JAM CM Journal of Applied Mathematics and Computational Mechanics 2025, 24(4), 5-25
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2025.4.01 e-ISSN 2353-0588

ADMISSIBILITY OF FRECHÉT SPACES

***Maciej Ciesielski*[1], *Grzegorz Lewicki*[2]**

[1] *Institute of Mathematics, Poznan University of Technology*
Poznań, Poland
[2] *Department of Mathematics and Computer Science, Jagiellonian University*
Kraków, Poland
maciej.ciesielski@put.poznan.pl, grzegorz.lewicki@im.uj.edu.pl

Received: 11 July 2025; Accepted: 9 October 2025

Abstract. The aim of this paper is to show the admissibility of some classes of Frechét spaces (see Definition 2.3), which generalizes the particular case given for modular function spaces E_ρ. As an application, we show the admissibility of a large class of modular spaces equipped with F-norms, as determined in Theorem 4.1. We also provide applications by proving fixed point theorems in F-admissible spaces (see Theorems 3.5 and 3.6). We would like to add that, in particular, Theorems 3.5 and 3.6 can be applied to find the existence of solutions of integral and differential equations in modular spaces (see Theorem 4.7, Remark 4.8 and Remark 4.9). It is worth noticing that F-norms introduced in Theorem 4.1 generalize the classical Luxemburg F-norm.

MSC 2010: *46A80, 46E30*
Keywords: *admissible space, modular function spaces, Frechét spaces, fixed point theory*

1. Introduction

The notion of admissibility, introduced by Klee in [1], allows one to approximate the identity on compact sets by finite-dimensional maps. Recall that locally convex spaces are admissible [2]. It is worth mentioning that an extensive body of literature is devoted to prove that particular classes of non-locally convex function spaces are admissible e.g. [3–5]. It is important to notice that not all non-locally convex spaces are admissible. In [6], Cauty provides an example of a metric linear space in which the admissibility fails.

The aim of this paper is to prove the admissibility of a large class of Frechét spaces introduced in Definition 2.3 (so-called F*-admissible spaces*; see Theorem 3.4). In particular, this generalizes earlier results obtained in [7] for modular function spaces introduced by Kozłowski in [8]. As an application, we prove two fixed point theorems in F-admissible spaces (see Theorem 3.5 and Theorem 3.6).

Next, we apply these results to the large class of modular spaces equipped with F-norms introduced in Theorem 4.1. The main interest of the admissibility of modular spaces lies in the possibility of applying the result to the fixed point theory. The fixed point theory in modular spaces, initiated in 1990 by Khamsi et al. [9], is a rather recent topic in the theory of nonlinear operators, see e.g. [10–13]. The advantage of the theory is that even though a metric may not be defined, many problems in metric fixed point theory can still be possibly formulated in modular spaces. It is worth mentioning that the fixed point theory has crucial applications in engineering, especially in solving controllability problems appearing in dynamical systems (for more details, see [14]). We hope that our fixed point theorems can be applied to find the existence of solutions of some integral and differential equations in modular spaces (see Theorem 4.7, Remark 4.8 and Remark 4.9).

Now we recall the definition of admissibility.

DEFINITION 1.1 [1] Let E be a Hausdorff topological vector space. A subset Z of E is said to be admissible if for every compact subset K of Z and for every neighborhood V of zero in E there exists a continuous mapping $H : K \to Z$ such that dim(span $[H(K)]) < +\infty$ and $x - Hx \in V$ for every $x \in K$. If $Z = E$ we say that the space E is admissible.

2. Preliminary results

We start with the following definitions.

DEFINITION 2.1 Let X be a linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A mapping $|\cdot|_F : X \to [0, +\infty)$ is said to be an F-norm if it satisfies the following conditions:

(i) $|x|_F = 0$ if and only if $x = 0$;

(ii) $|x|_F = |ax|_F$ for all $x \in X$, $a \in \mathbb{K}$, $|a| = 1$;

(iii) $|x+y|_F \leq |x|_F + |y|_F$ for all $x, y \in X$;

(iv) $|\lambda_n x_n - \lambda x|_F \to 0$ whenever $|x_n - x|_F \to 0$ and $\lambda_n \to \lambda$ for any $x \in X$, $(x_n) \subset X$, $\lambda \in \mathbb{K}$ and $(\lambda_n) \subset \mathbb{K}$.

The space X equipped with F-norm is called an F-space. An F-space is called a Frechét space if X is a complete metric space with respect to the metric introduced by F-norm.

DEFINITION 2.2 Let (T, Σ, μ) be a measure space with a σ-finite measure μ and let $(W, \|\cdot\|)$ be a Banach space. Unless we say otherwise, by a *partition* of a set T, we mean a collection of pairwise disjoint subsets of T. A function $s : T \to W$ is called a simple function if $s = \sum_{i=1}^{n} w_i \chi_{E_i}$ where $\{E_i\}_{i=1}^{n}$ is a partition of T, and for $i = 1, \ldots, n$,

$E_i \in \Sigma$, $\mu(E_i) > 0$, $w_i \in W$. A function $f : T \to W$ is called *strongly measurable* if there exists a sequence of simple functions $\{s_n\}$ such that $\lim_n s_n(t) = f(t)$ μ-a.e. [15, 16]. By $L_o(T)$, we denote the set of all strongly measurable functions from T into W (with equality μ-a.e.).

DEFINITION 2.3 Let (T, Σ, μ) be a measure space with a σ-finite measure μ, and let $(W, \|\cdot\|)$ be a Banach space. Let

$$S_F = \{s \in L_o(T) : s \text{ is a simple function }, \mu(supp(s)) \in [0, +\infty)\}.$$

A Frechét space $(X, |\cdot|_F)$ is called *F-admissible F-space* if the following conditions are satisfied:

(a) $X \subset L_o(T)$;

(b) $S_F \subset X$;

(c) If $f \in L_o(T)$, $g \in X$ and $\|f(t)\| \leq \|g(t)\|$ μ-a.e., then $f \in X$ and $|f|_F \leq |g|_F$;

(d) For any $\{w_n\} \subset W$ and $A \in \Sigma$, $\mu(A) < \infty$, if $\|w_n\| \to 0$ then $|w_n \chi_A|_F \to 0$;

(e) For any $\{A_n\} \subset \Sigma$ and $w \in W \setminus \{0\}$, we get

$$|w\chi_{A_n}|_F \to 0 \quad \text{if and only if} \quad \mu(A_n) \to 0;$$

(f) If $f \in X$, $\{f_n\} \subset X$, $f_n \to f$ μ-a.e. and $\|f_n(t)\| \leq \|f(t)\|$ for any $t \in T$, then

$$|f_n - f|_F \to 0.$$

EXAMPLE 2.4 The obvious example of F-norm space is $X = L^p(W)$ space [17], where W is a Banach space, equipped with the F-norm given by

$$\|f\|_{L^p(W)} = \begin{cases} \int_T |f(t)| d\mu(t), & \text{for} \quad 0 < p < 1, \\ \left(\int_T |f(t)|^p d\mu(t) \right)^{1/p}, & \text{for} \quad 1 \leq p < \infty, \end{cases}$$

for each $f \in L^p(W)$. Then, $L^p(W)$ is called the Lebesgue-Bochner space [15].

Now we state a vector version of Egoroff's Theorem for $L_o(T)$. Although the proof is standard, we provide it for the sake of completeness and the reader's convenience.

LEMMA 2.5 Let (T, Σ, μ) be a measure space, $0 < \mu(T) < \infty$, and let W be a Banach space. Let the sequence $\{f_n\} \subset L_o(T)$ satisfy the Cauchy condition with respect to the convergence μ-a.e. Then, for any $m > 0$, there exists $A_m \in \Sigma$, $\mu(A_m) < m$ such that for any $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that

$$\sup\{\|f_q(t) - f_p(t)\| : t \in T \setminus A_m\} \leq \varepsilon$$

provided $p, q \geq n_o$.

