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# MODELLING OF SHOCK PHENOMENA FOR PREDICTING SHIPPING CONTAINER DAMAGES IN TRANSPORT

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**Abstract.** The paper presents the modelling of shock phenomena for predicting potential damage to shipping containers and transported cargo. The container design consists of an outer and inner frame connected by wire rope shock isolators. Drop test simulations were conducted using SolidWorks Motion and Matlab. In SolidWorks, a full 3D rigid-body model has been tested, whereas in Matlab a simplified two-degree-of-freedom model has been considered. The displacements and accelerations of the model elements have been analysed based on flat drop tests. The presented approach is the first step to eliminate the physical drop tests, which, while effective, are costly and resource intensive.

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**Keywords:** shipping container, drop test, numerical modelling, shock isolation

#### 1. Introduction

Living in a fast-changing world that is open to global markets requires safe, efficient, and reliable transportation of highly advanced and sensitive technical devices – such as medical instruments, aircraft engine modules, microscopes, and many more [1, 2]. These devices are typically designed and manufactured in highly developed countries and must be delivered to all corners of the world using available means of transport. Numerous studies confirm that during parcel and container handling, significant shock loads can occur, even in standard delivery systems, which may lead to severe damage to the contents if not properly protected [3].

To meet the transportation requirements of sensitive goods, specially designed shipping containers equipped with shock isolation systems are utilised [4]. Although transport cycles have been shortened in recent times, delivering advanced and sensitive cargo to the final customer still takes several days. During transit, multiple transshipments may occur, exposing the containers to unpredictable accidental shock loads. Before the new design of shipping container is permitted for international

transport it shall be validated and qualified by a series of tests including impact and drop test to confirm their ability for safe and reliable shipment [5-7]. Drop tests can be performed locally or outsourced to companies specialised in testing shipping structures, however, both options are expensive and require significant effort to manufacture the object to be tested, set up the test bench and postprocess data [8-11]. In contrast to this approach, the following research presents methods for modeling shock phenomena to predict damage to shipping containers under harsh transport conditions [12].

The aim of this work is to develop and compare two numerical models to predict the dynamic response of shock-isolated shipping containers during free-fall tests onto a flat surface. Furthermore, a secondary objective is to assess the effectiveness of the isolation system in limiting payload overloads and cargo displacement. The feasibility of conducting virtual testing is demonstrated by implementing a numerical method governing the equations of motion and commercial SolidWorks Motion software to investigate the G loads transferred to the transported object and its displacements. The obtained results and the adopted approach can serve as a basis for developing a framework suitable for implementation in private sector companies, which could effectively replace traditional physical testing methods, thus reducing the overall costs associated with the commercialization of such products [13-15].

# 2. Container design and drop test configuration

In order to reproduce the behavior of the container during a fall, a simplified geometric model was developed. The geometry of the proposed transport container is shown in Figure 1. It consists of outer and inner frames made of S235JR steel, connected by wire rope shock isolators. Both frames are cube-shaped, with external dimensions of 0.7 m and 0.52 m, respectively. The outer frame weighs 60 kg, and the inner frame weighs 25 kg. A 20 kg adjustable structure is mounted at the midheight of the inner frame to support the load. For the tests, a payload mass of 35 kg was assumed.

International transportation industry standards [16-21] provide framework guidelines for performing impact or drop tests for many configurations. In this study, one such case was analyzed, namely the drop test, which involves dropping a container from a height of 0.2 m and examining the system response. The impact velocity in this case reaches 1.98 m/s. Theoretically, the probability of local plastic deformation is negligible due to the distribution of the impact force across a broad surface area. To ensure this outcome, the container base must be precisely aligned parallel to a flat surface, and its structural design and centre of gravity must facilitate uniform contact during impact. Under optimal conditions, rotational dynamics are unlikely, and linear displacement of the internal frame reduces the probability of interaction with the outer frame, thereby mitigating additional stress on the contents.

