Journal of Applied Mathematics and Computational Mechanics 2025, 24(1), 78-88
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2025.1.07 e-ISSN 2353-0588

# SOME REMARKS ON LOCAL OPERATORS IN GENERALIZED HÖLDER FUNCTION SPACES

***Małgorzata Wróbel***

*Department of Mathematics, Czestochowa University of Technology*
*Czestochowa, Poland*
*malgorzata.wrobel@pcz.pl*

Received: 4 December 2024; Accepted: 18 March 2025

**Abstract.** We discuss the form of local operators, sometimes called operators with memory, acting between spaces of generalized Hölder functions defined on the compact metric spaces and taking values in the special norm spaces. Using the McShane and the Minty extension theorems, we show that in some cases the operators of such a type become Nemytskij composition operators. Moreover, the uniformly bounded as well as the uniformly continuous local operators acting between different Hölder function spaces are investigated.

***MSC 2010:*** *47H30, 54C35*
***Keywords:*** *locally defined operator, Nemytskij composition operator, uniformly bounded operator, uniformly continuous operator, generalized Hölder function*

## 1. Introduction

Let $(X,\rho)$ be a metric space and $Y$, $Z$ be arbitrary sets. By $\mathscr{G}=\mathscr{G}(X,Y)$ and $\mathscr{H}=\mathscr{H}(X,Z)$, let us denote two classes of functions $\varphi : X\to Y$ and $\psi : X\to Z$, respectively. A mapping $K:\mathscr{G}\to\mathscr{H}$ is said to be a *locally defined operator* (briefly, a *local operator* or $(\mathscr{G},\mathscr{H})$*–local*) if for any open set $U\subset X$ and for any functions $f,g\in\mathscr{G}$ the following implication holds:

$$f\big|_U = g\big|_U \Rightarrow K(f)\big|_U = K(g)\big|_U.$$

The goal of this paper is to show that in the cases where $\mathscr{G}$ and $\mathscr{H}$ are two classes of generalized Hölder functions defined on metric spaces and taking values in the special norm spaces, every locally defined operator is a Nemytskij composition operator (Corollary 1, Remark 4). Moreover, in Section 4, under the additional assumption that local operators are uniformly bounded, applying the Matkowski result, we observe that the generators of operators of such a type must be affine with respect to the second variable (Theorem 2, Remark 6). This extends the main result of [1] and [2] for $k=0$.

The theory of the Nemytskij operators are closely connected with the theory of integral equations, differential equations, variational calculus or with the optimization theory. The Volterra integral operators on Banach spaces, like the Holder spaces, are used in the mathematical modelling of phenomena where the memory effects play a key role [3]. Similarly, the classical properties of the Nemytskij operators play an essential role in the partial differential equations and arise, for example, in the theory of the nonlinear diffusion process, in differential geometry and in the nuclear or chemical reactor theory [4].

## 2. Preliminaries

In the sequel, by a $\phi$-function we mean a function $\phi : [0,\infty) \to [0,\infty)$ such that $\phi$ is right continuous at 0, $\phi(0) = 0$, and both functions $\phi$ and $(0,\infty) \ni t \to \dfrac{t}{\phi(t)}$ are increasing. A standard example is $\phi(t) = t^{\alpha}$ with $\alpha \in (0,1]$.

**Remark 1** ([1], Remark 1) Every $\phi$-function is continuous and subadditive, i.e.,

$$\phi(s+t) \leq \phi(s) + \phi(t), \quad s,t \in [0,\infty).$$

Let $(X,\rho_X)$ be a metric space and $(Y,|\cdot|_Y)$ be a real normed space. Given a $\phi$-function, we define the space of Hölder functions $H_\phi((X,\rho_X),(Y,|\cdot|_Y))$, briefly $H_\phi(X,Y)$, as the family of all functions $f : X \to Y$ such that

$$H_\phi(f) := \sup\left\{\frac{\omega(f,s)}{\phi(s)} : \ s > 0\right\} < \infty,$$

where

$$\omega(f,s) := \sup\{|f(x) - f(\bar{x})|_Y : x,\bar{x} \in X, \ \rho_X(x,\bar{x}) \leq s\}$$

is the modulus of continuity of the function $f$. Of course, every function $f \in H_\phi(X,Y)$ is continuous.

