation of semantic models of space and time (Kripke's semantic model, the semantic model of time).

When we also consider the chronological sequence of these aspects along with the previously mentioned private correspondence between the authors of the respective logical formalisms (Kripke, Prior), it becomes fairly clear to identify the origin and inspiration behind their foundational ideas. This allows us to speak not only of a parallel but also of a strong interconnection of the above aspects. However, as we can observe when considering the nonlinear branching model of time, this interconnection begins to take on characteristics of absorption of the modal aspect. Moreover, it complicates the use of a very natural representation of time through linear orderings on sets of natural, integer, rational, or real numbers. Thus, when utilizing a nonlinear model of time, temporality expresses not only temporal but also spatial, i.e., modal dependencies. As a result, this approach to the formalization of time has met with considerable criticism, for instance, in the works of Rescher and Urquhart [20] and Belnap [21].

Therefore, in the following section, we focus on the interconnection of modality and temporality within the analysis of the target logical apparatus of this work – Tichý's Transparent Intensional Logic (TIL) which, by combining the modal and temporal approaches to semantic interpretation, creates a comprehensive spatio-temporal logical calculus.

3. Transparent Intensional Logic

TIL [22] is a unique methodology of logical analysis of natural language [23, 24] developed by Czech logician Pavel Tichý. The key work in this field is his monograph *The Foundations of Frege's Logic* [15], published in the late 1980s, which represents the culmination of his efforts that began in the 1970s with the work *Smysl a procedura*[1]. The ongoing relevance of this logical framework is evidenced by sustained scientific interest and a growing number of recent publications addressing various aspects of TIL, from its theoretical foundations [26–28] through practical implementations [29, 30], to the development of alternative variants [31, 32].

Another notable feature is that, unlike most systems, TIL is not defined via a standard syntactic-semantic correspondence. Rather, it is a directly interpreted, purely semantic framework grounded in typed λ-calculus. Following our earlier work [33], it can be described as a *complex procedural-semantic modal temporal partial typed* λ*-calculus* based on algorithmically structured procedures called constructions. This stands in contrast to the classical λ-calculus, which is traditionally viewed as a model of computation based on mathematical functions mapping inputs to outputs in a set-theoretical manner.

The following section focuses specifically on TIL's simultaneous treatment of modality and temporality, which offers a solution to the problems mentioned at the end of the previous section and provides the foundation for defining the key object of this work – Tichý-inspired logical spacetime.

3.1. Modality and temporality in TIL

Modality in TIL is not part of its temporality, as both aspects are captured in distinct ways, i.e. modality through the set of possible worlds and temporality using the standard strictly linearly ordered set of time moments. We note that instead of the commonly used term time instant, the term time moment is used here deliberately; for more information, the reader is referred to the work of Belnap [21].

The integration and interconnection of modal and temporal aspects in TIL occurs via higher-order functions as follows. The modal relativity of Fregean sense (inten-

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[1] Translated as *Sense and Procedure* in the collection of Tichý's papers [25].

sion) of some linguistic expression is captured by a function from the set of possible worlds. This function often provides a chronology, a function from the set of time moments that captures the temporal relativity of the intension of that expression. Only by subsequent selection and gradual application of a specific possible world w' and a specific time moment t' can we obtain the Fregean meaning (extension) of that expression.

The pair (w', t'), which we refer to as a *worldmoment* or simply *w-moment*, following our work [34], can therefore be considered as a *state* (spacetime specification), representing another result of the aforementioned integration of modality and temporality. Since this work focuses on Tichý-inspired logical spacetime, the next section goes deeper into its formalization and construction based on a more primitive notion of possible world, the insights from which are used in the subsequent structural analysis of the logical spacetime.

3.2. Tichý-inspired logical spacetime

Since w-moments represent individual states of logical spacetime, the next step is to explain how they are organized into larger structures – possible worlds. To this end, we need to clarify and formalize the notion of a possible world, which serves as the fundamental building block for the subsequent construction of logical spacetime.

Definition 3 A possible world is a chronologically ordered sequence of w-moments, whose first element is referred to as the initial w-moment.

The Definition 3 differs from the standard one (e.g., in Tichý's work [15]) only by introducing one additional constraint, namely, unilateral limitation of the possible world by the initial w-moment. The justification for this constraint can be grounded, following Hintikka [35], in modern physics. In particular, current physical cosmology widely accepts that both the universe and time itself have a beginning (e.g., the Big Bang singularity). The assumption of an initial temporal boundary, therefore, motivates the unilateral limitation of possible worlds by their respective initial w-moments. Moreover, this restriction is also natural in computer science, where systems are typically modeled as evolving from an explicitly defined initial state. We note that possible worlds do not necessarily have a single common initial w-moment. Instead, there is a set of initial w-moments, and some of them may be shared among multiple possible worlds.

