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ON A ONE-DIMENSIONAL DIFFUSION MODEL WITH DISCONTINUOUS MOTION CHARACTERISTICS AND A MOVING, STICKY, AND SEMI-PERMEABLE MEMBRANE

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Abstract. This paper is devoted to the construction of a one-dimensional Feller process with continuous trajectories, which arises from solving the problem of pasting together two diffusion processes at a certain point on the real line. It is assumed that the location of this point depends on time, and that one version of the general Feller-Wentzell-type conjugation condition is prescribed there. We focus on the case in which the boundary condition involves only terms corresponding to boundary effects of the diffusion process, such as delay and partial reflection. To solve the problem, we apply analytical methods based on the classical potential theory. The resulting process can serve as a one-dimensional mathematical model of a physical diffusion phenomenon in a medium with a moving, sticky, and semi-permeable membrane.

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1. Introduction

The aim of this paper is to construct and investigate a two-parameter Feller semi--group corresponding to a specific inhomogeneous Markov process on the real line. This process arises from pasting together, at a certain point on the line, two ordinary diffusion processes defined in corresponding subdomains. It is assumed that the position of this point is determined by a given function, which, as well as the behavior of the process itself, depends on the time variable. At the point that serves as the common boundary of the subdomains, two conjugation conditions are imposed for the semigroup. One of these corresponds to the Feller condition for a Markov process. The other represents a particular case of the general Feller-Wentzell-type boundary condition for one-dimensional diffusion processes (see [1]). The case considered here involves only those terms of the prescribed condition that correspond to such boundary effects of the diffusion process as partial reflection and delay.

To solve the stated problem, analytical methods are employed. Under this approach, the question of the existence of the desired semigroup leads to a corresponding conjugation problem for a one-dimensional (in the spatial variable) backward parabolic Kolmogorov equation with discontinuous coefficients, where the Feller-Wentzell-type conjugation condition involves first-order derivatives in both variables. The classical solvability of the parabolic conjugation problem in the space of continuous functions, under certain assumptions on the input data, is established here by means of the boundary integral equations method, using fundamental solutions of uniformly parabolic operators and the associated potentials they generate (see, for example, [2-5]). The resulting Markov process, constructed in this manner, may serve as a one-dimensional mathematical model of diffusion in media with moving membranes (see [6, 7]).

It should be noted that a similar problem was previously considered in [8] for the case where the diffusion processes to be pasted together coincide with parts of the same Wiener process. In [9], the problem was studied (also in a more general formulation) under the assumption that the pasting point of the diffusion processes is a fixed point on the real line. As for the application of other approaches and methods for constructing one-dimensional models of diffusion in media with different types of membranes, they are partially reflected in [7, 10, 11] (see also the references therein). We also draw attention to potential practical applications of the results obtained. For example, [12] demonstrates how a particular case of the model developed here can be used to study problems in high-energy astrophysics, in particular for solving the so-called non-stationary kinetic equation that describes the acceleration of charged particles in the vicinity of strong shock waves.

2. Formulation of the conjugation problem for the backward Kolmogorov equation with discontinuous coefficients

Let Q denote the domain in the space $\mathbb{R}^2$ of points (s,x), bounded by the lines $s=0$ and $s=T$, i.e.,

$$Q=\{(s,x): \quad 0<s<T, -\infty<x<\infty\}.$$

We consider two subdomains $Q_t^{(i)} \subset Q$, $i=1,2$, defined by

$$Q_t^{(i)}=\{(s,x): \quad 0\le s<t\le T, x\in D_s^{(i)}\},$$

where $D_s^{(1)}=(-\infty,g(s))$, $D_s^{(2)}=(g(s),\infty)$, $s\in[0,T]$, and $g(s)$ is a given continuous function. We denote $Q_t=Q_t^{(1)}\cup Q_t^{(2)}$, and $D_s=D_s^{(1)}\cup D_s^{(2)}$. Let $\overline{G}$ be the closure of G, and $C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$ with the norm

$$\|\varphi\| = \sup_{x\in\mathbb{R}} |\varphi(x)|.$$

We use the Hölder spaces $H^{\frac{k+\alpha}{2},k+\alpha}(\overline{Q})$ and $H^{k+\frac{\alpha}{2}}([0,T])$, $\alpha \in (0,1)$, $k = 0,1$, for functions defined on $\overline{Q}$ and $[0,T]$, respectively, as introduced in [3, Ch. I, §1].