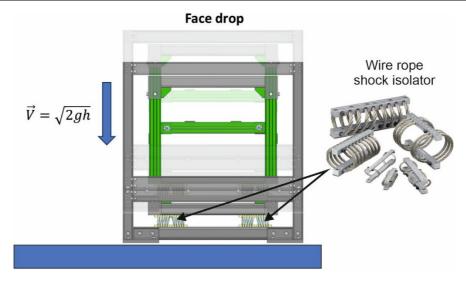


Fig. 1. Schematic representation of the shipping container and face drop test scenario

Therefore, particular attention was paid in the conducted research to the resulting accelerations and the potential closure of shock isolators, which could cause a secondary shock wave and rebound effect. For this purpose, numerical models were developed in SolidWorks Motion and Matlab, which allowed for comparison and verification of the proposed solutions.

## 3. Numerical modelling methods

Two numerical models were developed to analyze the behavior of the container during a fall and to evaluate the effectiveness of the shock isolation system. The first one was implemented based on a 3D model in the SolidWorks Motion environment and takes into account the rigid-body dynamics of the entire structure. The second model was a simplified model with 2-DOF that was implemented in Matlab. The use of two independent simulation tools allowed for the verification of the obtained results and the identification of key dynamic phenomena affecting the safety of the transported cargo.

### 3.1. 3D simulation using SolidWorks Motion

The geometric model developed in SolidWorks is shown in Figure 2. The load was represented by a solid element attached to the internal frame at a central point. The Motion solver takes into account material properties, mass, and inertia, but assumes that all solid elements in the model are rigid bodies. Therefore, the shock isolator was replaced with a linear spring and damper, provided by the simulation environment. It should be noted that these elements do not take into account lateral

stiffness (shear/roll), but this does not affect the performance of the face drop scenario. According to the catalog data for the ENIDINE WR5-800 shock isolator, the stiffness of the linear spring (compression) was set at 36800 N/m, and the damping constant at 5520 kg/s (15 % of the stiffness).

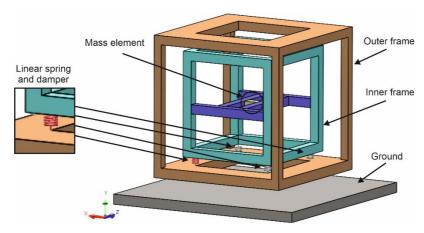


Fig. 2. 3D model of the container prepared in SolidWorks

As the structure falls under the influence of gravity, it impacts the ground. This requires determining the contact properties. SolidWorks can automatically perform this by defining the frame material and the ground onto which it falls. This allows us to determine the stiffness of the interaction between the two parts in collision (k), the maximum damping coefficient  $(c_{\text{max}})$ , and the penetration depth of one geometry into the other  $(p^e)$ . Impact force  $(F_n)$  is defined as:

$$F_n = k p^e + step(p, 0, 0, d_{\text{max}}, c_{\text{max}}) \frac{dp}{dt},$$
 (1)

where  $d_{\max}$  is the positive value of the penetration limit at which the solver applies the maximum attenuation coefficient  $c_{\max}$ , and  $\frac{dp}{dt}$  is the penetration volume at the contact point. The step function is a piecewise linear function built into SolidWorks Motion that allows one to determine the effective contact damping coefficient as a function of penetration according to the formula:

$$step(p, 0, 0, d_{max}, c_{max}) = \begin{cases} 0, & p \le 0, \\ c_{max} \frac{p}{d_{max}}, & 0 (2)$$

In SolidWorks Motion, the rigid integration method GSTIFF, developed by Gear [16], was chosen for the simulation. This method uses a variable-order backward difference formula with a variable step size. If the step size changes suddenly during

integration, the method will introduce a small error. Sudden changes in step size occur when there are discontinuous forces, discontinuous motions, or abrupt events such as contacts in the model [17].

The general form of the Backward Differentiation Formula (BDF) on which the GSTIFF solver is based is described by the equation [18]:

$$\sum_{j=0}^{r} \alpha_j \gamma_{p+j} = h \beta f(\tau_{p+r}, \gamma_{p+r}), \tag{3}$$

where r is the order of the method, p is the time step index,  $\gamma$  is the state variable,  $\alpha_j$  are the BDF coefficients depending on the order r,  $\gamma_{p+j}$  are the approximated values of the solution at previous time steps, h is the integration time step,  $\beta$  is the weighting coefficient for the right-hand side of the equation,  $\tau_{p+r}$  is the time corresponding to the last point in the approximation and  $\gamma_{p+r}$  is the estimated value of the solution at time  $\tau_{p+r}$ .