Since we have defined a possible world as the chronology, it is reasonable to approach its formal representation based on a mapping from the standard strictly linearly ordered subset of the set of all time moments T_w to the subset of the set of all w-moments S_w of a specific possible world w:

$$w : T_w \to S_w. \tag{1}$$

The mapping (1) is bijective. Each w-moment of a possible world represents the state of affairs at a particular time moment, ensuring that this mapping hits every element of S_w, i.e., the mapping is surjective. Furthermore, w-moments corresponding to distinct time moments must differ, because time is a valid state descriptor, and any variation in time necessarily implies a difference in the described objects, i.e., the w-moments. This point is supported by Peregrin [12], and it ensures the injectivity of the mapping.

Since the mapping (1) is bijective, it induces an isomorphic strict linear ordering on the set S_w, which we denote as the *partial accessibility relation* $\prec_w$,

$$(T_w, <) \cong (S_w, \prec_w). \tag{2}$$

Therefore, a possible world w can equivalently be represented as that ordered structure $(S_w, \prec_w)$. Building on this representation, we are now ready to formally introduce the notion of a logical spacetime, which aggregates all possible worlds into a single ordered structure.

Definition 4 A logical spacetime is an ordered pair $LST = (S, \prec)$, where:

- S is the set of w-moments of all possible worlds,

$$S = \bigcup_{w \in W} S_w, \tag{3}$$

- $\prec$ is a binary accessibility relation on the set S, defined as follows:

$$s_1 \prec s_2 \Leftrightarrow \exists w \in W \; s_1 \prec_w s_2. \tag{4}$$

The last notion to introduce is a complete possible world. This concept prevents the consideration of possible worlds with unnatural jumps (discontinuities) and from now on, the notion of possible world is understood to refer specifically to its complete variant.

Definition 5 Let w be a specific possible world within a logical spacetime LST, i.e., w is a substructure of LST. Possible world w is called complete if and only if its initial w-moment is the minimal element of LST and w is a substructure of LST^-, where LST^- is the transitive reduction of LST.

Finally, two additional points can be stated in the form of the following assumptions.

ASSUMPTION 1 The time in LST is a strictly linearly ordered discrete set of time moments T.

ASSUMPTION 2 The accessibility relation $\prec$ on S is backward linear, formally

$$\forall s_1, s_2, s_3 \in S \; (s_1 \prec s_3 \wedge s_2 \prec s_3 \Rightarrow s_1 = s_2 \vee s_1 \prec s_2 \vee s_2 \prec s_1).$$

The former assumption, particularly the part concerning the strictness of the ordering relation, may be viewed as an analogy to the Heraclitean principle *Panta rhei*. In the present context, this principle reflects an idea that time continually flows and time flow cannot be stopped in some moment. However, the second assumption follows Kripke's earlier appeal to Prior's approach[2].

4. Results

In this section, we focus on two aspects of the structural analysis of logical spacetime, namely absolute and relative analysis. While the first one results in a description of the relational properties of *LST*, the second describes the relationship with Kripke's framework, redefined based on Tichý's chronological understanding of possible worlds.

4.1. Properties of Tichý-inspired logical spacetime

When revealing individual relational properties of *LST*, we proceed exclusively in a deductive manner, based only on already defined assumptions and statements derived from them, with the aim of revealing its natural internal structure.

Lemma 1 *The accessibility relation $\prec$ on S is irreflexive.*

PROOF From the isomorphism (2), it is evident that the partial accessibility relation $\prec_w$ of each possible world w from the set W is irreflexive. Since the accessibility relation $\prec$ on S is defined as the aggregation of all partial accessibility relations $\prec_w$ according to the defining condition (4), the resulting relation $\prec$ must also be irreflexive. □

Lemma 2 *The accessibility relation $\prec$ on S is asymmetric.*

PROOF Now we give an indirect proof, specifically a proof by contradiction.

Assume that the accessibility relation $\prec$ on S is not asymmetric. (5)

We can then make the following judgment:

$$\underbrace{s_1 \prec s_2} \wedge \underbrace{s_2 \prec s_1} \tag{6}$$

$$\Swarrow \qquad\qquad \Searrow$$

$$\exists t_1, t_2 \in T \left(\exists w_1, w_2 \in W \left(\underbrace{w_1(t_1) \prec_{w_1} w_1(t_2)} \wedge \underbrace{w_2(t_2) \prec_{w_2} w_2(t_1)}\right)\right) \tag{7}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$t_1 < t_2 \qquad\quad \wedge \qquad\quad t_2 < t_1 \tag{8}$$

---

[2] Given the significance of the second assumption, we will revisit it at the end of section 5 and offer an idea of its more objective defense.