In the strip Q, we consider two second-order parabolic operators with bounded and continuous coefficients:

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s,x) \frac{\partial^2}{\partial x^2} + a_i(s,x) \frac{\partial}{\partial x}, \quad i = 1,2.$$

We consider the problem of finding a function

$$u(s,x,t) = u_i(s,x,t), \quad (s,x) \in \overline{Q}_t^{(i)}, i = 1,2,$$

such that in each domain $Q_t^{(i)}$, $i = 1,2$, the function u_i satisfies the equation

$$\frac{\partial u_i}{\partial s} + L_s^{(i)} u_i = 0, \quad (s,x) \in Q_t^{(i)}, i = 1,2, \tag{1}$$

and the following initial condition holds at $s = t$:

$$u_i\big|_{s=t} = \varphi(x), \quad x \in \overline{D}_t^{(i)}, i = 1,2. \tag{2}$$

On the boundary $x = g(s)$, the functions u_1 and u_2 are required to satisfy the following conjugation conditions:

$$B_1 u \equiv u_1(s,g(s),t) - u_2(s,g(s),t) = 0, \quad 0 \le s \le t \le T \tag{3}$$

$$\begin{aligned} B_2 u \equiv \sigma(s) \frac{\partial u}{\partial s}(s,g(s),t) - q_1(s) \frac{\partial u_1}{\partial x}(s,g(s),t) \\ + q_2(s) \frac{\partial u_2}{\partial x}(s,g(s),t) = 0, \quad 0 \le s < t \le T. \end{aligned} \tag{4}$$

Here, φ, σ, q_1 and q_2 are given continuous functions. In addition, we assume that $\varphi \in C_b(\mathbb{R})$, and the coefficients of operator B_2 satisfy the conditions:

$$\sigma(s) > 0, \quad q_i(s) \ge 0, \quad i = 1,2, s \in [0,T]. \tag{5}$$

The derivative $\dfrac{\partial u}{\partial s}(s,g(s),t)$ in (4) is understood as the limit of the function

$$\frac{\partial u}{\partial s}(s,x,t) = \frac{\partial u_i}{\partial s}(s,x,t), \, (s,x) \in Q_t^{(i)}, \, i = 1,2,$$

as $(s,x) \to (s,g(s))$ from within the domain $Q_t^{(i)}$. It is assumed that

$$\frac{\partial u_1}{\partial s}(s,g(s),t) = \frac{\partial u_2}{\partial s}(s,g(s),t).$$

The derivatives $\partial u_i/\partial x(s,g(s),t)$, $i=1,2$, in (4) are interpreted analogously, but without assuming their continuity at $x=g(s)$. We also make use of the derivative

$$\frac{\partial}{\partial s}[u(s,g(s),t)] = \frac{\partial}{\partial s}[u_i(s,g(s),t)], \quad i=1,2,$$

which, under the assumption that the function $g(s)$ is differentiable, is related to

$$\frac{\partial u}{\partial s}(s,g(s),t) = \frac{\partial u_i}{\partial s}(s,g(s),t),\, i=1,2,$$

via the formula

$$\frac{\partial}{\partial s}[u_i(s,g(s),t)] = \frac{\partial u_i}{\partial s}(s,g(s),t) + \frac{\partial u_i}{\partial x}(s,x,t)\bigg|_{x=g(s)} \cdot g'(s), \quad i=1,2. \tag{6}$$