PROOF Fix $m > 0$ and $k \in \mathbb{N} \setminus \{0\}$. Define

$$A_{i,j,k} = \{t \in T : \|f_i(t) - f_j(t)\| < 1/k\} \text{ and } B_{p,k} = \bigcap_{i+j \geq p} A_{i,j,k}.$$

Notice that for any $p \in \mathbb{N}$, $B_{p,k} \subset B_{p+1,k}$. Moreover,

$$\bigcup_{p=1}^{\infty} B_{p,k} = T.$$

Indeed, if $t \in T$, then there exists $p_o \in \mathbb{N}$ such that for $i, j \geq p_o$, $\|f_i(t) - f_j(t)\| < 1/k$. Hence, $t \in B_{p_o,k}$. Since $\mu(T) < \infty$, for any $k \in \mathbb{N} \setminus \{0\}$ there exists $p(k)$ such that $\mu(T \setminus B_{p(k),k}) < \frac{1}{2^k}$. Fix $k_o \in \mathbb{N}$ such that $\sum_{j=k_o}^{\infty} \mu(T \setminus B_{p(j),j}) < m$, and put

$$A_m = \bigcup_{j=k_o}^{\infty} (T \setminus B_{p(j),j})$$

It is clear that $\mu(A_m) < m$. Moreover $T \setminus A_m = \bigcap_{j=k_o}^{\infty} B_{p(j),j}$. Now fix $\varepsilon > 0$ and $j_o \in \mathbb{N}$, $j_o \geq k_o$ such that $\frac{1}{j_o} < \varepsilon$. Observe that if $t \in T \setminus A_m$, then $t \in B_{p(j_o),j_o}$. Hence, for any $i, j \geq p(j_o)$

$$\sup\{\|f_i(t) - f_j(t)\| : t \in T \setminus A_m\} < \frac{1}{j_o} < \varepsilon,$$

which proves our claim. ■

Reasoning in the same way as in Lemma 2.5 we can prove

LEMMA 2.6 Let (T, Σ, μ) be a measure space, $0 < \mu(T) < \infty$, and let W be a Banach space. Let $\{f_n\} \subset L_o(T)$ and $f \in L_o(T)$. Assume that $f_n(t) \to f(t)$ μ-a.e. Then, for any $m > 0$, there exists $A_m \in \Sigma$, $\mu(A_m) < m$ such that for any $\varepsilon > 0$ there exists $n_o \in \mathbb{N}$ such that

$$\sup\{\|f_n(t) - f(t)\| : t \in T \setminus A_m\} \leq \varepsilon$$

provided $n \geq n_o$.

THEOREM 2.7 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $\mu(T) < \infty$. Let for $M > 0$,

$$W^M = \{s \in S_F : \sup\{\|s(t)\| : t \in T\} \leq M\}.$$

Let $K = \{T_1, ..., T_k\}$ be an arbitrary partition of T and

$$S^K = \{s \in S_F : s = \sum_{j=1}^{k} w_i \chi_{T_i} : w_i \in W\}.$$

Define $P_K : S_F \to S^K$ by

$$P_K(s) = \sum_{i=1}^{k} z_i(s) \chi_{T_i},$$

where for $s = \sum_{j=1}^{l} w_j \chi_{S_j}$, and $i = 1, ..., k$,

$$z_i(s) = \frac{\sum_{j=1}^{l} w_j \chi_{S_j \cap T_i} \mu(S_j \cap T_i)}{\mu(T_i)}.$$

Assume that $\{s_n\} \subset W^M$ satisfies the Cauchy condition with respect to the convergence μ-a.e. Then, for any $\varepsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for $n, m \geq n_o$ and any partition K

$$|P_K(s_n) - P_K(s_m)|_F \leq \varepsilon.$$

PROOF Fix $\varepsilon > 0$. By Lemma 2.5 for any $p \in \mathbb{N} \setminus \{0\}$, there exists a sequence $\{A_p\} \subset T$ such that $\mu(A_p) \to 0$, and $\{s_n\}$ satisfies the Cauchy condition with respect to the uniform convergence on $T \setminus A_p$. Fix a partition K. Observe that for any $p \in \mathbb{N} \setminus \{0\}$,

$$|P_K(s_n) - P_K(s_m)|_F = |P_K(s_n - s_m)\chi_{T \setminus A_p} + P_K(s_n - s_m)\chi_{A_p}|_F$$

$$\leq |P_K(s_n - s_m)\chi_{T \setminus A_p}|_F + |P_K(s_n - s_m)\chi_{A_p}|_F.$$

Fix $w \in W$ such that $\|w\| = 2M$. By F-admissibility of X, there exists $p_o \in \mathbb{N} \setminus \{0\}$, such that $|w\chi_{A_{po}}|_F \leq \varepsilon/2$. Since for any $n \in \mathbb{N}$ we have $s_n \in W^M$, by definition of P_K, we get $P_K(s_n) \in W^M$. Hence, for any $t \in T$, $n, m \in \mathbb{N}$

$$\|P_K(s_n - s_m)\chi_{A_{po}(t)}\| \leq \|w\chi_{A_{po}}(t)\|.$$

By F-admissibility of X, for any $n, m \in \mathbb{N}$

$$|P_K(s_n - s_m)\chi_{A_{po}}|_F \leq |w\chi_{A_{po}}|_F < \varepsilon/2.$$

Now fix $z \in W \setminus \{0\}$. By F-admissibility of X, there exists $k_o \in \mathbb{N}$ such that $|\frac{z}{k_o}\chi_T|_F < \varepsilon/2$. By definition of A_{p_o} and Lemma 2.5, there exists $n_o \in \mathbb{N}$ such that $n, m \geq n_o$

$$\sup\{\|P_K(s_n - s_m)(t)\| : t \in T \setminus A_{p_o}\} \leq \frac{\|z\|}{k_o}.$$

By F-admissibility of X for $n, m \geq n_o$,

$$|P_K(s_n - s_m)\chi_{T\setminus A_{p_o}}|_F \leq |\frac{z}{k_o}\chi_{T\setminus A_{p_o}}|_F \leq |\frac{z}{k_o}\chi_T|_F < \varepsilon/2.$$

Observe that for any partition K and $n, m \in \mathbb{N}$,

$$\sup\{\|P_K(s_n - s_m)(t)\| : t \in T\} \leq \sup\{\|s_n - s_m\| : t \in T\}.$$

Hence, the choice of n_o is independent of K. The proof is complete. ■

LEMMA 2.8 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $\mu(T) < \infty$. Let $f \in X$, and let $\sup\{\|f(t)\| : t \in T\} = M < \infty$. Then, there exists a sequence $\{w_n\} \subset W^{4M}$ such that $w_n \to f$ μ-a.e. Moreover, $|w_n - f|_F \to 0$.