Individual steps of this proof can be described as follows:

(6) – Initial assumption.

(7) – If there is an accessibility relation between two w-moments, then according to the defining condition (4), there must exist a partial accessibility relation in some possible world between these w-moments. Due to generality, consider possible worlds w_1 for the first conjunct and w_2 for the second conjunct. It is also important to note that the same w-moments in the first and second conjunct must correspond to the same time moment.

(8) – From the isomorphism (2), it can be concluded that $t_1 < t_2$ and $t_2 < t_1$, which is a contradiction.

The negation (5) does not hold; therefore, Lemma 2 is true. □

Lemma 3 *The accessibility relation $\prec$ on S is anti-symmetric.*

PROOF For the proof of Lemma 3, we combine Lemma 2 with the fact that asymmetric relations are anti-symmetric. □

Lemma 4 *The accessibility relation $\prec$ on S is transitive.*

PROOF We provide a direct proof, starting with the following initial assumption:

$$s_1 \prec s_2 \wedge s_2 \prec s_3. \tag{9}$$

Based on the defining condition (4), the existence of an accessibility relation between two w-moments implies the existence of a partial accessibility relation of a certain possible world between these w-moments. Therefore from assumption (9), the claim (10) follows, from which the application of the isomorphism (2) leads to the conclusion (11). Due to generality, we again consider possible world w_1 for the first conjunct and w_2 for the second conjunct.

$$\exists t_1, t_2, t_3 \in T \left(\exists w_1, w_2 \in W \left(\underbrace{w_1(t_1) \prec_{w_1} w_1(t_2)}_{\Downarrow} \wedge \underbrace{w_2(t_2) \prec_{w_2} w_2(t_3)}_{\Downarrow}\right)\right) \tag{10}$$

$$t_1 < t_2 \qquad \wedge \qquad t_2 < t_3 \tag{11}$$

First structural label: From assumption (9), it is also evident that w-moment s_2 is not the minimal element of S. Therefore in possible world w_2, there must exist a w-moment $s_x = w_2(t_x)$, for which the following holds:

$$
\begin{aligned}
& s_x \prec_{w_2} s_2 \text{ and thus } s_x \prec s_2 && (12)\\
& \Rightarrow w_2(t_x) \prec_{w_2} w_2(t_2) && (13)\\
& \Rightarrow t_x < t_2 \Rightarrow t_x < t_3 && (14)\\
& \Rightarrow w_2(t_x) \prec_{w_2} w_2(t_3) && (15)\\
& \Rightarrow s_x \prec_{w_2} s_3 \text{ and thus } s_x \prec s_3 && (16)
\end{aligned}
$$

The individual derivation steps can be described as follows:

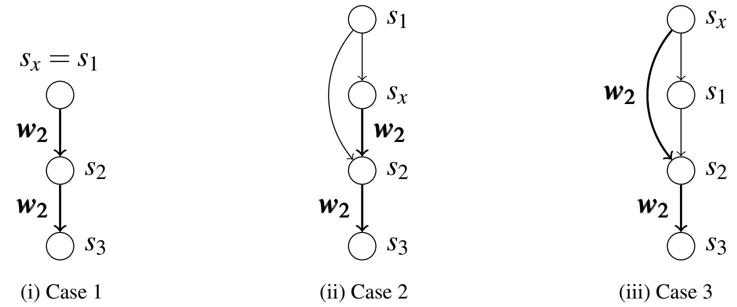
(13) – Rewriting w-moments using possible world w_2.

(14) – Application of the isomorphism (2) and using the transitivity of the ordering relation on the set of time moments.

(15) – Backward application of the the isomorphism (2).

(16) – Conclusion. □

Second structural label: Based on Assumption 2 (on backward linearity), we can now proceed with a case analysis, graphically presented in Figure 1.



(i) Case 1 (ii) Case 2 (iii) Case 3

Fig. 1. Individual cases from Lemma 2

- *Case $s_1 = s_x$:*

$$
\begin{aligned}
& s_x \prec_{w_2} s_3 \\
& \Rightarrow s_1 \prec_{w_2} s_3 && (17)\\
& \Rightarrow s_1 \prec s_3. && (18)
\end{aligned}
$$

- *Case $s_1 \prec s_x$:*
  In this case, we must realize that the construction of this part of the proof resembles the structure of this proof itself, starting from its *first structural label*, as the same holds for s_x as it did for s_2.

  Therefore, we only need to prove the termination of this branch. In our initial choice of auxiliary w-moment s_x, we were bounded by w-moments s_1 and s_2.