Recall (see [1,6,7]) that the conjugation problem (1)-(4) represents, in terms of semigroups, the so-called problem of pasting together two given diffusion processes on the real line, with a boundary condition additionally prescribed at the point of pasting together these processes, which is a particular case of the general Feller--Wentzell-type boundary condition for one-dimensional diffusion processes. As is well known, this condition describes all possible extensions of a diffusion process after it reaches the boundary of the domain. If we assume that, based on the solution $u(s,x,t)$ of the problem (1)-(4), one can define a two-parameter semigroup of operators T_{st}, $0 \le s \le t \le T$, acting on the space $C_b(\mathbb{R})$ and yielding a certain Markov process on $\mathbb{R}$ (see Theorem 2), then the fact that the function $u(s,x,t) = T_{st}\varphi(x)$ satisfies equation (1) implies that the parts of this process inside the domains $D_s^{(i)}$ coincide with the diffusion processes defined there by generating differential operators $L_s^{(i)}$, $i=1,2$. Moreover, the initial condition (2) corresponds to the equality $T_{ss}=I$, where I denotes the identity operator. Next, the conjugation condition (3) reflects the Feller property of the resulting Markov process, while the relation (4) represents the aforementioned version of the general Feller-Wentzell-type conjugation condition. In our setting, this condition involves only terms that characterize specific properties of the diffusion processes at the point where they are joined – namely, delay (represented in (4) by the derivative of u with respect to s, multiplied by the coefficient σ) and partial reflection (represented by the one-sided derivatives of u with respect to x, with coefficients q_1 and q_2).

We further assume that the following conditions hold for the coefficients of the operators $L_s^{(i)}$, $i=1,2$, and B_2, as well as for the function g:

I. There exist constants b and B such that for all $(s,x)\in\overline{Q}$,

$$b \le b_i(s,x) \le B, \quad i=1,2.$$

II. The functions $a_i(s,x)$ and $b_i(s,x)$, $i=1,2$, belong to the Hölder space $H^{\frac{\alpha}{2},\alpha}(\overline{Q})$ and $H^{\frac{1+\alpha}{2},1+\alpha}(\overline{Q})$, respectively.

III. The functions $\sigma(s)$ and $q_i(s)$, $i = 1,2$, belong to the Hölder space $H^{\frac{\alpha}{2}}([0,T])$ and satisfy conditions (5).

IV. The function $g(s)$ belongs to the Hölder class $H^{1+\frac{\alpha}{2}}([0,T])$.

Let $G_i(s,x,t,y)$, $0 \le s < t \le T$, $x,y \in \mathbb{R}$, $i = 1,2$, be the fundamental solution of the operator $\partial/\partial s + L_s^{(i)}$ (see [3, Ch. IV, §11], [7, Eq. (1.8)]):

$$G_i(s,x,t,y) = Z_{i0}(s,x,t,y) + Z_{i1}(s,x,t,y), \quad i = 1,2,$$

where

$$Z_{i0}(s,x,t,y) = [2\pi b_i(t,y)(t-s)]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(y-x)^2}{2b_i(t,y)(t-s)}\right\},$$

and Z_{i1} is an integral term that admits a weaker singularity than Z_{i0} as $s \to t$. The functions G_i and Z_{i1} satisfy the following estimates for $0 \le s < t \le T$, $x,y \in \mathbb{R}$, $i = 1,2$:

$$|D_s^r D_x^p G_i(s,x,t,y)| \le C(t-s)^{-\frac{1+2r+p}{2}} \cdot \exp\left\{-c\frac{(y-x)^2}{t-s}\right\}, \tag{7}$$
$$|D_s^r D_x^p Z_{i1}(s,x,t,y)| \le C(t-s)^{-\frac{1+2r+p-\alpha}{2}} \cdot \exp\left\{-c\frac{(y-x)^2}{t-s}\right\},$$

where C and c are positive constants, and r, p are non-negative integers such that $2r + p \le 2$. Here, D_s^r and D_x^p denote partial derivatives of order r and p with respect to s and x, respectively.

Using the fundamental solution $G_i(s,x,t,y)$, $i = 1,2$, along with given functions $\varphi(x)$, $x \in \mathbb{R}$, and $g(s)$, $s \in [0,T]$, and a density function $V_i(s,t)$, $0 \le s < t \le T$, $i = 1,2$, we define the following integrals:

$$u_{i0}(s,x,t) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy, \quad i = 1,2, \tag{8}$$

$$u_{i1}(s,x,t) = \int_s^t G_i(s,x,\tau,g(\tau))V_i(\tau,t)d\tau, \quad i = 1,2. \tag{9}$$