PROOF Since $f \in L_o(T)$, there exists a sequence $\{s_n\} \subset S_F$ such that $s_n \to f$ μ-a.e. Let $s_n = \sum_{j=1}^{k_n} s_{n,j}\chi_{A_{n,j}}$. Set $w_{n,j} = s_{n,j}$ if $\|s_{n,j}\| \leq 4M$ and $w_{n,j} = \frac{2Ms_{n,j}}{\|s_{n,j}\|}$ in the opposite case. Define

$$w_n = \sum_{j=1}^{k_n} w_{n,j}\chi_{A_{n,j}}.$$

Let $t \in T$. Then, there exists exactly one $j \in \{1, ..., k_n\}$ such that $t \in A_{n,j}$. If $\|s_{n,j}\| > 4M$, then

$$\|(s_{n,j} - f)(t)\| - \|(w_{n,j} - f)(t)\| \geq \|s_{n,j}(t)\| - \|f(t)\| - (\|w_{n,j}(t)\| + \|f(t)\|)$$

$$\geq \|s_{n,j}(t)\| - \|w_{n,j}(t)\| - 2\|f(t)\| > 0.$$

Hence, $w_n \to f$ μ-a.e. Now we show that $|w_n - f|_F \to 0$. Applying Lemma 2.6 and using the same method as in the proof of Theorem 2.7, we get that for any $\varepsilon > 0$ and $k \in \mathbb{N} \setminus \{0\}$ there exists $A_k \subset T$, and $n_o \in \mathbb{N}$ such that for $n \geq n_o$

$$|(f - w_n)\chi_{A_k}|_F \leq \varepsilon/2 \text{ and } |(f - w_n)\chi_{T\setminus A_k}|_F \leq \varepsilon/2.$$

Hence, $|f - w_n|_F \to 0$. ■

The following result is a more general case of the well-known theorem stating that norm convergence implies convergence of some subsequence μ-a.e. (for more details see [18]).

LEMMA 2.9 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $\mu(T) < \infty$. Assume that $\{f_n\} \subset X$, $f \in X$ and $|f_n - f|_F \to 0$. Then, there exists a subsequence $\{n_k\}$ such that f_{n_k} converges to f μ-a.e.

PROOF Assume that $|f_n - f|_F \to 0$. First, we show that f_n converges to f in measure. Assume that this is not true. Passing to a convergent subsequence, if necessary, we can assume that there exist $d > 0$ and $e > 0$ such that

$$\mu(A_{d,n}) = \mu(\{t \in T : \|f_n(t) - f(t)\| \geq d\}) > e$$

for all $n \in \mathbb{N}$. Fix $z \in W$ such that $\|z\| = d$, and let $g_n = z\chi_{A_{d,n}}$. Note that for any $t \in T$,

$$\|(f - f_n)(t)\| \geq \|(f - f_n)\chi_{A_{d,n}}(t)\| \geq \|g_n(t)\|.$$

Since X is an F-admissible F-space $|g_n|_F$ does not converge to 0 and consequently $|f - f_n|_F$ does not converge to 0; a contradiction. Now, select for any $k \in \mathbb{N} \setminus \{0\}$, $n_k \in \mathbb{N}$ such that $\mu(A_{1/k,n_k}) \leq \dfrac{1}{2^k}$. Let for any $m \in \mathbb{N}$, $B_m = \displaystyle\bigcup_{k=m}^{\infty} A_{1/k,n_k}$. Notice that

$$\mu(B_m) \leq \sum_{k=m}^{\infty} \mu(A_{1/k,n_k}) \leq \frac{1}{2^{m-1}}.$$

Let $B = \displaystyle\bigcap_{m=1}^{\infty} B_m$. It is clear that $\mu(B) = \displaystyle\lim_{m\to\infty} \mu(B_m) = 0$. Let $t \in T \setminus B$. Then, there exists m_o such that $t \notin B_{m_o}$. Hence, for any $k \geq m_o$ $t \notin A_{1/k,n_k}$. Consequently, for any $k \geq k_o$ $\|f(t) - f_n(t)\| < \dfrac{1}{k_o}$, which shows our claim. ■

LEMMA 2.10 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $\mu(T) < \infty$. Fix $f \in X$ with $\sup\{\|f(t)\| : t \in T\} = M < \infty$. Let $\{w_n\} \subset W^{4M}$ be so chosen that $w_n \to f$ μ-a.e. Let K be a fixed partition of T, and let P_K be the operator from Theorem 2.7. Then, there exists the limit of $P_K(w_n)$ as $n \to \infty$, independently of the choice of a sequence (w_n). Define $P_K(f)$ for any $f \in X \subset L_o(T)$ by

$$P_K(f) = \lim_{n\to\infty} P_K(w_n).$$

PROOF Assume that $\{w_n\} \subset W^{4M}$ converges to f μ-a.e. (by Lemma 2.8 such a sequence exists). Then, $\{w_n\}$ satisfies the Cauchy condition with respect to the μ-a.e. convergence. By Theorem 2.7, the sequence $\{P_K(w_n)\}$ satisfies the Cauchy condition with respect to the convergence in X. Since X is complete, there exists $\displaystyle\lim_{n\to\infty} P_K(w_n)$. If $\{z_n\} \subset W^{4M}$ is the other sequence converging to f μ-a.e., then considering a sequence $s_{2n} = w_n$ and $s_{2n+1} = z_n$, we get that the obtained limit is independent of the choice of $\{w_n\} \subset W^{4M}$. ■

LEMMA 2.11 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $0 < \mu(T) < \infty$. Let $\{f_n\} \subset X$ and $f \in X$. Assume that there exists $M > 0$ such that $\sup\{\|f_n(t)\| : t \in T, n \in \mathbb{N}\} \leq M$ and $\sup\{\|f(t)\| : t \in T\} \leq M$. Let K be a fixed partition of T. If $|f_n - f|_F \to 0$, then $|P_K(f_n) - P_K(f)|_F \to 0$.

PROOF By definition of P_K, for any $n \in \mathbb{N}$, there exists $\{s_{k,n}\} \subset W^{4M}$ such that $|P_K(f_n) - P_K(s_{k,n})|_F \to_k 0$. Hence, for any $n \in \mathbb{N}$, there exists $k_n \in \mathbb{N}$ such that

$$|P_K(f_n) - P_K(s_{k_n,n})|_F \leq 1/(2n) \text{ and } |f_n - s_{k_n,n}|_F < 1/(2n).$$

Since $|f_n - f|_F \to, 0$, $|f - s_{k_n,n}|_F \to_n 0$. By definition of P_K,

$$|P_K(f) - P_K(s_{k_n,n})|_F \to_n 0.$$

Hence,

$$0 \leq |P_K(f) - P_K(f_n)|_F \leq |P_K(f) - P_K(s_{k_n,n})|_F + |P_K(f_n) - P_K(s_{k_n,n})|_F$$

$$\leq 1/(2n) + |P_K(f) - P_K(s_{k_n,n})|_F.$$

Hence, $|P_K(f_n) - P_K(f)|_F \to 0$, as required. ■

DEFINITION 2.12 Let $0 < \mu(T) < \infty$, and let $K = \{K_1, ...K_n\}$ and $G = \{G_1, ..., G_m\}$, denote two partitions of T, $K_i, G_j \in \Sigma$, for $i = 1, ..., n$ and $j = 1, ..., m$. We say that $K \leq G$, if for any $i = 1, ..., n$,

$$K_i = \sum_{j=1}^{m_i} G_{ij}$$

with $G_{ij} \in G$ for $j = 1, ..., m_i$ and $1 \leq m_i \leq m$.

LEMMA 2.13 Let $s \in W^M \setminus \{0\}$, and let $0 < \mu(T) < \infty$. Let $K = \{K_1, ..., K_n\}$ be a partition associated with s, i.e.

$$s = \sum_{j=1}^{n} w_j \chi_{K_j}.$$

If $G = \{G_1, ..., G_k\}$ is a partition of T with $\mu(G_i) > 0$ for $i = 1, ..., k$ and $K \leq G$ then $P_G(s) = s$.