Without loss of generality, we can also speak of bounds by time moments t_1 and t_2 (where $t_1 < t_2$); for the next choice, the bounds would be t_1 and t_x (where $t_1 < t_x < t_2$), etc. However, since we are dealing with discrete time, there is only a finite number of time moments between any two moments, and thus a finite number of corresponding w-moments. This guarantees the termination of this branch of the proof by exhausting the set (19).

$$\{w_2(t) \in S \mid t \in T \wedge t_1 < t < t_2\} \tag{19}$$

- *Case* $s_x \prec s_1$:
  According to Definition 5 of a complete possible world, after applying the transitive reduction to the fragment of the logical spacetime from Figure 1(iii), the continuity of the possible world w_2 requires the existence of a w-moment $s_y = w_2(t_y)$, for which the following holds:

$$s_x \prec_{w_2} s_y \wedge s_y \prec_{w_2} s_2. \tag{20}$$

This situation is also illustrated by the Figure 2.

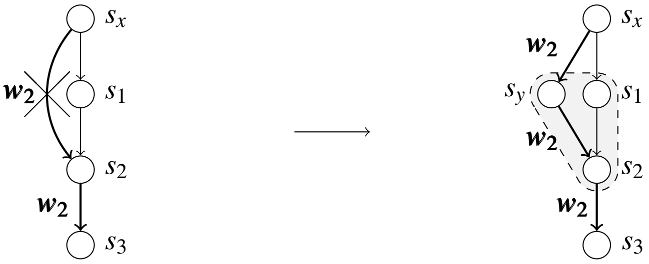


Fig. 2. Transitive reduction of the fragment of logical spacetime from Figure 1(iii) and its consequences

If we focus on the highlighted group of three w-moments on the right-hand side of Figure 2, another opportunity for applying Assumption 2 (on backward linearity) can be fairly easily identified, specifically based on Lemma 9 (on preservation of the backward linearity). However, this part of the proof is analogous to the construction of this proof itself from its *second structural label*, since the same holds for s_y as it did for s_x.

As in the previous case, it is therefore sufficient to proceed with proving the termination of this recursive branch. In our initial choice of auxiliary w-moment s_y, we were bounded by w-moments s_x and s_1. Without loss of generality, we can also speak of bounds by time moments t_x and t_1 (where $t_x < t_1$); for the next choice, the bounds would be t_y and t_1 (where $t_x < t_y < t_1$), etc. However, since we are dealing with discrete time, there is only a finite number of time moments between any two moments, and thus a finite number of corresponding w-moments. This guarantees the termination of this branch of the proof by exhausting the set (21).

$$\{w_2(t) \in S \mid t \in T \ \wedge \ t_x < t < t_1\} \tag{21}$$

Lemma 5 *The accessibility relation $\prec$ on S is acyclic, formally*

$$\forall n \in \mathbb{N} \left(\nexists s_1, s_2, \ldots, s_n \in S(s_1 \prec s_2 \wedge \ldots \wedge s_{n-1} \prec s_n \wedge s_n \prec s_1) \right).$$

Before proving Lemma 5 by mathematical induction, Lemma 6 needs to be already proved.

Lemma 6 *Let R be a transitive binary relation defined on a set A. Then for all $n \in \mathbb{N}$, $n > 2$, it holds that if there exists a directed cycle of length n in (A,R), then there must also exist a directed cycle of length $n-1$ in (A,R), formally*

$$\begin{aligned} &\exists a_1, a_2, \ldots, a_n \in A \ (a_1 R a_2 \wedge \ldots \wedge a_{n-1} R a_n \wedge a_n R a_1) \Rightarrow \\ &\exists a_1, a_2, \ldots, a_{n-1} \in A \ (a_1 R a_2 \wedge \ldots \wedge a_{n-2} R a_{n-1} \wedge a_{n-1} R a_1). \end{aligned}$$

PROOF Let (A,R) be a structure that contains a directed cycle $\vec{C_n}$ of length n. By removing the k-th element from this cycle, a new directed cycle $\vec{C}_{n-1}$ of length $n-1$ is formed based on the Lemma 4 (on transitivity), as illustrated in the following Figure 3.

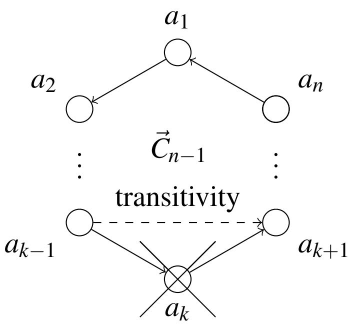


Fig. 3. Visualization of the proof principle

Based on the above Lemma 6, we can now proceed to the proof of Lemma 5.

PROOF (PROOF OF LEMMA 5) We prove this using mathematical induction.