In other words, $f \in H_\phi(X,Y)$ if and only if the quantity

$$\sup\left\{\frac{|f(x) - f(\bar{x})|_Y}{\phi(\rho_X(x,\bar{x}))} : x,\bar{x} \in X, \ x \neq \bar{x}\right\}$$

is finite or, equivalently, if and only if there exists a constant $H_\phi(f) \geq 0$ such that

$$|f(x) - f(\bar{x})|_Y \leq H_\phi(f)\phi(\rho_X(x,\bar{x})); \quad x,\bar{x} \in X. \tag{1}$$

Given $x_0 \in X$, we can introduce $\|\cdot\|_{H_\phi(X,Y),x_0} : H_\phi(X,Y) \to [0,\infty)$ by

$$\|f\|_{H_\phi(X,Y),x_0} := |f(x_0)|_Y + H_\phi(f), \quad f \in H_\phi(X,Y). \tag{2}$$

By ([5], Remark 2), the pair $\left(H_\phi((X,Y),\|\cdot\|_{H_\phi(X,Y),x_0}\right)$ is a normed space. Moreover, for any $x_0,x_1 \in X$, the norms $\|\cdot\|_{H_\phi(X,Y),x_0}$ and $\|\cdot\|_{H_\phi(X,Y),x_1}$ are equivalent. Thus, to simplify the notation, we shall write $\|f\|_{H_\phi(X,Y)}$ instead of $\|f\|_{H_\phi(X,Y),x_0}$, for all $f \in H_\phi(X,Y)$.

Fix $x_0 \in X$. Let us notice that if we assume, additionally, that the metric space $(X,\rho_X)$ is bounded, then, for all $f \in H_\phi(X,Y)$ and $x \in X$, we get

$$\begin{aligned}
|f(x)|_Y &\leq |f(x_0)|_Y + |f(x) - f(x_0)|_Y \\
&= |f(x_0)|_Y + \frac{|f(x) - f(x_0)|_Y}{\phi(\rho_X(x,x_0))}\phi(\rho_X(x,x_0)) \\
&\leq |f(x_0)|_Y + H_\phi(f)\phi(\rho_X(x,x_0)) \leq |f(x_0)|_Y + H_\phi(f)\phi(diamX) \\
&\leq |f(x_0)|_Y + H_\phi(f) + \left(H_\phi(f) + |f(x_0)|_Y\right)\phi(diamX) \\
&= \|f\|_{H_\phi(X,Y)}(1 + \phi(diamX)),
\end{aligned}$$

where *diamX* denotes the diameter of the metric space $X$. Hence, the quantity

$$\|f\|_\infty ;= \sup\{|f(x)|_Y : x \in X\}, \tag{3}$$

is finite and the following holds

$$\|f\|_\infty \leq \|f\|_{H_\phi(X,Y)}(1 + \phi(diamX)), \tag{4}$$

for all $f \in H_\phi(X,Y)$.

Thus a functional $\|\cdot\|^\infty_{H_\phi(X,Y)} : H_\phi(X,Y) \to [0,\infty)$, where

$$\|f\|^\infty_{H_\phi(X,Y)} := \|f\|_\infty + H_\phi(f), \quad f \in H_\phi(X,Y), \tag{5}$$

is well defined. By [5], it is a norm.