PROOF Notice that

$$P_G(s) = \sum_{i=1}^{k} z_i(s) \chi_{G_i},$$

where for $s = \sum_{j=1}^{n} w_j \chi_{K_j}$, and $i = 1, ..., k$,

$$z_i(s) = \frac{\sum_{j=1}^{n} w_j \chi_{K_j \cap G_i} \mu(K_j \cap G_i)}{\mu(G_i)}.$$

Since $K \leq G$, there exists exactly one $j_i \in \{1, ..., n\}$ such that $G_i \subset K_{j_i}$. Hence,

$$z_i(s) = \frac{w_{j_i} \chi_{G_i \cap K_{j_i}} \mu(G_i \cap K_{j_i})}{\mu(G_i)} = w_{j_i} \chi_{G_i}.$$

Let $t \in T$. Then, there exists exactly one $i \in \{1, ..., k\}$, such that $t \in G_i$. Hence, $s(t) = w_{j_i}$, which shows that $P_G(s) = s$. ■

LEMMA 2.14 Let $0 < \mu(T) < \infty$. Let for $n \in \mathbb{N}$, $K^n = \{K_{1,n}..., K_{k,n}\}$ be a partition of T. Then, there exists a sequence $\{S^n\}$ of partitions of T such that $S^i \leq S^j$ for $i \leq j$ and for any $n \in \mathbb{N}$, $K^n \leq S^j$ for $j \geq n$.

PROOF Let $S^1 = K^1$. Put $S^2 = \{K_1 \cap K_2 : K_1 \in K^1, K_2 \in K^2\}$. It is clear that S^2 is a partition of T and $K^i \leq S^2$ for $i = 1, 2$. Now assume that we have constructed $S^1, ..., S^n$. Define $S^{n+1} = \{S \cap K : S \in S^n, K \in K^{n+1}\}$. It is clear that S^{n+1} is a partition of T, $S^i \leq S^{n+1}$ for $i = 1, ..., n$ and $K^i \leq S^{n+1}$ for $i = 1, ..., n+1$. By our construction, for any $n \in \mathbb{N}$ and $j \geq n$, $S^n \leq S^j$. Also $K^n \leq S^j$, for $j \geq n$, as required. ■

THEOREM 2.15 Let $X \subset L_o(T)$ be an F-admissible F-space. Assume that $0 < \mu(T) < \infty$. Let $Z \subset X$ be a compact set. Assume that there exists $M > 0$ such that $\sup\{\|f(t)\| : f \in Z, t \in T\} \leq M$. Then, for any $\varepsilon > 0$ there exists K a partition of T such that

$$\sup\{|P_K(f) - f|_F : f \in Z\} < \varepsilon.$$

PROOF Since Z is compact, there exists a countable, dense set $A = \{g_n\} \subset Z$. Fix for any $n \in \mathbb{N}$ a sequence $\{s_{k,n}\} \subset W^{4M}$ such that $s_{k,n} \to_k g_n$ μ-a.e. (by Lemma 2.8 such a sequence exists). Let $\{U_{k,n}\}$ be the sequence of partitions associated with $\{s_{k,n}\}$. Let $\phi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a fixed bijection. Put $K_n = U_{\phi(n)}$. By Lemma 2.14 there exists a sequence of partitions $\{S^n\}$ such that $S^i \leq S^j$ for $i \leq j$ and for any $n \in \mathbb{N}$, $K_n \leq S^j$ for $j \geq n$. Let $P_n = P_{S^n}$ be the projection associated with S^n by Theorem 2.7. Since $L_o(S)$ is the space of strongly measurable functions with equality μ-a.e. (see Definition 2.2), we can assume that for any $S \in S^n$ we have $\mu(S) > 0$, which shows that definition of P_n is correct (compare with Theorem 2.7). We show that for any $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that

$$\sup\{|P_n(f) - f|_F : f \in Z\} < \varepsilon.$$

Assume, on the contrary, that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there exists $f_n \in Z$ such that

$$|f_n - P_n(f_n)|_F \geq \varepsilon.$$

Passing to a converging subsequence, if necessary, we may assume that $|f_n - f|_F \to 0$ for some $f \in Z$. Again, by Lemma 2.9, passing to a convergent subsequence, if necessary, we may assume that $f_n \to f$ μ-a.e. Let for any $n \in \mathbb{N}$, $\{w_{k,n}\} \subset W^{4M}$ be such that $|f_n - w_{k,n}|_F \to_k 0$ and $\|w_{k,n}(t) - f_n(t)\| \to 0$ μ-a.e. Since $\{g_n\}$ is a dense subset of Z, by Lemma 2.9, we can assume that for any $n \in \mathbb{N}$, $\{w_{k,n}\} \subset \bigcup_{m=1}^{\infty} \{s_{k,m}\}$. Since $f_n \to f$ μ-a.e. and $w_{k,n} \to_k f_n$ μ-a.e., by Lemma 2.9, we can select for any $n \in \mathbb{N}$ $w_{k_n,n}$ such that

$$|f - w_{k_n,n}|_F \to_n 0,\ f(t) - w_{k_n,n}(t) \to 0\ \mu\text{-a.e.},$$

$|P_n(f_n) - P_n(w_{k_n,n})| \leq \varepsilon/8$, and $w_{k_n,n}$ satisfies the Cauchy condition with respect to the convergence μ-a.e. Fix $n \in \mathbb{N}$ such that $|f_n - f|_F \leq \varepsilon/8$.

$$|f_n - P_n(f_n)|_F \leq |f - f_n|_F + |f - P_n(f)|_F + |P_n(f_n) - P_n(f)|_F.$$

By definition of P_n, $\{s_{k,n}\}$ and $\{w_{k,n}\}$, there exists $m_o \in \mathbb{N}$ such that for $m \geq m_o$, $|f - w_{k_m,m}|_F \leq \varepsilon/8$, $w_{k_m.m} = P_n(w_{k_m,m})$ and $|P_n(f) - P_n(w_{k_m,m})| < \varepsilon/8$. Hence,

$$|f - P_n(f)|_F \leq |f - w_{k_m,m}|_F + |w_{k_m,m} - P_n(f)|_F$$

$$= |f - w_{k_m,m}|_F + |P_n(w_{k_m,m}) - P_n(f)|_F \leq \varepsilon/4.$$

Also

$$|P_n(f_n) - P_n(f)|_F \leq |P_n(f_n) - P_n(w_{k_n,n})|_F$$

$$+ |P_n(w_{k_n,n}) - P_n(w_{k_m.m})|_F + |P_n(f) - P_n(w_{k_m,m})|_F.$$

Since $\{w_{k_n,n}\}$ satisfies the Cauchy condition with respect to the μ-a.e. convergence, by Theorem 2.7, for $n, m \geq n_o$

$$|P_n(w_{k_n,n}) - P_n(w_{k_m,m})|_F \leq \varepsilon/8.$$

Consequently, we get $|P_n(f_n) - f_n|_F < \varepsilon$ for sufficiently large $n \in \mathbb{N}$. Hence, we obtain a contradiction and complete the proof. ■

3. Admissibility

In this section, we prove the admissibility of any F-admissible F-space. We begin with the following proposition by showing that the space of simple functions generated by a fixed partition K of T is admissible.

PROPOSITION 3.1 Let $K = \{K_1, ..., K_n\}$ be a finite partition of T such that $\mu(K_i) > 0$ for any $i \in \{1, ..., n\}$. Then, the subspace

$$S_K = \left\{ s \in X : s = \sum_{i=1}^{n} w_i \chi_{K_i}, \;\; w_i \in W \right\}$$

of X is admissible.

PROOF Let Z be a compact subset of S_K and $\varepsilon > 0$ be given. For each $g \in Z$ we can write

$$g = \sum_{i=1}^{n} w_i(g) \chi_{K_i}$$

for suitable elements $w_i(g)$ of the Banach space W.