1. *Base case 1*
   First, we prove that Lemma 5 holds for $n = 1$, which means that *LST* contains no loops. This statement holds trivially based on Lemma 1 (on irreflexivity):

$$\nexists s_1 \in S s_1 \prec s_1.$$

2. *Base case 2*
   Now, we prove that Lemma 5 holds for $n = 2$. This statement holds trivially based on Lemma 2 (on asymmetry):
   $$\nexists s_1, s_2 \in S(s_1 \prec s_2 \wedge s_2 \prec s_1).$$
3. *Inductive step*
   Finally, it remains to prove that for all $n \in \mathbb{N}$, $n > 2$, the following holds:
   $$\begin{aligned}&\nexists s_1, s_2, \ldots, s_{n-1} \in S\ (s_1 \prec s_2 \wedge \ldots \wedge s_{n-2} \prec s_{n-1} \wedge s_{n-1} \prec s_1) \Rightarrow\\ &\nexists s_1, s_2, \ldots, s_n \in S\ (s_1 \prec s_2 \wedge \ldots \wedge s_{n-1} \prec s_n \wedge s_n \prec s_1).\end{aligned}$$
   We only need to realize that this statement is a contraposition statement to Lemma 6, and therefore it holds.

Now, we can conclude that the Lemma 5 holds. □

4.2. Isomorphism of logical spacetime and Kripke frame

Since the logical spacetime also operates with the notion of a possible world, we now define its corresponding Kripke frame.

Definition 6 Let *LST* be a logical spacetime, and a possible world be any complete temporal sequence of w-moments starting with the initial w-moment. The Kripke frame is then the ordered pair $KF^{TIL} = (W, \subset)$, where:

- W is the set of all possible worlds of *LST*,
- the accessibility relation on the set W is a proper inclusion relation $\subset$.

As the title of this section suggests, the aim here is to demonstrate the isomorphism of the logical spacetime *LST* and the redefined Kripke frame KF^{TIL}, using the following series of lemmas and their proofs.

Lemma 7 *If the relational structures (A_1, R_1) and (A_2, R_2) are isomorphic, then their transitive closures (A_1, R_1^+) and (A_2, R_2^+) are also isomorphic, formally*

$$(A_1, R_1) \cong (A_2, R_2) \Rightarrow (A_1, R_1^+) \cong (A_2, R_2^+). \tag{22}$$

PROOF From the assumption that $(A_1, R_1) \cong (A_2, R_2)$, it is clear that there exists a bijective function $\varphi : A_1 \to A_2$ such that

$$\forall u, v \in A_1\ \big((u, v) \in R_1 \Leftrightarrow (\varphi(u), \varphi(v)) \in R_2\big).$$

Since $R_1 \subseteq R_1^+$ and $R_2 \subseteq R_2^+$, it follows that, for elements from R_1 and R_2, it also holds that

$$\forall u, v \in A_1\ \big((u, v) \in R_1^+ \Leftrightarrow (\varphi(u), \varphi(v)) \in R_2^+\big),$$

with the same function φ. Therefore, it is sufficient to prove the validity of this statement (the implications of both directions) for the elements added by the transitive closure.

- *Right implication*

$$\begin{aligned} & R_1 \not\ni (u,v) \in R_1^+ && (23)\\ & \Rightarrow \exists a_1,\dots,a_n \in A_1 \big((u,a_1) \in R_1 \wedge \dots \wedge (a_n,v) \in R_1\big) && (24)\\ & \Rightarrow \big((\varphi(u),\varphi(a_1)) \in R_2 \wedge \dots \wedge (\varphi(a_n),\varphi(v)) \in R_2\big) && (25)\\ & \Rightarrow (\varphi(u),\varphi(v)) \in R_2^+ && (26) \end{aligned}$$

- *Left implication*

$$\begin{aligned} & R_2 \not\ni (\varphi(u),\varphi(v)) \in R_2^+ && (27)\\ & \Rightarrow \exists a_1,\dots,a_n \in A_1 \big((\varphi(u),\varphi(a_1)) \in R_2 \wedge \dots \wedge (\varphi(a_n),\varphi(v)) \in R_2\big) && (28)\\ & \Rightarrow \big((u,a_1) \in R_1 \wedge \dots \wedge (a_n,v) \in R_1\big) && (29)\\ & \Rightarrow (u,v) \in R_1^+ && (30) \end{aligned}$$

Individual steps of the proofs above can be described as follows:

(23), (27) – Initial assumption of proving implication.

(24), (28) – If in the transitive closure of a relation there exists a pair that is not part of the original relation, then there must exist a path between these elements in the original relation.

(25), (29) – Use of the assumption of the proven statement.

(26), (30) – The desired conclusion. □

Based on the above, it is clear that $(A_1,R_1^+) \cong (A_2,R_2^+)$.