The function u_{i0} is referred to as the Poisson potential with kernel G_i and density φ, while u_{i1} is called the parabolic simple-layer potential with kernel G_i and density V_i. We now recall several known properties of these functions, assuming that $\varphi \in C_b(\mathbb{R})$, $g \in H^{1+\frac{\alpha}{2}}([0,T])$, and the density V_i is continuous in $\tau \in [0,t)$ and may admit a weak singularity as $\tau \to t$, with exponent $\mu \ge -1/2$, as follows (see [2, Ch. XXII, §8], [3, Ch. IV, §14, 15], [5, Sec. 2], [6, Ch. II, §3]):

1) The potential u_{i0}, as a function of (s,x), is a solution of the Cauchy problem for equation (1) in the domain $(s,x) \in [0,t) \times \mathbb{R}$, with the initial condition (2)

for $x \in \mathbb{R}$. Moreover, it satisfies the estimate (for $0 \le s < t \le T, x \in \mathbb{R}$, and $2r + p \le 2$)

$$|D_s^r D_x^p u_{i0}(s,x,t)| \le C\|\varphi\|(t-s)^{-\frac{2r+p}{2}}.$$

2) The potential u_{i1}, as a function of (s,x), is bounded and continuous for $0 \le s < t \le T, x \in \mathbb{R}$. It satisfies equation (1) in the domain $(s,x) \in [0,t) \times (\mathbb{R} \setminus g(s))$ and vanishes at $s = t, x \in \mathbb{R}$.

3) The derivative of the function u_{i1} with respect to x (the so-called conormal derivative) satisfies the jump relation:

$$\lim_{x \to g(s) \pm 0} \frac{\partial u_{i1}}{\partial x}(s,x,t) = \mp \frac{V_i(s,t)}{b_i(s,g(s))} + \int_s^t \frac{\partial G_i}{\partial x}(s,g(s),\tau,g(\tau))V_i(\tau,t)d\tau. \quad (10)$$

The existence of the integral on the right-hand side of (10) follows from the inequality

$$\left| \frac{\partial G_i}{\partial x}(s,g(s),\tau,g(\tau)) \right| \le C(\tau - s)^{-1+\frac{\alpha}{2}}, \quad i = 1,2.$$

3. Solution of the parabolic conjugation problem

In this section, we prove the following existence and uniqueness theorem:

Theorem 1 *Suppose that assumptions I-IV hold and that $\varphi \in C_b(\mathbb{R})$. Then there exists a unique solution of the problem* (1)-(4)*, continuous in $\overline{Q}_t$, for which the following estimate holds:*

$$|u(s,x,t)| \le C\|\varphi\|, \quad (s,x) \in \overline{Q}_t, \quad (11)$$

where C is a positive constant. □

PROOF We first prove the existence of a classical solution $u(s,x,t) = u_i(s,x,t)$, $(s,x) \in \overline{Q}_t^{(i)}$, $i = 1,2$, of the problem (1)-(4). The functions $u_i(s,x,t)$ are sought in the form

$$u_i(s,x,t) = u_{i0}(s,x,t) + u_{i1}(s,x,t), \quad (s,x) \in \overline{Q}_t^{(i)}, i = 1,2, \quad (12)$$

where u_{i0} and u_{i1} are defined by formulas (8) and (9), respectively. Here, φ is the function from the initial condition (2), and V_i, $i = 1,2$, are unknown functions to be determined. Assume that the functions V_i, $i = 1,2$, satisfy the conditions under which properties 2) and 3) hold for u_{i1}, $i = 1,2$. Together with property 1), this implies that to determine V_i, $i = 1,2$, it remains only to apply the conjugation conditions (3) and (4), which any solution of the problem (1)-(4) must satisfy. To this end, we introduce the function $\upsilon(s,t) = u(s,g(s),t) = u_i(s,g(s),t)$, $i = 1,2$. Using the conjugation condition (4), together with formulas (6), (8), (9) and (10), we obtain the following relation for $\frac{\partial \upsilon}{\partial s}$:

$$\frac{\partial \upsilon}{\partial s} = \Phi_0(s,t), \quad s \in [0,t), \tag{13}$$

where

$$\Phi_0(s,t) = \sum_{j=1}^{2} \gamma_j(s) \frac{\partial u_{j0}}{\partial x}(s,g(s),t) + \sum_{j=1}^{2} \frac{q_j(s)}{\sigma(s) b_j(s,g(s))} V_j(s,t)$$
$$+ \sum_{j=1}^{2} \gamma_j(s) \int_s^t \frac{\partial G_j}{\partial x}(s,g(s),\tau,g(\tau)) V_j(\tau,t) d\tau,$$
$$\gamma_j(s) = \frac{1}{2} g'(s) + (-1)^{j-1} \frac{q_j(s)}{\sigma(s)}, \quad j = 1,2.$$