First we show that for any $i = 1, ..., n$ the mapping $g \to w_i(g)$ is continuous with respect to $|\cdot|_F$. Assume on the contrary that there exist $i \in \{1, ..., n\}$, $g_k, g \in Z$ and $d > 0$ such that $\|w_i(g_k) - w_i(g)\| \geq d$. Fix $z \in W$, $\|z\| = d$. Let $f = z\chi_{K_i}$. Observe that for any $t \in T$,

$$\|(g_k - g)(t)\| \geq \|(g_k - g)\chi_{K_i}(t)\| \geq \|f(t)\|.$$

Since $\mu(K_i) > 0$, and X is an F-admissible F-space, $|f|_F > 0$, which leads to a contradiction. Consequently, for any fixed $i = 1, ..., n$, the set $C_i = \{w_i(g) : \;\; g \in Z\}$ is a compact subset of W, and $C = \bigcup_{i=1}^{n} C_i$ is a compact subset of W too.

Let $\delta > 0$ be fixed. Then, by the admissibility of the Banach space W, there exist a finite dimensional space $Z_\delta = \text{span}[z_1, ..., z_m]$ in W and a continuous mapping $H_\delta : \; C \to Z_\delta$ such that

$$\|w - H_\delta(w)\| \leq \delta \;\; \text{for all} \;\; w \in C. \tag{1}$$

Then, for each $i \in \{1, ..., n\}$, $g \in S_K$ and for suitable $w_j^i(g) \in \mathbb{R}$, $j = 1, ..., m$ we can write

$$H_\delta(w_i(g)) = \sum_{j=1}^{m} w_j^i(g) z_j.$$

As no confusion can arise, we denote again by $H_\delta : Z \to S_K$ the continuous mapping defined by

$$H_\delta(g) = \sum_{i=1}^{n} H_\delta(w_i(g))\chi_{K_i} = \sum_{i=1}^{n}\Big(\sum_{j=1}^{m} w_j^i(g)z_j\Big)\chi_{K_i}.$$

Then, $H_\delta(S_K) \subseteq \text{span}[\chi_{K_i}z_j, i = 1,...,n;\ j = 1,...,m]$ and $\dim(\text{span}[H_\delta(S_K)]) < +\infty$. On the other hand, for each $g \in Z$, we have

$$\begin{aligned} |g - H_\delta(g)|_F &= |\sum_{i=1}^{n} w_i(g)\chi_{K_i} - \sum_{i=1}^{n}(\sum_{j=1}^{m} w_j^i(g)z_j)\chi_{K_i}|_F \\ &\le \sum_{i=1}^{n} |(w_i(g) - \sum_{j=1}^{m} w_j^i(g)z_j)\chi_{K_i}|_F. \end{aligned} \tag{2}$$

Since X is an F-admissible F-space

$$\sum_{i=1}^{n} |(w_i(g) - \sum_{j=1}^{m} w_j^i(g)z_j)\chi_{K_i}|_F \le \sum_{i=1}^{n} |\delta\chi_{K_i}|_F.$$

Since $|\cdot|_F$ is an F-norm, for any $\varepsilon > 0$ we can find $\delta > 0$ such that $\sum_{i=1}^{n} |\delta\chi_{K_i}|_F < \varepsilon$. This shows the admissibility of Z in X. ■

In order to prove our main result of this paper Theorem 3.4, we need the following two lemmas.

LEMMA 3.2 Let $Z \subset X$ be a compact set. Set $F_n(f) = f\chi_{T_n}$, where $T = \bigcup_{n=1}^{\infty} T_n$, $\mu(T_n) < \infty$, $T_n \subset T_{n+1}$ for any $n \in \mathbb{N}$. Then, for any $f, g \in X$ and $n \in \mathbb{N}$,

$$|F_n(f) - F_n(g)|_F \le |f - g|_F.$$

Moreover, for any $\varepsilon > 0$, there exists $n_o \in \mathbb{N}$ such that for each $n \ge n_o$ we have

$$\sup\{|f - F_n(f)|_F : f \in Z\} < \varepsilon.$$

PROOF Let $f, g \in X$. Observe that for any $t \in T$

$$\|(f - g)(t)\| \ge \|F_n(f)(t) - F_n(g)(t)\|.$$

Hence, by F-admissibility of X,

$$|F_n(f) - F_n(g)|_F \le |f - g|_F.$$

Notice that for any $f \in X$, $F_n(f) \to f$ μ-a.e. and $\|F_n(f)(t)\| \le \|f(t)\|$ for any $t \in T$. By F-admissibility of X, $|f - F_n(f)|_F \to 0$. Now assume that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, we have

$$|f_n - F_n(f_n)|_F \ge \varepsilon.$$

By compactness of Z, we can assume that $|f_n - f|_F \to 0$, for some $f \in Z$. Fix $n_o \in \mathbb{N}$ such that for $n \geq n_o$, $|f - F_n(f)|_F < \varepsilon/3$ and $|f - f_n|_F < \varepsilon/3$. Hence, for any $n \geq n_o$,

$$|f_n - F_n(f_n)|_F \leq |f_n - f|_F + |f - F_n(f_n)|_F$$

$$\leq |f_n - f|_F + |F_n(f) - F_n(f_n)|_F + |f - F_n(f)|_F$$

$$\leq 2|f_n - f|_F + |f - F_n(f)|_F < \varepsilon,$$

a contradiction. ■

Let $a > 0$. Now we denote by R_a the radial projection, of the Banach space W onto its closed ball $B_a(W)$ of radius a, defined for $w \in W$ by

$$R_a w = \begin{cases} w & \text{if } \ \|w\| \leq a \\ a\,\dfrac{w}{\|w\|} & \text{if } \ \|w\| > a. \end{cases}$$

Then, we define the mapping $T_a : X \to X$ by setting for $t \in T$

$$(T_a(f))(t) = R_a(f(t)).$$

LEMMA 3.3 For any $a > 0$ and $f, g \in X$,

$$|T_a(f) - T_a(g)|_F \leq 2|f - g|_F.$$

Moreover, for any $\varepsilon > 0$, and for any compact subset Z of X, there exists $a > 0$ such that

$$\sup\{|f - T_a(f)|_F : f \in Z\} < \varepsilon.$$

PROOF Fix $a > 0$. Note that by definition of T_a for any $t \in T$

$$\|T_a(f) - T_a(g)\| \leq 2\|f(t) - g(t)\|$$

as R_a is a Lipschitz mapping with constant 2 [19]. Hence, by F-admissibility of X,

$$|T_a(f) - T_a(g)|_F \leq 2|f - g|_F.$$

Now fix $Z \subset X$ compact and $f \in Z$. Note that for any $t \in T$, $\lim_{n\to\infty} T_n(f)(t) = f(t)$ and $\|T_n(f)(t)\| \leq \|f(t)\|$ for any $n \in \mathbb{N}$. By F-admissibility of X,

$$|f - T_n(f)|_F \to 0. \tag{3}$$

To prove our second assert, assume by contradiction that there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$, there exists $f_n \in Z$ such that

$$|f_n - T_n(f_n)|_F > \varepsilon.$$

Without loss of generality, passing to a convergent subsequence if necessary, we can assume that there exists $f \in X$ such that $|f_n - f|_F \to 0$. Fix $n \geq n_o$ such that for $n \geq n_o$ $|f - f_n|_F < \varepsilon/4$ and $|f - T_n(f)|_F < \varepsilon/4$. Notice that for $n \geq n_o$,

$$\begin{aligned} |f_n - T_n(f_n)|_F &\leq |f - T_n(f)|_F + |T_n(f) - T_n(f_n)|_F + |f - f_n|_F \\ &\leq 3|f_n - f||_F + |f - T_n(f)||_F. \end{aligned}$$

Hence, $|f_n - T_n(f_n)|_F < \varepsilon$ for $n \geq n_o$, a contradiction. ■

We are now in the position to prove our main result.