**Remark 2** Let $(Y,|\cdot|_Y)$ be a real normed space and $\phi : [0,\infty) \to [0,\infty)$ be a $\phi$-function. If a metric space $(X,\rho_X)$ is bounded, then the norms $\|\cdot\|_{H_\phi(X,Y)}$ and $\|\cdot\|^\infty_{H_\phi(X,Y)}$ are equivalent. □

Indeed, from (2)-(5) it follows that

$$\|f\|_{H_\phi(X,Y)} \leq \|f\|^\infty_{H_\phi(X,Y)} \leq \|f\|_{H_\phi(X,Y)}(2 + \phi(diamX)),$$

for all $f \in H_\phi(X,Y)$.

**Lemma 1** *Let a metric space $(X,\rho_X)$ be bounded, and let a $\phi$-function be given. Then, for all $f,g \in H_\phi(X,\mathbb{R})$, the following holds*

$$\|fg\|^\infty_{H_\phi(X,\mathbb{R})} \leq \|f\|^\infty_{H_\phi(X,\mathbb{R})}\|g\|^\infty_{H_\phi(X,\mathbb{R})}. \tag{6}$$

PROOF Take any $f,g \in H_\phi(X,\mathbb{R})$. Without loss of generality, we may assume that the quantities $\|f\|^\infty_{H_\phi(X,\mathbb{R})}$, $\|g\|^\infty_{H_\phi(X,\mathbb{R})}$, $H_\phi(f)$, $H_\phi(g)$ are strictly positive. Since, for all $x, \bar{x} \in X$, by (1), we have

$$\begin{aligned}
\frac{|(fg)(x) - (fg)(\bar{x})|}{\phi(\rho_X(x,\bar{x}))} &= \frac{|f(x)g(x) - f(x)g(\bar{x}) + f(x)g(\bar{x}) - f(\bar{x})g(\bar{x}))|}{\phi(\rho_X(x,\bar{x}))} \\
&\leq |f(x)| \frac{|g(x) - g(\bar{x})|}{\phi(\rho_X(x,\bar{x}))} + |g(\bar{x})| \frac{|f(x) - f(\bar{x})|}{\phi(\rho_X(x,\bar{x}))} \\
&\leq \|f\|_\infty H_\phi(g) + \|g\|_\infty H_\phi(f),
\end{aligned}$$

therefore

$$H_\phi(fg) \leq \|f\|_\infty H_\phi(g) + \|g\|_\infty H_\phi(f).$$

Hence, we obtain the following estimates

$$\begin{aligned}
\|fg\|^\infty_{H_\phi(X,\mathbb{R})} &\leq \|fg\|_\infty + \|f\|_\infty H_\phi(g) + \|g\|_\infty H_\phi(f) \\
&\leq \|f\|_\infty \big(\|g\|_\infty + H_\phi(g)\big) + \|g\|_\infty H_\phi(f) \\
&\leq \|f\|_\infty \|g\|^\infty_{H_\phi(X,\mathbb{R})} + \|g\|^\infty_{H_\phi(X,\mathbb{R})} H_\phi(f) \\
&\leq \|f\|^\infty_{H_\phi(X,\mathbb{R})} \|g\|^\infty_{H_\phi(X,\mathbb{R})},
\end{aligned}$$

which gives (6), and the proof is completed. ■

**Remark 3** If a metric space $(X, \rho_X)$ is bounded and $(Y, |\cdot|_Y)$ is complete, then the pair $\Big(H_\phi((X,Y), \|\cdot\|_{H_\phi(X,Y)}\Big)$ or, equivalently, $\Big(H_\phi((X,Y), \|\cdot\|^\infty_{H_\phi(X,Y)}\Big)$ is a Banach space [6]. Moreover, $\Big(H_\phi((X,\mathbb{R}), \|\cdot\|_{H_\phi(X,\mathbb{R})}\Big)$ is a Banach algebra. □