### 3.2. 2-DOF lumped-parameter model implemented in Matlab

In the second case, a 2-DOF system was formulated (Fig. 3), in which the outer and inner frames were replaced by masses  $m_1$  and  $m_2$ , respectively. Masses  $m_1$  and  $m_2$  move relative to the ground according to coordinates  $x_1(t)$  and  $x_2(t)$ . Mass  $m_2$  is connected to mass  $m_1$  via spring  $k_1$  and damper  $c_1$ , which represent a shock isolator. To ensure the model is consistent with the SolidWorks model, the stiffness of this spring and the damping coefficient were assumed to be the sums of four such quantities. The contact point of mass  $m_1$  with the ground is replaced by spring k and damper c. During the calculations, the values of these parameters were defined depending on the type of ground onto which the container falls.

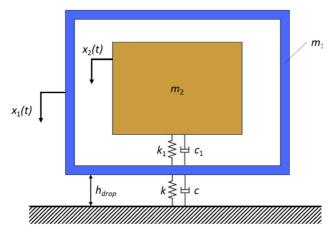


Fig. 3. 2-DOF representation of the container and shock isolation system

The equations of motion taking into account the impact for such a system take the form:

$$m_1 \ddot{x}_1 = -m_1 g + F_S + F_{cont} \,, \tag{4}$$

$$m_2\ddot{x}_2 = -m_2g - F_S \,, \tag{5}$$

where:

$$F_s = k_1(x_2 - x_1 - l_0) + c_1(\dot{x}_2 - \dot{x}_1), \qquad (6)$$

$$F_{cont} = \begin{cases} -kx_1 - c\dot{x}_1, & x_1 \le 0\\ 0, & x_1 > 0 \end{cases}$$
 (7)

and  $l_0$  is the initial spring height of 0.046 m.

The solution to the problem was obtained based on the development of appropriate scripts and functions in Matlab and the use of the ready-made function *ode45*, which applies the fourth-order Runge-Kutta method to solve the initial problem.

When analyzing the single-impact phenomenon (contact without multiple reflections) assuming a linear Kelvin-Voigt model for both the insulators and the substrate contact, the equations of motion of the two-degree-of-freedom system can be reduced to a system of linear differential equations with constant coefficients. The resulting displacement and velocity response will be a combination of exponential functions (or damped oscillations). The initial conditions for this phenomenon are defined at the moment of first contact, while the loss of contact condition corresponds to the situation when the contact deformation reaches zero again. Due to the model comparison and the possibility of extension to multiple contact events or different substrates, only the numerical approach (ode45) was used in this work without an analytical solution for a single contact phase.

The key assumptions of the 2-DOF lumped-parameter model are presented below:

- both the outer and inner frames are treated as rigid bodies with masses  $m_1$  and  $m_2$ .
- the impact isolators are modeled using the Kelvin-Voigt model with stiffness k and damping c,
- the ground contact is modeled as a unilateral Kelvin-Voigt element, with stiffness  $k_g$  and damping  $c_g$ , active only for positive penetration (p > 0); these parameters are selected depending on the ground material and correspond to the configuration adopted in the 3D model,
- motion is analyzed only in the vertical axis, ignoring sliding friction and small frame rotations,
- the initial conditions result from the free fall height, taking into account the effects of gravity,
- the model ignores material and geometric nonlinearity and the absence of hysteresis.
- numerical integration is performed using the *ode45* function based on the fourth-order Runge-Kutta method.

## 4. Simulation results and model comparison

The face-drop scenario was conducted for three different cases, in which a steel structure falls onto a steel, aluminum, or rubber substrate, respectively. This required setting appropriate contact stiffness values (spring stiffness and damping coefficient). In SolidWorks, this operation was accomplished by selecting appropriate materials, while in numerical modeling, appropriate values had to be assigned to variables. The following values were assumed:

- steel steel:  $k = 1.0 \times 10^8$  N/m, c = 49915.66 Ns/m,
- steel aluminium:  $k = 3.33 \times 10^7 \text{ N/m}, c = 27948.85 \text{ Ns/m},$
- steel rubber:  $k = 2.86 \times 10^6$  N/m, c = 490.33 Ns/m.

In order to ensure a clear comparison of the numerical approaches used, Table 1 lists the key simulation parameters, while Table 2 presents the main differences between the implemented computational models.