THEOREM 3.4 Any F-admissible F-space is admissible.

PROOF Fix Z a compact set in X, and $\varepsilon > 0$. Since Z is compact, by Lemma 3.2, there exists $n \in \mathbb{N}$ such that

$$\sup\{|F_n(f) - f|_F : f \in Z\} \leq \varepsilon/4.$$

Moreover, F_n is continuous. By Lemma 3.3 applied to the compact set $F_n(Z)$, there exists $a > 0$ satisfying

$$\sup\{|T_a F_n(f) - F_n(f)|_F : f \in Z\} \leq \varepsilon/4.$$

Since T_a is a continuous mapping, by Theorem 2.15 applied to $(T_a \circ F_n)(Z)$ and $T_n \in \Sigma$, $\mu(T_n) < \infty$, there exists $k \in \mathbb{N}$ with

$$\sup\{|P_k T_a F_n(f) - T_a F_n(f)|_F : f \in Z\} \leq \varepsilon/4.$$

Notice that by Lemma 2.11, P_k is a continuous mapping for any $k \in \mathbb{N}$. Hence, by Proposition 3.1 applied to $W = (P_k \circ T_a \circ F_n)(Z)$ there exits $H_\varepsilon : W \to E_\rho$ such that span$[H_\varepsilon(W)]$ is finite-dimensional and

$$\sup\{|H_\varepsilon P_k T_a F_n(f) - P_k T_a F_n(f)|_F : f \in Z\} \leq \varepsilon/4.$$

Notice that the continuous mapping $H = H_\varepsilon \circ P_k \circ T_a \circ F_n$ satisfies dim[span$[H(Z)] < \infty$. Moreover, by the above facts, for any $f \in Z$

$$|f - H(f)|_F \leq |F_n(f) - f|_F + |T_a(F_n(f)) - F_n(f)|_F$$

$$+|P_k T_a F_n(f) - T_a F_n(f)|_F + |H_\varepsilon P_k T_a F_n(f) - P_k T_a F_n(f)|_F \leq \varepsilon,$$

and the admissibility of X is proved. ■

Now we present two important consequences of admissibility and Theorem 3.4. We formulate a version of the fixed point property for compact continuous mappings in any admissible F-space.

THEOREM 3.5 Let X be an admissible F-space. Let $T : X \to X$ be a compact and continuous mapping. Then, there exists $f \in X$ such that $T(f) = f$.

PROOF The proof which will be presented works for any admissible Hausdorff topological vector space, and it is well-known. We present it for a sake of completeness. Since T is a compact mapping, the set $Z = cl[T(X)] \subset X$ is a compact set. Hence, by Theorem 3.4 for any $\varepsilon > 0$, there exists a continuous mapping $H_\varepsilon : Z \to X$ such that $\dim[\mathrm{span}[H_\varepsilon(Z)] < \infty$ and $\sup_{f\in Z} |f - H_\varepsilon(f)|_F \le \varepsilon$. Let $T_\varepsilon = H_\varepsilon \circ T$. Notice that $T_\varepsilon(X) = H_\varepsilon(T(X)) \subset H_\varepsilon(Z)$ and consequently, since $conv(H_\varepsilon(Z)) \subset X$,

$$T_\varepsilon[conv(H_\varepsilon(Z))] \subset T_\varepsilon(X) \subset H_\varepsilon(Z) \subset conv(H_\varepsilon(Z)).$$

Also T_ε is a continuous map. Since $\dim[\mathrm{span}[H_\varepsilon(Z)] < \infty$, by the Carathéodory Theorem, the set $conv(H_\varepsilon(Z))$ is a compact set. By the Brouwer Theorem, there exists $f_\varepsilon \in X$ such that $T_\varepsilon(f_\varepsilon) = f_\varepsilon$. Hence, for any $n \in \mathbb{N}$,

$$|T(f_{1/n}) - f_{1/n}|_F = |T(f_{1/n}) - T_{1/n}(f_{1/n})|_F = |T(f_{1/n}) - (H_{1/n} \circ T)(f_{1/n})|_F \le 1/n,$$

since $T(f_{1/n}) \in Z$. By the compactness of Z, we can assume that $\lim_{n\to\infty} |T(f_{1/n}) - f|_F = 0$ for some $f \in Z$. Hence, by the above estimate

$$|f_{1/n} - f|_F \le |T(f_{1/n}) - f_{1/n}|_F + |f - T(f_{1/n})|_F.$$

Hence, $\lim_{n\to\infty} |f_{1/n} - f|_F = 0$. By the continuity of T, $\lim_{n\to\infty} |T(f_{1/n}) - T(f)|_F = 0$, which gives that $T(f) = f$. ■

THEOREM 3.6 Let X be an admissible F-space, and let $C \subset X$ be a nonempty set. Assume that C is a retract of X, i.e. there exists a continuous mapping $P : X \to C$ such that $Pc = c$ for any $c \in C$. Let $T : C \to C$ be a continuous compact mapping. Then, there exists $x \in C$ such that $Tx = x$.

PROOF Applying Theorem 3.5 to the mapping $T \circ P$, we get that there exists $x \in X$ such that $(T \circ P)x = x$. Since $(T \circ P)x \in C$, $x \in C$. Consequently $Px = x$ and $Tx = x$, as required. ■

4. Applications to modular spaces

In this section, we show that a large class of modular spaces satisfies the requirements of Definition 2.3. First we recall a notion of modular.
Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of complex, reals, nonnegative reals and positive integers, respectively. Let us denote by $(e_i)_{i=1}^n$ a standard basis in $\mathbb{R}^n$. Let X be a linear space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A function $\rho : X \to [0, +\infty]$ is called a *semimodular* if there holds for arbitrary $x, y \in X$:

(a) $\rho(dx) = 0$ for any $d \geq 0$ implies $x = 0$ and $\rho(0) = 0$;

(b) $\rho(dx) = \rho(x)$, for any $d \in \mathbb{K}$, $|d| = 1$;

(c) $\rho(ax+by) \leq \rho(x) + \rho(y)$ for any $a,b \geq 0$, $a+b = 1$.

If we replace (c) by:

$(c1)$ there exists $s \in (0,1]$ such that $\rho(ax+by) \leq a^{1/s}\rho(x) + b^{1/s}\rho(y)$ for any $a,b \geq 0$, $a^{1/s}+b^{1/s} = 1$,

then ρ is called *s-convex* (*convex* if $s = 1$).

Let $X_\rho = \{x \in X : \rho(tf) < \infty : \text{ for some } t \in \mathbb{R}\}$. Then, X_ρ is called a modular space (for more details see [20]).

THEOREM 4.1 Let X be a linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Fix $n \geq 2$, and let ρ_i be a semimodular defined on X for any $i \in \{1,...,n-1\}$. Put $\rho = \max_{1 \leq i \leq n-1}\{\rho_i\}$. Assume that $\varphi : \mathbb{R}^n \to [0,+\infty)$ is an F-norm such that for any $x = (x_1,x_2,...,x_n) \in (\mathbb{R}_+)^n$ and $y = (y_1,y_2,...,y_n) \in (\mathbb{R}_+)^n$ if $x_j \leq y_j$ for $j = 1,...,n$, then

$$\varphi(x) \leq \varphi(y). \tag{4}$$

Let us define for $x \in X_\rho$,

$$|x|_\varphi = \inf_{k>0}\left\{\varphi\left(ke_1 + \sum_{i=2}^{n}\rho_{i-1}\left(\frac{x}{k}\right)e_i\right)\right\}.$$

Then, $|\cdot|_\varphi$ is an F-norm in X_ρ.