Lemma 8 *If the transitive reductions of the relational structures* (A_1,R_1) *and* (A_2,R_2) *with transitive relations* R_1 *and* R_2 *are isomorphic, then the original relational structures are also isomorphic.*

PROOF We prove this directly.

$$\begin{aligned} & (A_1,R_1^-) \cong (A_2,R_2^-) && (31)\\ & \Rightarrow (A_1,(R_1^-)^+) \cong (A_2,(R_2^-)^+) && (32)\\ & \Rightarrow (A_1,R_1) \cong (A_2,R_2) && (33) \end{aligned}$$

Individual steps of the proof above can be described as follows:

(31) – Initial assumption.

(32) – Application of Lemma 7.

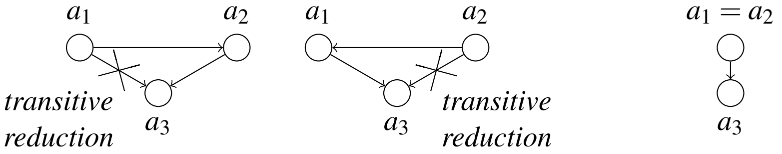
(33) – From the definition of transitive reduction, we have $(R_1^-)^+ = R_1^+$ and $(R_2^-)^+ = R_2^+$. Given the assumption of the transitivity of relations, we also have $R_1^+ = R_1$ and $R_2^+ = R_2$, leading us to the desired conclusion. $\square$

Lemma 9 *If a relational structure (A,R) is backward linear, then its transitive reduction (A,R^-) is also backward linear.*

Proof From the assumption, it holds

$$\forall a_1,a_2,a_3 \in A \ \big((a_1,a_3) \in R \wedge (a_2,a_3) \in R \Rightarrow (a_1,a_2) \in R \vee (a_2,a_1) \in R \vee a_1 = a_2\big)$$

The core of this statement is an implication, which means that if its antecedent is false, the core of the statement is true. Based on this, we can divide the set of triples a_1,a_2,a_3 into those for which the core of the statement is vacuously true and those for which it is non-vacuously true. It should be clear that the transitive reduction of the relational structure (by removing elements from R) cannot affect the vacuous truth of the core of the statement for the first group of triples[3]. Therefore, we only need to focus on the second group, within which only three cases can occur, as shown in the following Figure 4.



(i) Case 1 (ii) Case 2 (iii) Case 3

Fig. 4. Visualization of the proof principle

For the fragments of logical spacetime from Figures 4(i) and 4(ii), the statement's core holds vacuously after transitive reduction, whereas in case from Figure 4(iii), the reduction leaves its validity unchanged. Based on this, we can conclude the validity of Lemma 9.

Lemma 10 *Let $\vec{G}$ be a digraph corresponding to the relational structure LST, and let $\vec{G}^-$ be its transitive reduction. Then $\vec{G}^-$ contains no cycles.*

Proof We now give an indirect proof, specifically proof by contradiction.

$$\text{Let us assume that } \vec{G}^- \text{ contains cycle } C. \tag{34}$$

Since $\vec{G}^- \subseteq \vec{G}$, it is clear that based on Lemma 5 (on acyclicity) C must be an undirected cycle. This cycle must then contain at least one source and one sink, as illustrated in the following Figure 5.

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[3] The vacuous truth of this statement can only be affected by adding elements to R.

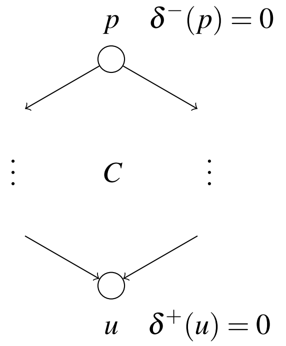


Fig. 5. Undirected cycle in $\vec{G}$

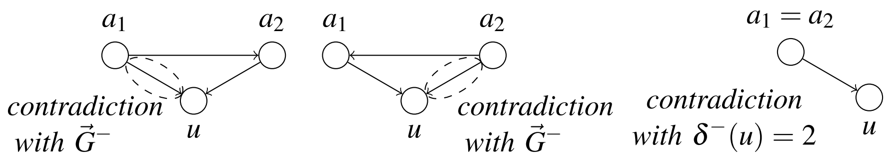
Let's examine the sink vertex of the cycle C more closely. Since every vertex in the cycle has a degree of 2, and u is one of the vertices in the cycle, it must satisfy the equality (35). This equality can be rewritten using the definition of the vertex degree to the equality (36), and from this, using the definition of the sink as a vertex with a zero out-degree, we obtain the final equality (37):

$$\delta(u) = 2, \tag{35}$$

$$\delta^+(u) + \delta^-(u) = 2, \tag{36}$$

$$\delta^-(u) = 2. \tag{37}$$

Based on the Lemma 9 (on preservation of the backward linearity), for the sink u, one of the following cases depicted in Figure 6 must occur.