We treat equation (13) as an autonomous ordinary differential equation for the function $\upsilon(s,t)$, which satisfies the initial condition

$$\lim_{s \uparrow t} \upsilon(s,t) = \varphi(g(t))$$

Solving equation (13) with the initial condition above, we obtain

$$\upsilon(s,t) = \varphi(g(t)) - \int_s^t \Phi_0(\lambda,t) d\lambda. \tag{14}$$

Thus, we obtain three different expressions for the function $\upsilon(s,t) = u(s,g(s),t)$: the representation (14), and two expressions from (12), in which one must substitute $x = g(s)$ and take into account condition (3). Equating the right-hand sides of the expressions for $\upsilon(s,t)$ and $u_1(s,g(s),t)$, and then for $\upsilon(s,t)$ and $u_2(s,g(s),t)$, we obtain the following system of integral equations for the unknown functions V_i, $i = 1,2$:

$$\int_s^t G_i(s,g(s),\tau,g(\tau)) V_i(\tau,t) d\tau + \sum_{j=1}^{2} \int_s^t K_j(s,\tau) V_j(\tau,t) d\tau = \Phi_i(s,t), \tag{15}$$

where $0 \le s < t \le T$, $i = 1,2$, and

$$K_j(s,\tau) = \int_s^\tau \gamma_j(\lambda) \frac{\partial G_j}{\partial x}(\lambda,g(\lambda),\tau,g(\tau)) d\lambda + \frac{q_j(\tau)}{\sigma(\tau) b_j(\tau,g(\tau))}, \quad j = 1,2,$$

$$\Phi_i(s,\tau) = \varphi(g(t)) - u_{i0}(s,g(s),t) - \sum_{j=1}^{2} \int_s^t \gamma_j(\lambda) \frac{\partial u_{j0}}{\partial x}(\lambda,g(\lambda),t) d\lambda, \quad i = 1,2.$$

The system (15) consists of two Volterra integral equations of the first kind. We now transform it into a system of Volterra integral equations of the second kind. To this end, we introduce the integro-differential operator $\mathscr{E}$, defined by

$$\mathscr{E}(s,t)f = \sqrt{\frac{2}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho - s)^{-\frac{1}{2}} f(\rho,t)d\rho, \quad 0 \le s < t \le T.$$

Applying the operator $\mathscr{E}$ to both sides of each equation in (15), we find that this system of equations for V_i, $i = 1,2$, can be rewritten as the following system of Volterra integral equations of the second kind:

$$V_i(s,t) = \Psi_i(s,t) + \sum_{j=1}^{2}\int_s^t K_{ij}(s,\tau)V_j(\tau,t)d\tau, \quad i = 1,2, \tag{16}$$

where

$$\Psi_i(s,t) = -\sqrt{b_i(s,g(s))}\cdot\mathscr{E}(s,t)\Phi_i, \quad i = 1,2,$$

$$\mathscr{E}(s,t)\Phi_i = \frac{1}{\sqrt{2\pi}}\int_s^t (\rho - s)^{-\frac{3}{2}}[\Phi_i(\rho,t) - \Phi_i(s,t)]d\rho$$

$$- \sqrt{\frac{2}{\pi}}(t-s)^{-\frac{1}{2}}\Phi_i(s,t), \quad i = 1,2,$$

$$K_{ii}(s,\tau) = \sqrt{\frac{2b_i(s,g(s))}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho - s)^{-\frac{1}{2}}\Big(\big[Z_{i0}(\rho,g(\rho),\tau,g(\tau))$$

$$- Z_{i0}(\rho,g(\tau),\tau,g(\tau))\big] + Z_{i1}(\rho,g(\rho),\tau,g(\tau)) + K_i(\rho,\tau)\Big)d\rho, \quad i = 1,2,$$

$$K_{ij}(s,\tau) = \sqrt{\frac{2b_i(s,g(s))}{\pi}}\frac{\partial}{\partial s}\int_s^t (\rho - s)^{-\frac{1}{2}}K_j(\rho,\tau)d\rho, \quad i = 1,2,\ j = 1,2,\ i \ne j.$$