Given two $\phi$-functions $\phi$ and $\psi$, we write $\phi \preccurlyeq \psi$ if $\phi(t) = O(\psi(t))$, for $t \to 0$ (where $O$ is Landau's symbol), i.e., if there exist numbers $P > 0$ and $\delta > 0$ such that

$$\phi(t) \leq P\psi(t), \quad 0 < t \leq \delta. \tag{7}$$

**Lemma 2** *Let $(X, \rho_X)$ be a bounded metric space and $(Y, |\cdot|_Y)$ be a real normed space. Suppose that $\phi$ and $\psi$ are two $\phi$-functions. From $\phi \preccurlyeq \psi$ it follows that $H_\phi(X,Y) \subset H_\psi(X,Y)$, and there exists a positive constant $\gamma$ such that*

$$\|f\|_{H_\psi(X,Y)} \leq \gamma \|f\|_{H_\phi(X,Y)}, \qquad f \in H_\phi(X,Y). \tag{8}$$

PROOF By the assumption, there exist positive numbers $P$ and $\delta$ such that (7) is fulfilled. Take any $f \in H_\phi(X,Y)$. Then, there exists a constant $H_\phi(f)$ satisfying (1) for all $x, \bar{x} \in X$. Hence, in the case where $\rho_X(x,\bar{x}) \leq \delta$, $x, \bar{x} \in X$, we have

$$|f(x) - f(\bar{x})|_Y \leq H_\phi(f)\phi(\rho_X(x,\bar{x})) \leq PH_\phi(f)\psi(\rho_X(x,\bar{x})). \tag{9}$$

Now, suppose that $x, \bar{x} \in X$ and $\rho_X(x,\bar{x}) > \delta$. Since $\psi$ is monotonically increasing,

$$\psi(\rho_X(x,\bar{x})) > \psi(\delta). \tag{10}$$

Then, by (1), (10), and the boundedness of $X$, we obtain

$$
\begin{aligned}
|f(x)-f(\overline{x})|_Y &= \frac{|f(x)-f(\overline{x})|_Y}{\phi(\rho_X(x,\overline{x}))}\frac{\phi(\rho_X(x,\overline{x}))}{\psi(\rho_X(x,\overline{x}))}\psi(\rho_X(x,\overline{x})) \\
&\leq H_\phi(f)\frac{\phi\,(diamX)}{\psi\,(\delta)}\psi(\rho_X(x,\overline{x})),
\end{aligned}
$$

for $x,\overline{x}\in X$ such that $\rho_X(x,\overline{x})>\delta$. Thus, putting

$$\overline{\gamma}=\overline{\gamma}(\phi,\psi):=\max\left\{P,\ \frac{\phi\,(diamX)}{\psi\,(\delta)}\right\},$$

by (9), it follows that

$$|f(x)-f(\overline{x})|_Y\leq\overline{\gamma}H_\phi(f)\psi(\rho_X(x,\overline{x})),$$

for all $x,\overline{x}\in X$. Therefore, $f\in H_\psi(X,Y)$, by (1) with $H_\psi(f):=\gamma H_\phi(f)$, and, setting $\gamma=\overline{\gamma}+1$, (8) is fulfilled, which gives a required claim. ■

## 3. Main results

We will start with the following definitions.

**Definition 1** Let $\mathscr{G}=\mathscr{G}(X,Y)$ and $\mathscr{H}=\mathscr{H}(X,Z)$ be two function spaces. We say that an operator $K:\mathscr{G}\to\mathscr{H}$ has a *point-memory property*, if for every $f,g\in\mathscr{G}$ and $x_0\in X$, from $f(x_0)=g(x_0)$ it follows that $K(f)(x_0)=K(g)(x_0)$. □

**Definition 2** Let $X,Y,Z$ be arbitrary nonempty sets and let $h:X\times Y\to Z$. Denote by $Y^X$ the family of all functions $f:X\to Y$. A mapping $H:Y^X\to Z^X$ defined by

$$H(f)(x)=h(x,f(x)),\quad f\in Y^X\quad(x\in X),$$

is called a *Nemytskij (composition, superposition) operator*, and the function $h$ is referred to as a *generator of* $H$. □