(i) Case 1 (ii) Case 2 (iii) Case 3

Fig. 6. Case analysis

While in the first two cases from Figures 6(i) and 6(ii), we encountered a contradiction with the assumption that $\vec{G}^-$ is a transitive reduction of $\vec{G}$, in the case from Figure 6(iii) we came to a contradiction, with the elementary property of a circle.

The negated statement (34) does not hold; therefore, the Lemma 10 is true. □

At this point, we have all the necessary to formulate the final theorem on the isomorphism of *LST* and KF^{TIL}, and subsequently prove it. Before we do so, however, we take a closer look at the necessary precondition for the validity of this theorem, which is formulated as the following Lemma 11.

Lemma 11 *The number of possible worlds in a finite logical spacetime LST is equal to the number of its w-moments.*

PROOF From Lemma 10 (on the absence of cycles) it follows that the digraph $\vec{G}^-$ is a directed forest. So let's assume that this forest consists of n components – directed trees $\vec{T}_1, \vec{T}_2, \ldots, \vec{T}_n$[4]. The following then applies to the number of possible worlds of LST.

$$\begin{aligned}
|W| &= \sum_{i=1}^{n} |W_i| && (38)\\
&= \sum_{i=1}^{n} \left(1 + \delta^-(s_0^i) + \delta^-(s_1^i) + \ldots + \delta^-(s_{|S_i|-1}^i)\right) && (39)\\
&= n + \sum_{i=1}^{n} \sum_{s \in S_i} \delta^-(s) && (40)\\
&= n + \sum_{i=1}^{n} | \prec_i^- | && (41)\\
&= n + \sum_{i=1}^{n} (|S_i| - 1) && (42)\\
&= n + |S| - n && (43)\\
&= |S| && (44)
\end{aligned}$$

The individual steps of the proof of this equality can be described as follows:

> (38) – Since the sets of possible worlds in the individual components of the digraph $\vec{G}^-$ are disjoint (due to the discontinuity of these components), the total number of possible worlds is equal to the sum of the possible worlds of its individual components $\vec{T}_1, \vec{T}_2, \ldots, \vec{T}_n$.
>
> (39) – Therefore, over each component $\vec{T}_i$, it is necessary to perform the sum of its possible worlds, which can be realized gradually based on the length of the possible world as follows. First, we determine the number of possible worlds of length one that consist exclusively of the initial w-moment s_0^i; this number is exactly one. Subsequently, we add the number of possible worlds of length two equal to the outer degree of the initial w-moment s_0^i, etc.
>
> (40) – Selection of the constant before summation and rewriting the sum of the sequence using summation.
>
> (41) – Application of the relationship between the sum of the outer degrees of the graph's vertices and the number of edges in the graph.

---

[4] This is a generalization of *LST*, within which there is not only a single initial w-moment, but a set of initial w-moments.

(42) – Application of the relationship between the number of edges and vertices of a tree.

(43) – Removing summation.

(44) – The desired conclusion. □

Theorem 1 *The logical spacetime LST and the Kripke frame KF^{TIL} are isomorphic structures, formally*

$$\exists \varphi \in S \leftrightarrow W \left(\forall s_1, s_2 \in S \left(s_1 \prec s_2 \Leftrightarrow \varphi(s_1) \subset \varphi(s_2) \right) \right).$$

PROOF From the transitivity of the relation of proper inclusion $\subset$ and from Lemmas 4 (on transitivity) and 10 (on absence of cycle), it follows that it is sufficient to prove the isomorphism of the transitive reductions of the logical spacetime *LST* and the Kripke frame KF^{TIL}, i.e., the statement

$$\exists \varphi \in S \leftrightarrow W \left(\forall s_1, s_2 \in S \left(s_1 \prec^- s_2 \Leftrightarrow \varphi(s_1) \subset^- \varphi(s_2) \right) \right).$$

Let φ be a mapping that assigns to each w-moment in the set S a possible world terminating in this w-moment. From Lemma 10 (on absence of cycle), it is clear that this is a bijection. Now we prove the validity of the above statement (the implications of both directions) for this mapping φ.