Unit Parameter Value Symbol Outer frame mass 60  $m_1$ kg Inner frame mass  $m'_2$ 45 kg Pavload mass  $m_p$ 35 kg Total mass of inner frame + payload 60  $m_2$ kg Isolator stiffness (per isolator) 36800  $k_i$ N/mIsolator damping (per isolator) 5520 Ns/m $c_i$ Equivalent isolator stiffness (4 isolators) 147200 N/m  $k_1$ Equivalent isolator damping (4 isolators) 22080 Ns/m  $c_1$ Initial spring height 0.046  $l_0$ Drop height 0.2 h m  $m/s^2$ Gravity acceleration 9.81 g

Table 1. Model parameters used in simulations

Table 2. Main differences between simulation methods

Feature	SolidWorks Motion (3D model)	Matlab (2-DOF model)
Model dimensionality	Full 3-D rigid-body system	Planar lumped-parameter system
Frame-frame isolation	4 separate spring/dampers	Single equivalent spring/damper
Ground contact	Built-in contact model with material-dependent $k$ , $c$	One-sided Kelvin-Voigt model
Contact stiffness/damping	Automatically derived from material data	Manually assigned
Integration method	GSTIFF (variable-order BDF, variable step)	ode45 (4th order Runge-Kutta)
Geometry	CAD-based solids	Mass points
Solver control	SolidWorks internal timestep control	User-defined tolerances

Based on the conducted analyses, the mass displacements  $m_1$  (external container) (Fig. 4), the spring deflection (Fig. 5) and the acceleration values of both containers (Figs. 6-8) were obtained.

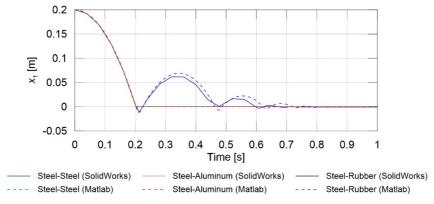


Fig. 4. Displacement of the external container

Analyzing the displacements of the external container (Fig. 4), a very close agreement of the results obtained in SolidWorks Motion and Matlab is observed. When the container falls onto a sufficiently stiff surface (aluminum, steel), there is no rebound of the system, and at approximately 0.2 s, the displacements of the external frame are equal to 0. However, in the case of contact with a rubber substrate, minimal penetration and additional rebounds of the structure from the ground are observed.

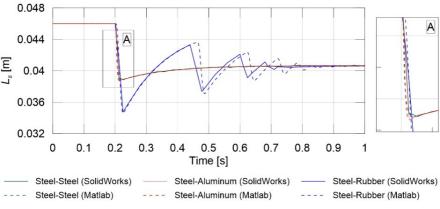


Fig. 5. Spring length (deflection)

Analyzing the spring deflection (Fig. 5), it can be concluded that a fall onto a rigid surface causes rapid damping of vibrations, while a fall onto a more compliant surface causes the inner container to oscillate. With a 35 kg load, the maximum deflection is approximately 0.011 m, and importantly, the springs do not fully compress (their length does not reach zero).

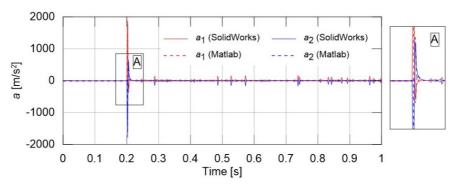


Fig. 6. Accelerations in the system (steel – steel)

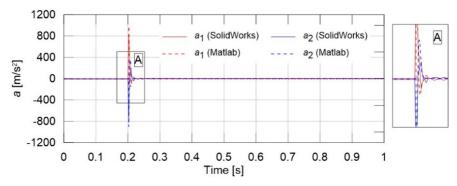


Fig. 7. Accelerations in the system (steel – aluminum)

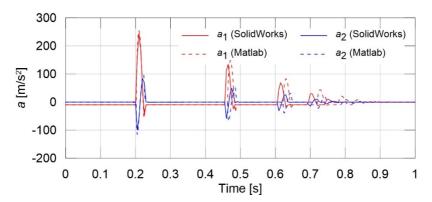


Fig. 8. Accelerations in the system (steel – rubber)

The situation differs in the case of acceleration. The stiffer the ground onto which the container falls, the higher the resulting acceleration. Maximum values are achieved at the moment of contact between the container and the ground, after which the accelerations quickly approach zero. Only in the case of a rubber substrate does the structure exhibit rebounds. A fall onto a steel base causes the outer container to

reach an acceleration  $(a_1)$  of almost 2000 m/s<sup>2</sup>. The acceleration of the inner container relative to the outer container  $(a_2)$  is always smaller and occurs in antiphase.