**Theorem 1** *Let $\mathscr{G}=\mathscr{G}(X,Y)$ and $\mathscr{H}=\mathscr{H}(X,Z)$ be two function spaces such that all constant functions defined on X are contained in $\mathscr{G}$. If an operator K mapping $\mathscr{G}$ into $\mathscr{H}$ has a point-memory property, then K is a Nemytskij superposition operator, i.e., there exists a unique function $h:X\times Y\to Z$ such that, for all $f\in\mathscr{G}$,*

$$K(f)(x)=h(x,f(x)),\quad x\in X.$$

PROOF Given $y_0\in Y$, let us define a function $P_{y_0}:X\to Y$ by

$$P_{y_0}(x):=y_0,\quad x\in X. \tag{11}$$

Of course $P_{y_0}$, as a constant function, by assumption, belongs to $\mathscr{G}$. To define the function $h : X \times Y \to Z$, fix arbitrarily $x_0 \in X,\ y_0 \in Y$, and put

$$h(x_0, y_0) := K(P_{y_0})(x_0). \tag{12}$$

Since, by (11), for all functions $f$,

$$f(x_0) = P_{f(x_0)}(x_0),$$

by the point-memory property of $K$, we have

$$K(f)(x_0) = K(P_{f(x_0)})(x_0) = h(x_0, f(x_0)).$$

As the uniqueness of the function $h$ is obvious, the proof is completed. ∎

**Theorem 2** *Let $(X, \rho_X)$ be a compact metric space, $(Z, |\cdot|_Z)$ be a real normed space, and let $\phi$, $\psi$ be two $\phi$-functions. Then, every locally defined operator $K : H_\phi(X, \mathbb{R}) \to H_\psi(X, Z)$ has a point-memory property.* □

PROOF We have to show that for all $f, g \in H_\phi(X, \mathbb{R})$ and $x_0 \in X$, if

$$f(x_0) = g(x_0), \tag{13}$$

then

$$K(f)(x_0) = K(g)(x_0). \tag{14}$$

To this end fix arbitrarily $x_0 \in X$ and take a pair of functions $f, g \in H_\phi(X, \mathbb{R})$ fulfilling (13). Clearly, if $x_0$ is an isolated point of $X$, we get (14) directly from the definition of a local operator. Assume that $x_0$ is a cluster point of $X$. Choose a one-to-one sequence of points $x_n \in X$ such that $\lim\limits_{n\to\infty} x_n = x_0$ and

$$n \neq m \Rightarrow \rho_X(x_n, x_m) > \frac{1}{2} \max\{\rho_X(x_n, x_0), \rho_X(x_m, x_0)\}. \tag{15}$$

Put

$$r_n := \frac{1}{6} \rho_X(x_n, x_0), \quad n \in \mathbb{N}. \tag{16}$$

Then, for all

$$x \in B(x_n, r_n), \quad y \in B(x_m, r_m); \quad n \neq m,\ n, m \in \mathbb{N}, \tag{17}$$

the following inequalities are fulfilled:

$$\rho_X(x, x_0) \leq 7\rho_X(x, y), \quad \rho_X(y, x_0) \leq 7\rho_X(x, y) \tag{18}$$

(the symbol $B(x, r)$ denotes the ball centered at $x$ and the radius $r$).

Indeed, by the triangle inequality, (15) and (16), for all $x, y$ satisfying (17), we get

$$\begin{aligned}&\frac{1}{2}\max\{\rho_X(x_n,x_0),\rho_X(x_m,x_0)\}\le\rho_X(x_n,x_m)\\&\le\rho_X(x_n,x)+\rho_X(x,y)+\rho_X(y,x_m)\le r_n+\rho_X(x,y)+r_m\\&=\frac{1}{6}\rho_X(x_n,x_0)+\rho_X(x,y)+\frac{1}{6}\rho_X(x_m,x_0)\le\rho_X(x,y)+\frac{2}{6}\max\{\rho_X(x_n,x_0),\rho_X(x_m,x_0)\},\end{aligned}$$