- *Right implication*:

$$s_1 \prec^- s_2 \tag{45}$$

$$\mathit{Let}\ \varphi(s_2) = w_2\ \mathit{then}\ \exists w_x \in W w_2 = (w_x, s_2) \tag{46}$$

$$\Rightarrow w_x \subset^- w_2 \tag{47}$$

$$\Rightarrow \varphi^{-1}(w_x) \prec^- s_2 \tag{48}$$

$$\Rightarrow \underbrace{s_1 \prec^- \varphi^{-1}(w_x)}_{\Searrow} \vee \underbrace{\varphi^{-1}(w_x) \prec^- s_1}_{\Swarrow} \vee s_1 = \varphi^{-1}(w_x) \tag{49}$$

$$\mathit{contradiction\ with} \prec^-$$

$$\Rightarrow \varphi(s_1) = w_x \tag{50}$$

$$\Rightarrow \varphi(s_1) \subset^- \varphi(s_2) \tag{51}$$

Individual steps of the proof above are the following:

(45) – Initial assumption.

(46), (47) – The w-moment s_2 is indeed not an initial w-moment, as it is not the minimal element of S, which follows from the assumption (45). Therefore, there must exist a non-empty possible world w_x, which is a proper subset of the possible world w_2, such that extending it with the w-moment s_2 results in the possible world w_2.

(48) – For the last w-moment of the possible world w_x, it must hold that $\varphi^{-1}(w_x) \prec^- s_2$.

(49) – Based on Assumption 2 (on backward linearity) and Lemma 9 (on preservation of the backward linearity), for the w-moment $\varphi^{-1}(w_x)$, one of three cases must apply. Since the first two cases, where $s_1 \prec^- \varphi^{-1}(w_x)$ or $\varphi^{-1}(w_x) \prec^- s_1$, lead to a contradiction with the assumption that $\prec^-$ is a transitive reduction of $\prec$, it follows that $s_1 = \varphi^{-1}(w_x)$.

(50) – Rewriting of the equality from step (49).

(51) – By applying the first equality from step (46) and the equality from step (50) within the statement (47), we obtain the desired conclusion.

- *Left implication*:

$$\varphi(s_1) \subset^- \varphi(s_2) \tag{52}$$

$$\textit{Let } \varphi(s_1) = w_1 \textit{ and } \varphi(s_2) = w_2 \textit{ then } \exists s_x \in S\; w_2 = (w_1, s_x) \tag{53}$$

$$\Rightarrow \varphi^{-1}(w_1) \prec^- s_x \tag{54}$$

$$\Rightarrow \varphi^{-1}(w_2) = s_x \Rightarrow \varphi(s_x) = w_2 \Rightarrow s_x = s_2 \tag{55}$$

$$\Rightarrow s_1 \prec^- s_2 \tag{56}$$

Individual steps of the proof above can be described as follows:

(52) – Initial assumption.

(53) – From the assumption (52), it follows that the possible world w_1 is a proper subset of the possible world w_2, and extending the possible world w_1 with the w-moment s_x results in the possible world w_2.

(54) – For the last w-moment of the possible world w_1, it must hold that $\varphi^{-1}(w_1) \prec^- s_2$.

(55) – From the step (53), it follows that the last w-moment of the possible worlds w_2 is s_2, but also s_x, which implies the identity of these w-moments.

(56) – By applying the resulting equality from step (55) within the statement from step (54), we obtain the desired conclusion. □

5. Discussion and conclusion

Since this work has been primarily focused on the logical formalization of the notions of space and time within the framework of Transparent Intensional Logic (TIL), we began by analyzing the standard semantic models employed in modal and temporal logical systems, namely the Kripke model and Prior's temporal model. Tichý's critique of the Kripke model posed a challenge for our subsequent effort to define a logical spacetime that would serve as a well-founded structural analog to the Kripke frame.

To address this, we formally defined the accessibility relation in logical spacetime based on the chronological ordering of w-moments (partial accessibility relations) within individual possible worlds. Then we proceeded with its structural analysis, revealing the internal relational properties of the logical spacetime. Unlike Kripke's approach [4], which merely classifies categories of frames by the properties of their accessibility relations and associates them with particular axiom systems, our method enabled a deductive characterization of the inherent relational features of the logical spacetime itself.

In the light of TIL's original focus on the logical analysis of natural language, the absence of an extensional semantic model has traditionally been considered a justified limitation. However, this also significantly hindered its practical applicability, despite the fact that TIL, at its core, is a partial, hyperintensional variant of typed λ-calculus. The introduction of the extensional model in the form of logical spacetime thus represents a breakthrough, as it provides a concrete structural foundation for the execution, i.e., interpretation of TIL's constructions. This, in turn, enables novel applications of the system in areas such as formal verification and model checking, where rigorous semantic models are crucial.

Finally, in alignment with Kripke's original suggestion regarding the integration of backward linearity into Prior's time model, we adopted the assumption of backward linearity for logical spacetime. This relational property, however, exhibits clear analogies with the so-called general principle of persistence, which we defined as a generalization of Kripke's original principle of persistence. Another future research direction will therefore focus on this connection by examining how the structure of logical spacetime determines the logical principles valid within it.

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