PROOF Since $x \in X_\rho$, there exists $u > 0$ such that $\rho_i(x/u) < \infty$, for $i = 1,...,n-1$. Hence, $|x|_\varphi < \infty$. Obviously, $|0|_\varphi = 0$. Now we show that $|x|_\varphi > 0$ for $x \neq 0$. First assume that $\rho(x) \in \{0,+\infty\}$. Since $x \neq 0$, $\rho(x/k_o) = +\infty$ for some $k_o > 0$. Notice that for $k \geq k_o$,

$$\varphi\left(ke_1 + \sum_{i=2}^{n}\rho_{i-1}\left(\frac{x}{k}\right)e_i\right) \geq \varphi(k_o,0,...,0) > 0.$$

If $k < k_o$, then

$$\varphi\left(ke_1 + \sum_{i=2}^{n}\rho_{i-1}\left(\frac{x}{k}\right)e_i\right) \geq \varphi\left(\sum_{i=2}^{n}\rho_{i-1}\left(\frac{x}{k_o}\right)e_i\right) = +\infty.$$

If there exists $k_o > 0$, such that $0 < \rho(x/k_o) < \infty$, then for $k < k_o$,

$$\varphi\left(ke_1 + \sum_{i=2}^{n}\rho_{i-1}\left(\frac{x}{k}\right)e_i\right)$$

$$\geq \varphi\left(ke_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{x}{k_o}\right)e_i\right) \to_k \varphi\left(\sum_{i=2}^{n} \rho_{i-1}\left(\frac{x}{k_o}\right)e_i\right) > 0.$$

If $k \geq k_o$, then

$$\varphi\left(ke_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{x}{k_o}\right)e_i\right) \geq \varphi(k_o e_1, 0, ..., 0) > 0.$$

Hence, $|x|_\varphi > 0$. By defintion of modular, $|ax|_\varphi = |x|_\varphi$ for $a \in \mathbb{K}$, $|a| = 1$. Now we show that $|x+y|_\varphi \leq |x|_\varphi + |y|_\varphi$. To do that, fix $\varepsilon > 0$, $u > 0$ and $v > 0$ such that

$$\varphi\left(ue_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{x}{u}\right)e_i\right) \leq |x|_\varphi + \varepsilon$$

and

$$\varphi\left(ve_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{y}{v}\right)e_i\right) \leq |y|_\varphi + \varepsilon.$$

Notice that

$$\varphi\left((u+v)e_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{x+y}{u+v}\right)e_i\right)$$

$$= \varphi\left((u+v)e_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{ux}{(u+v)u} + \frac{vy}{(u+v)v}\right)e_i\right)$$

$$\leq \varphi\left((u+v)e_1 + \sum_{i=2}^{n} \left(\left(\rho_{i-1}\left(\frac{x}{u}\right) + \rho_{i-1}\left(\frac{y}{v}\right)\right)e_i\right)\right)$$

$$\leq \varphi\left(ue_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{x}{u}\right)e_i\right) + \varphi\left(ve_1 + \sum_{i=2}^{n} \rho_{i-1}\left(\frac{y}{v}\right)e_i\right)$$

$$\leq |x|_\varphi + |y|_\varphi + 2\varepsilon.$$

Hence, $|x+y|_\varphi \leq |x|_\varphi + |y|_\varphi$, as required. ■

Now, applying Theorem 4.1 we get the following theorem.

THEOREM 4.2 Let ρ_1 and $f : \mathbb{R}^2 \to \mathbb{R}$ be as in Theorem 4.1, and let ρ_1 be a semi-modular. Then, the function

$$|x|_\varphi = \inf_{k>0}\{\varphi(ke_1 + \rho_1(x/k)e_2)\}$$

is an F-norm on X_{ρ_1}.

PROOF It is necessary to apply Theorem 4.1 for $n = 2$ ■

REMARK 4.3 Observe that if φ is equal to the maximum norm on $\mathbb{R}^2$, ρ is a semimodular and $x \in X_\rho$, then

$$|x|_\varphi = \inf\{u > 0 : \rho(x/u) \leq u\},$$

which means that $|\cdot|_\varphi$ coincides with the classical F-norm on X_ρ.
Indeed, let $\varphi(u,v) = max\{|u|,|v|\}$. Let

$$|x|_L = \inf\{u > 0 : \rho(x/u) \leq u\}.$$

We show that $|x|_\varphi = |x|_L$. Notice that for any $\varepsilon > 0$,

$$\rho\left(\frac{x}{|x|_L+\varepsilon}\right) \leq |x|_L + \varepsilon.$$

Hence, $\varphi(|x|_L+\varepsilon, \rho(\frac{x}{|x|_L+\varepsilon})) = |x|_L+\varepsilon$, which shows that $|x|_\varphi \leq |x|_L$. If $|x|_\varphi < |x|_L$ for some $x \in X_\rho$, then there exist $u > 0$ and $\delta > 0$ such that

$$\varphi(u, \rho(x/u)) < |x|_L - \delta.$$

This shows that $|u| \leq \|x\|_L - \delta$ and consequently

$$\rho\left(\frac{x}{|x|_L-\delta}\right) \leq \rho\left(\frac{x}{u}\right) < |x|_L - \delta,$$

which leads to a contradiction with definition of $|\cdot|_L$.

It is worth noticing that, recently in [21], similar problems devoted to s-norms generated by s-convex semimodulars have been investigated. We present all the details for the sake of completeness and the reader's convenience. Namely, the following results were established.

THEOREM 4.4 [21] Let X be a linear space over $\mathbb{K}$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, and let $s \in (0,1]$. Fix $n \geq 2$, and let ρ_i be a s-convex semimodular defined on X for any $i \in \{1,\ldots,n-1\}$. Put $\rho = \max_{1\leq i\leq n-1}\{\rho_i\}$. Assume that $\varphi : \mathbb{R}^n \to [0,+\infty)$ is a convex function such that $\varphi(x) = 0$ if and only if $x = 0$. Assume furthermore that for any $x = (1,x_2,\ldots,x_n) \in (\mathbb{R}_+)^n$ and $y = (1,y_2,\ldots,y_n) \in (\mathbb{R}_+)^n$ if $x_j \leq y_j$ for $j = 2,\ldots,n$, then

$$\varphi(x) \leq \varphi(y). \tag{5}$$

Let us define for $x \in X_\rho$,

$$|x|_\varphi = \inf_{k>0} \left\{ k\varphi \left(e_1 + \sum_{i=2}^{n} \rho_{i-1} \left(\frac{x}{k^{1/s}} \right) e_i \right) \right\}.$$

Then, $|\cdot|_\varphi$ is an s-norm (norm if $s = 1$) in X_ρ.

THEOREM 4.5 [21] Let ρ_1 and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ be as in Theorem 4.4, and let ρ_1 be an s-convex semimodular. Then, the function

$$\|x\|_\varphi = \inf_{k>0} \{k\varphi(1, \rho_1(x/k^{1/s}))\}$$

is an s-convex norm (norm if $s = 1$) on X_{ρ_1}.

It is worth recalling one more result that was presented in [21] and comparing it with Remark 4.3.

REMARK 4.6 [21] Observe that if φ is equal to the maximum norm on $\mathbb{R}^2$, ρ is a convex semimodular, and $x \in X_\rho$, then

$$\|x\|_\varphi = \inf\{u > 0 : \rho(x/u) \leq 1\},$$

which means that $\|\cdot\|_\varphi$ coincides with the classical Luxemburg norm on X_ρ. If φ is equal to the l_1-norm on $\mathbb{R}^2$ and ρ is a convex semimodular then $\|\cdot\|_\varphi$ is equal to the classical Orlicz-Amemiya norm on X_ρ, which shows that the notion of $\|\cdot\|_\varphi$ is a natural generalization of two classical norms considered in semimodular spaces. Moreover, if φ is the l_p-norm on $\mathbb{R}^2$, $1 < p < \infty$ and ρ is a convex semimodular, then $\|\cdot\|_\varphi$ is equal to the p-Orlicz-Amemiya norm on X_ρ [22–24].