whence

$$\rho_X(x,y)\ge\frac{1}{6}\max\{\rho_X(x_n,x_0),\rho_X(x_m,x_0)\}. \tag{19}$$

Moreover, by (16) and (17),

$$\rho_X(x,x_0)\le\rho_X(x,x_n)+\rho_X(x_n,x_0)\le r_n+\rho_X(x_n,x_0)=\frac{7}{6}\rho_X(x_n,x_0),\quad n\in\mathbb{N},$$

and, analogously,

$$\rho_X(y,x_0)\le\frac{7}{6}\rho_X(x_m,x_0).$$

Hence, making use of (19), we obtain (18).

Putting

$$B_1:=\bigcup_{n\in\mathbb{N}}B(x_{2n-1},r_{2n-1}),\quad B_2:=\bigcup_{n\in\mathbb{N}}(x_{2n},r_{2n}),$$

define $\gamma: B_1\cup B_2\cup\{x_0\}\to\mathbb{R}$ by the formula

$$\gamma(x):=\begin{cases}f(x) & \text{if} \quad x\in B_1\\ f(x_0) & \text{if} \quad x=x_0\\ g(x) & \text{if} \quad x\in B_2\end{cases}.$$

We shall show that there exists $H_\phi(\gamma)\in\mathbb{R}_+$ such that

$$|\gamma(x)-\gamma(y)|\le H_\phi(\gamma)\phi(\rho_X(x,y)), \tag{20}$$

for all $x, y\in B_1\cup B_2\cup\{x_0\}$.

Since $f\in H_\phi(B_1\cup\{x_0\})$ and $g\in H_\phi(B_2\cup\{x_0\})$, there exist $H_\phi(f), H_\phi(g)\in\mathbb{R}_+$ such that

$$|f(x)-f(\bar{x})|\le H_\phi(f)\phi(\rho_X(x,\bar{x}));\quad x,\bar{x}\in B_1\cup\{x_0\}, \tag{21}$$

and

$$|g(y)-g(\bar{y})|\le H_\phi(g)\phi(\rho_X(y,\bar{y}));\quad y,\bar{y}\in B_2\cup\{x_0\}. \tag{22}$$

Of course, if $x,y\in B_1\cup\{x_0\}$ or $x,y\in B_2\cup\{x_0\}$, then (20), by (21) and (22), is obvious. Take $x\in B_1\cup\{x_0\}$ and $y\in B_2\cup\{x_0\}$.

Taking into account the definition of $\gamma$, (13), (21) and (22) with $\bar{x} = \bar{y} = x_0$, we have

$$\begin{aligned} |\gamma(x) - \gamma(y)| &\leq |\gamma(x) - \gamma(x_0)| + |\gamma(y) - \gamma(x_0)| \\ &= |f(x) - f(x_0)| + |g(y) - g(x_0)| \\ &\leq H_\phi(f)\phi(\rho_X(x,x_0)) + H_\phi(g)\phi(\rho_X(y,x_0)), \end{aligned}$$

and, consequently, by (18), the monotonicity and the subadditivity of $\phi$, we get

$$|\gamma(x) - \gamma(y)| \leq 7H_\phi(f)\phi(\rho_X(x,y)) + 7H_\phi(g)\phi(\rho_X(x,y)).$$

Thus, $\gamma \in H_\phi(B_1 \cup B_2 \cup \{x_0\})$ and fulfils (20) with a constant $H_\phi(\gamma) := 14\max\{H_\phi(f), H_\phi(g)\}$.