For the adopted model parameters, and based on the obtained simulation results, it can be clearly stated that the transported load remains safe. This is due to the fact that the springs did not close, which could cause a secondary shock wave and rebound effect, and the recorded acceleration values remain within acceptable limits.

The differences observed between the SolidWorks Motion and Matlab models result primarily from the method of modeling the systems. In the case of SolidWorks, a spatial arrangement was considered with springs/dampers placed at each corner connecting the inner container with the outer container. In Matlab, equivalent stiffness and equivalent damping were present in the planar system. Another element that introduces differences in the results is the integration methods used and the integration steps. The oscillations observed during the steel-on-steel impact case (Fig. 6) can be eliminated in SolidWorks by enabling penetration between contacting elements.

#### 5. Conclusions

This paper proposes two simulation models for conducting fall tests of specialized transport containers onto surfaces made of different materials. The container consisted of two frames connected by a shock isolation system.

The tests were conducted for a single configuration of impact and drop resistance tests, which are included in international industry standards for transport.

The basic assumption of face drop analysis is that container falls from a relatively low height, hence air resistance can be neglected. However, it should be noted that for containers with low weight, large surface area and dropped from heights exceeding 1.5 meters, air resistance should be taken into account as its effect could be noticeable in results.

Both considered models assume pure motion along the vertical axis only. In reality, during container handling, it is very difficult to achieve such an ideal case. The analyzed models are an approximation and simplification of container behavior during a drop. However, in reality, the mechanics of this phenomenon are much more complex and harder to evaluate with analytical approaches.

Industry standards evoke some other test cases for shipping containers like edge or corner drop. Nevertheless, modeling these scenarios in both SolidWorks and Matlab is difficult to implement due to solver limitation and motion equation complexity. The models presented in this study can serve as a foundation for further research exploring the other interesting drop studies like edge or corner drop which will represent even more realistic behavior of shipping containers exposed to accidental transportation loads.

The system's response was examined in SolidWorks Motion (rigid 3D model) and Matlab (planar model). These two parameters have the greatest impact on the safe transport of sensitive loads. They showed that in the model under consideration, the springs did not close, a point at which they could be damaged by simultaneous

contact between the inner and outer containers, generating a high rebound force. The second important parameter in the permissible standards was acceleration, which should not generate overloads affecting the transported elements. Both analyzed models demonstrated good sensitivity in terms of input parameters like mass, spring and substrate stiffness, and they can be easily scalable for simulating the real container structure response.

Summarizing the results presented in this paper, the following conclusion can be drawn:

- differences observed between the SolidWorks Motion and Matlab models result primarily from the method of modeling the systems and integration approach,
- displacement of the external container is insensitive to the adopted modeling approach and gives good correlation between SolidWorks Motion and Matlab,
- plotted spring deflection results show a good match for stiff substrate (steel, aluminum) and 0.02 s offset in time for rubber when comparing 3D and 2D models,
- the stiffer the ground onto which the container falls, the higher the resulting acceleration – maximum is registered for steel ground and reach nearly 200 G's acceleration,
- for the rubber substrate, a rebound effect is observed and the peak acceleration for the outer frame reaches 250 m/s<sup>2</sup>,
- the selected simulation method has a significant impact on the calculation time SolidWorks Motion needs about 4.27 s (at 100 frames per second) to solve the given problem, while MATLAB performs the calculations in about 0.25 s (without drawing graphs) or about 2.19 s (with plotting the results).

The next stage of work will be the development of computational models in software that will allow, in addition to kinematic tests, the determination of stresses and deformations in the structural elements of the container. Besides using more appropriate software for container drop analysis, the authors intend to build a laboratory station with acceleration and displacement sensors to conduct real drop tests and validate the numerical results.

Although the performed study is narrowed to face drop only, it can be a valuable source of information for industry applications which provide transport services for sensitive cargo like medical instruments or aerospace modules.

The authors aim to develop a methodology for conducting virtual container fall tests and to eliminate physical tests, which, while effective, are expensive and resource-intensive.

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