In addition, the following estimates hold for the functions Ψ_i and the kernels K_{ij}, $i,j = 1,2$,

$$|\Psi_i(s,t)| \le C\|\varphi\|(t-s)^{-\frac{1}{2}}, \quad 0 \le s < t \le T, i = 1,2, \tag{17}$$

$$|K_{ij}(s,\tau)| \le C(\tau - s)^{-1+\frac{\alpha}{2}}, \quad 0 \le s < t \le T, i = 1,2,\ j = 1,2. \tag{18}$$

Estimates (17) and (18) can be established using the same method as that used to derive inequalities (2.8) and (2.10) in [7].

The inequalities (17) and (18) ensure that the ordinary method of successive approximations can be applied to the system of integral equations (16) (see, e.g., [3, Ch. IV, §11]). As a result, we conclude that the system of equations (16) has a unique solution in the class of continuous functions for $0 \le s < t \le T$, which admits the representation

$$V_i(s,t) = \Psi_i(s,t) + \sum_{i,j=1}^{2} \int_s^t R_{ij}(s,\tau)\Psi_j(\tau,t)d\tau, \quad i = 1,2, \tag{19}$$

where the resolvent $R_{ij}(s,\tau)$ has a weak singularity of the form (18), and the same estimate (17) holds for the functions V_i as for Ψ_i, $i = 1,2$.

From estimates (7) (with $r = p = 0$) and (17) for V_i, $i = 1,2$, it follows that the integrals in the representation (12) exist, and that the functions u_i satisfy equation (1), the initial condition (2), and the inequality (11). We also verify that the derivative of the resulting function u with respect to s remains continuous across the boundary $x = g(s)$. This completes the proof of the existence of a classical solution of the problem (1)-(4). As for the uniqueness of this solution in the class of continuous functions, it follows from the maximum principle for parabolic equations in the case of the first boundary value problem (Cf. the proof of Theorem 2.2 in [7]). Theorem 1 is proved. ■

4. Construction of a diffusion process with discontinuous local characteristics and a moving, sticky, and semi-permeable membrane

Using the solution of the parabolic conjugation problem (1)-(4), we define a two--parameter family of linear operators $T_{st}, 0 \le s \le t \le T$, acting on $C_b(\mathbb{R})$ by the formula

$$T_{st}\varphi(x) = T_{st}^{(i0)}\varphi(x) + T_{st}^{(i1)}\varphi(x), \quad 0 \le s < t \le T, x \in \overline{D}_{is}, i = 1,2, \tag{20}$$

where

$$T_{st}^{(i0)}\varphi(x) = \int_{\mathbb{R}} G_i(s,x,t,y)\varphi(y)dy,$$

$$T_{st}^{(i1)}\varphi(x) = \int_s^t G_i(s,x,\tau,g(\tau))V_i(\tau,t,\varphi)d\tau,$$

and the densities V_i, $i = 1,2$, represent the solution of the form (19) of the system of equations (16), to which the original problem (1)-(4) is reduced. Moreover, $T_{tt} = I$, where I is the identity operator, and the estimate

$$|T_{st}\varphi(x)| \le C\|\varphi\|$$

holds for all $0 \le s \le t \le T, x \in \mathbb{R}$.

The availability of the integral representation for the family of operators $T_{st}, 0 \le s \le t \le T$, acting on the space $C_b(\mathbb{R})$, allows one to readily verify that they possess the following properties:

a) If $\varphi_n \in C_b(\mathbb{R})$, $n = 1, 2, \ldots$, $\sup_n \|\varphi_n\| < \infty$ and $\lim_{n\to\infty} \varphi_n(x) = \varphi(x)$ for all $x \in \mathbb{R}$, where $\varphi \in C_b(\mathbb{R})$, then $\lim_{n\to\infty} T_{st}\varphi_n(x) = T_{st}\varphi(x)$ for all $(s,x) \in \overline{Q}_t$.

b) $T_{st}\varphi(x) \geq 0$ for all $(s,x) \in \overline{Q}_t$ if $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$.

c) The operators T_{st} are contractive, i.e.,

$$\|T_{st}\varphi\| \leq \|\varphi\|$$

for any $\varphi \in C_b(\mathbb{R})$.

d) For all $0 \leq s \leq \tau \leq t \leq T$,

$$T_{st} = T_{s\tau} T_{\tau t}.$$

The proofs of properties a) – d), with obvious modifications, follow along the same lines as the corresponding ones for the Markov process constructed in [7, Sec. 3].