Now we state the following result.

THEOREM 4.7 Let ρ be a modular defined on $L_o(T)$, and let X_ρ be a modular space. Assume that ρ satisfies the requirements of Definition 2.3 (instead of $|\cdot|_F$). Then, $(X_\rho, |\cdot|_\varphi)$ is an admissible F-space for any F-norm $|\cdot|_\varphi$ given in Theorem 4.2 and Theorem 4.4.

PROOF Notice that if $\{w_n\} \subset W$, $\|w_n\| \to 0$, and $A \in \Sigma$, with $\mu(A) < \infty$, then for any $u > 0$, $\rho((w_n\chi_A)/u) \to 0$. By definition of $|\cdot|_\varphi$ this shows that $|w_n\chi_A|_\varphi \to 0$. Analogously, if $w \in W \setminus \{0\}$ and $\{A_n\} \subset \Sigma$, then for any $u > 0$ $\rho((w\chi_{A_n})/u) \to 0$ if and only if $\mu(A_n) \to 0$ and again by definition of $|\cdot|_\varphi$, the same condition is satisfied by $|\cdot|_\varphi$.
If $x \in X$, $\{x_n\} \subset X$, $x_n \to x$ μ-a.e. and $\|x_n(t)\| \leq \|x(t)\|$ for any $t \in T$, then for any $u > 0$, $\rho((x_n - x)/u) \to 0$, which shows that $|x_n - x|_\varphi \to 0$ by definition of $|\cdot|_\varphi$. This shows that the assumptions of Theorem 3.4 are satisfied by $(X_\rho, |\cdot|_\varphi)$. ∎

REMARK 4.8 Typical examples of modulars satisfying the requirements of Theorem 4.7 are vector-valued Musielak-Orlicz spaces determined by σ-finite measure spaces and modular function spaces [8]. It is clear that, in general, these spaces are not locally convex, (see e.g. [8], Theorem 3.4.1, p. 77). It is worth mentioning that Theorem 3.4 generalizes the main result [7] proved for Modular Function Spaces introduced by Kozłowski in [8], as well as the main result of [3] proved for spaces of vector-valued measurable functions.

REMARK 4.9 Let (T,Σ,μ) be a measure space, $\mu(T) < \infty$, and let W be a Banach space. Define on $L_o(T,\Sigma,\mu)$ the following F-norm

$$\|f\|_F = \int_T \frac{\|f(t)\|_W}{1+\|f(t)\|_W}\, d\mu(t).$$

Then, the space $(L_o(T,\Sigma,\mu), \|\cdot\|_F)$ is an admissible F-space.

5. Conclusions

The main results show that any Frechét space (defined in Definition 2.3) possesses the admissibility property, which entails the fixed point property in a modular space, despite the absence of a metric (which may be undefinable). However, the final conclusion devoted to the fixed point property is also presented as an application of the admissibility property in the special case for modular spaces equipped with an F-norm, which is metrizable.

Acknowledgement

The first author (Maciej Ciesielski) is supported by Poznan University of Technology, Poland, Grant No. 0213/SBAD/0122.

References

[1] Klee, V. (1960). Leray-Schauder theory without local convexity. *Math. Ann.*, *141*, 286-296.

[2] Nagumo, M. (1951). Degree of mapping in convex linear topological spaces. *Amer. J. Math.*, *73*, 497-511.

[3] Caponetti, D., Lewicki, G., Trombetta, A., & Trombetta, G. (2013). On the admissibility of the space $L_o(\mathscr{A},X)$ of vector-valued, measurable functions. *Bull. Korean Math. Soc.*, *50*, 6, 1915-1922.

[4] Riedrich, T. (1964). Die Räume $L^p(0,1)$ $(0 < p < 1)$ sind zulässig. *Bull. Korean Math. Soc.*, *13*, 1-6.

[5] Ishii, J. (1965). On the admissibility of function spaces. *J. Fac. Sci. Hokkaido Univ. Series I*, *19*, 49-55.

[6] Cauty, R. (1994). Un espace métrique linéaire qui n'est pas un rétracte absolu. *Fund. Math.*, *146*, 1, 85-99.

[7] Caponetti, D., & Lewicki, G. (2017). A note on admissibility of modular function spaces. *Journ. Mat. Anal. Appl.*, *448*, 1331-1342.

[8] Kozłowski, W.M. (1988). Modular function spaces, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 122, New York/Basel: Dekker.

[9] Khamsi, M.A., Kozłowski, W.M., & Reich, S. (1990). Fixed point theory in modular function spaces. *Nonlinear Anal.*, *14*, 11, 935-953.

[10] Khamsi, M.A. (1996). A convexity property in modular function spaces. *Math. Japonica*, *44*, 2, 269-279.

[11] Khamsi, M.A., & Kozłowski, W.M. (2010). On asymptotic pointwise contractions in modular function spaces. *Nonlinear Anal.*, *73*, 2957-2967.

[12] Khamsi, M.A., & Kozłowski, W.M. (2011). On asymptotic pointwise nonexpasive mappings in modular function spaces. *J. Math. Anal. Appl.*, *380*, 2, 697-708.

[13] Khamsi, M.A., & Kozłowski, W.M. (2015). Fixed Point Theory in Modular Function Spaces. Birkhäuser.

[14] Gao, H., & Zhang, B. (2006). Fixed points and controllability in delay systems. *Fixed Point Theory Appl.*, Art. ID 41480.

[15] Diestel, J., & Uhl, J.J. (1977). Vector measures. With a foreword by B.J. Pettis. Math. Surveys and Monographs, Vol. 15. Providence: American Mathematical Society.

[16] Albiac, F., & Kalton, N.J. (2016). Topics in Banach Space Theory. Second edition. Berlin: Springer.

[17] Chiţescu, I., Sfetcu, R.-C., & Cojocaru, O. (2019). Köthe-Bochner spaces: general properties. *Bull. Braz. Math. Soc. (N.S.)*, *50*, 2, 323-345.

[18] Bennett, C., & Sharpley, R. (1988). Interpolation of Operators. Pure and Applied Mathematics Series 129. Academic Press Inc.

[19] Dunkl, C.F., & Wiliams, K.S. (1964). A simple norm inequality. *Amer. Math. Monthly*, *71*, 53-54.

[20] Musielak, J. (1983). Orlicz spaces and modular spaces. Lecture Notes in Math. 1034, Berlin: Springer-Verlag.

[21] Ciesielski, M., & Lewicki, G. (2019). On a certain class of norms in semimodular spaces and their monotonicity properties. *J. Math. Anal. Appl.*, *475*, 1, 490-518.

[22] Cui, Y., Hudzik, H., & Wisła, M. (2015). Monotonicity properties and dominated best approximation problems in Orlicz spaces equipped with the p-Amemiya norm. *J. Math. Anal. Appl.*, *432*, 2, 1095-1105.

[23] Cui, Y., Hudzik, H., & Wisła, M. (2016). M-constants, Dominguez-Benavides coefficient, and weak fixed point property in Orlicz sequence spaces equipped with the p-Amemiya norm. *Fixed Point Theory Appl.*, Paper No. 89, 14 pp.

[24] Wisła, M. (2015). Geometric properties of Orlicz spaces equipped with p-Amemiya norms- -results and open questions. *Comment. Math.*, *55*, 2, 183-209.