Since, by Remark 1, $\phi \circ \rho_X$ is a metric, therefore, applying the McShane extension theorem, with $\|\cdot\| := \phi \circ \rho_X$ ([7], Theorem 1), we get the existence of a function $\bar{\gamma} \in H_\phi(X,\mathbb{R})$ such that

$$\bar{\gamma}|_{B_1} = f|_{B_1}, \quad \bar{\gamma}|_{B_2} = g|_{B_2},$$

and, by the definition of a local operator,

$$K(\bar{\gamma})|_{B_1} = K(f)|_{B_1}, \quad K(\bar{\gamma})|_{B_2} = K(g)|_{B_2}.$$

The continuity of the functions $K(\bar{\gamma}), K(f)$ and $K(g)$ at $x_0$ implies (14), which completes a proof. ■

By Theorems 1-2, we get the following

**Corollary 1** *Every locally defined operator $K : H_\phi(X,\mathbb{R}) \to H_\psi(X,Z)$, where $(X,\rho_X)$ is a compact metric space and $(Z,|\cdot|_Z)$ is a real normed space is a Nemytskij superposition operator.* □

**Remark 4** Let $\phi(t) = \phi_\alpha(t) = t^\alpha$, $t \geq 0$. Making use of Minty's extension theorem ([8], Theorem 1) and using analogous methods, it follows that Theorem 2 and Corollary 1 remain valid on replacing

1. $H_\phi(X,\mathbb{R})$ by $H_{\phi_\alpha}(X,Y)$, where $(Y,|\cdot|_Y)$ is a Hilbert space and $\alpha \in (0, \frac{1}{2}]$;
2. $H_\phi(X,\mathbb{R})$ by $H_{\phi_\alpha}(X,Y)$ and $(X,\rho_X)$ by $\left(X',|\cdot|_{X'}\right)$, where $\left(X',|\cdot|_{X'}\right)$, $(Y,|\cdot|_Y)$ are Hilbert spaces and $\alpha \in (0,1]$. □

## 4. A remark concerning uniform boundedness of local operators

In this section an important role plays the following

**Definition 3** ([5], Definition 1). Let $\mathscr{Y}$ and $\mathscr{Z}$ be metric (or normed) spaces. A mapping $H : \mathscr{Y} \to \mathscr{Z}$ is said to be *uniformly bounded*, if for any $t > 0$ there is a nonnegative real number $\gamma(t)$ such that for any set $B \subset \mathscr{Y}$ we have

$$\operatorname{diam} B \leq t \Rightarrow \operatorname{diam} H(B) \leq \gamma(t).$$

**Theorem 3** *Let $(X, \rho_X)$ be a compact metric space and $(Z, |\cdot|_Z)$ be a real normed space.*

*(i) If a locally defined operator K: $H_\phi(X,\mathbb{R}) \to H_\psi(X,Z)$, generated by a function $h : X \times \mathbb{R} \to Z$ such that for any $x \in X$ a function $h(x,\cdot) : \mathbb{R} \to Z$ is continuous with respect to the second variable, is uniformly bounded, then there exist $\alpha \in H_\psi(X, L(\mathbb{R},Z))$ and $\beta \in H_\psi(X,Z)$ such that*

$$K(f)(x) = \alpha(x) f(x) + \beta(x), \quad f \in H_\phi(X,\mathbb{R}) \quad (x \in X) \tag{23}$$

*(here $L(\mathbb{R},Z)$ denotes a normed space of all linear and continuous mappings $\alpha : \mathbb{R} \to Z$).*

*(ii) Conversely, let $\phi \preccurlyeq \psi$. If an operator $K : \mathbb{R}^X \to Z^X$ is defined by (23) for some functions $\alpha, \beta \in H_\phi(X,\mathbb{R})$, then it maps $H_\phi(X,\mathbb{R})$ into $H_\psi(X,\mathbb{R})$, is locally defined, and satisfies the global Lipschitz condition (so it is uniformly bounded).* □

Proof (i) This part of the theorem is an immediate consequence of Corollary 1 and the Matkowski result ([5], Theorem 4.3).