From properties a) – d), it follows that the operators T_{st} possess the semigroup property and generate a certain inhomogeneous Markov process on $\mathbb{R}$ (see, e.g., [6, Ch. I], [13, Ch. 2, §1]). We denote its transition probability by $P(s,x,t,dy)$, so that

$$T_{st}\varphi(x) = \int_{\mathbb{R}} \varphi(y) P(s,x,t,dy), \quad 0 \leq s < t \leq T, x \in \mathbb{R}, \varphi \in C_b(\mathbb{R}).$$

We also emphasize an additional property of the constructed Markov process, based on the integral representation of $T_{st}\varphi(x)$. Through direct computation, we show that the transition probability $P(s,x,t,dy)$ satisfies the following conditions:

i) $\sup_{x\in\mathbb{R}} \int_{\mathbb{R}} |y-x|^4 P(s,x,t,dy) \leq C(t-s)^2, \quad 0 \leq s < t \leq T,$

ii) $\lim_{\Delta s \downarrow 0} \frac{1}{\Delta s} \int_{\mathbb{R}} (y-x) P(s,x,s+\Delta s,dy) = \begin{cases} a_i(s,x), & s \in [0,T], x \in D_s^{(i)}, i = 1,2, \\ \dfrac{q_2(s) - q_1(s)}{\sigma(s)}, & s \in [0,T], x = g(s), \end{cases}$

iii) $\lim_{\Delta s \downarrow 0} \frac{1}{\Delta s} \int_{\mathbb{R}} (y-x)^2 P(s,x,s+\Delta s,dy) = \begin{cases} b_i(s,x), & s \in [0,T], x \in D_s^{(i)}, i = 1,2, \\ 0, & s \in [0,T], x = g(s). \end{cases}$

Condition i) implies that the trajectories of the constructed Markov process are continuous (see [4, Ch. V, §4], [6, Ch. I, §2]), while the existence of the limits in ii) and iii) indicates that this process can be interpreted as an ordinary diffusion process, albeit with discontinuous diffusion characteristics: the drift coefficient $a(s,x)$ and the diffusion coefficient $b(s,x)$, given by the right-hand sides of formulas in ii) and iii), respectively.

Thus, we have proved the following theorem:

Theorem 2 *Let the assumptions of Theorem 1 hold. Then the two-parameter Feller semigroup* T_{st}, $0 \leq s \leq t \leq T$, *defined by formulas* (19) *and* (20)*, describes an inhomogeneous Markov process on* $\mathbb{R}$ *whose transition probability* $P(s,x,t,dy)$ *satisfies conditions i) – iii).* □

Finally, we note that the continuous Markov process constructed in Theorems 1 and 2 can serve as a one-dimensional mathematical model of a physical diffusion phenomenon with a membrane located at the point of pasting together the original diffusion processes. Since the membrane's position on the real line is variable, and the constructed process possesses the properties of delay and partial reflection at the membrane, we refer to it as a moving, sticky, and semi-permeable membrane.

5. Conclusions

This paper presents the application of heat potential theory methods to the construction of a one-dimensional mathematical model of a physical diffusion phenomenon in a medium with a moving, sticky, and semi-permeable membrane. The model is derived by analytically solving the problem of pasting together two diffusion processes on the real line, with a version of the general Feller-Wentzell-type conjugation condition prescribed at their pasting point. It is assumed that the location of this point depends on time, and that the conjugation condition is local and involves only terms corresponding to such extensions of the diffusion process at the boundary point as delay and partial reflection.

The main steps in solving the problem under consideration are as follows:

- establishing, by the boundary integral equations method, the classical solvability of the conjugation problem for the backward Kolmogorov equation with discontinuous coefficients, to which the original problem is reduced;
- proving that the two-parameter Feller semigroup defined via the solution of this problem describes a certain Markov process on the real line;
- establishing several additional properties of the resulting process.

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