(ii) Since $\phi \preccurlyeq \psi$, therefore $H_\phi(X,\mathbb{R}) \subset H_\psi(X,\mathbb{R})$, by Lemma 2, and since $H_\psi(X,\mathbb{R})$ is a Banach algebra, by Remark 3, it follows that the operator given by (23) maps $H_\phi(X,\mathbb{R})$ into $H_\psi(X,\mathbb{R})$. Of course, every operator defined by (23) is locally defined.

Now, for all $f, g \in H_\phi(X,\mathbb{R})$, Remark 2, formulas (6), (8), and (23) lead to the following estimates

$$\begin{aligned}
\|K(f) - K(g)\|_{H_\psi(X,\mathbb{R})} &\leq \|K(f) - K(g)\|^\infty_{H_\psi(X,\mathbb{R})} \leq \|\alpha\|^\infty_{H_\psi(X,\mathbb{R})} \|f - g\|^\infty_{H_\psi(X,\mathbb{R})} \\
&\leq (2 + \phi(diamX)) \|\alpha\|^\infty_{H_\psi(X,\mathbb{R})} \|f - g\|_{H_\psi(X,\mathbb{R})} \\
&\leq \gamma(2 + \phi(diamX)) \|\alpha\|^\infty_{H_\psi(X,\mathbb{R})} \|f - g\|_{H_\phi(X,\mathbb{R})},
\end{aligned}$$

which shows that $K$ is Lipschitzian and a proof is completed. ■

**Remark 5** In a special case when a local operator $K : H_\phi(X,\mathbb{R}) \to H_\psi(X,\mathbb{R})$ is uniformly continuous (with respect to the norms) and $X \subset \mathbb{R}$ is a compact interval, the form of $K$ given by (23), for some $\alpha, \beta \in H_\psi(X,\mathbb{R})$, can be obtained directly from Corollary 1 and Matkowski's result ([9], Remark 5), without any additional assumptions on the generating function $h$. □

**Definition 4** ([10], Definition 6.26) We say that the pair $(\mathscr{G}(X,Y), \mathscr{H} = \mathscr{H}(X,Z))$ has *the uniform Matkowski property,* if the generator $h$ of the uniformly bounded

Nemytskij superposition operator $H : G(X,Y)$ and $H = H(X,Z)$ has the form

$$h(x,y) = \alpha(x)y + \beta(x), \quad x \in X,\ y \in Y,$$

for some $\alpha \in L(Y,Z)$ and $\beta \in H_\psi(X,Z)$. □

**Remark 6** The pair $\left(H_\phi(X,\mathbb{R}), H_\psi(X,Z)\right)$ has the uniform Matkowski property, where $(X,\rho_X)$ is a compact metric space and $(Z,|\cdot|_Z)$ is a real normed space. □

**Remark 7** Under the assumptions of Theorem 3, if a locally defined operator $K : H_\phi(X,\mathbb{R}) \to H_\psi(X,Z)$ is uniformly continuous, then there exist $\alpha \in H_\psi(X,L(\mathbb{R},Z))$ and $\beta \in H_\psi(X,Z)$ such that (23) is fulfilled. □

**Remark 8** Let $\phi_\alpha(t) = t^\alpha$, $t \geq 0$. Then, applying Matkowski's result [5] and Remark 4 again, we conclude that Theorem 3 (i) remains true if we

1. $H_\phi(X,\mathbb{R})$ replace by $H_{\phi_\alpha}(X,Y)$, where $(Y,|\cdot|_Y)$ is a Hilbert space and $\alpha \in (0,\frac{1}{2}]$;
2. $H_\phi(X,\mathbb{R})$ replace by $H_{\phi_\alpha}(X,Y)$ and $(X,\rho_X)$ by $\left(X',|\cdot|_{X'}\right)$, where $\left(X',|\cdot|_{X'}\right)$, $(Y,|\cdot|_Y)$ are Hilbert spaces and $\alpha \in (0,1]$. □

## 5. Conclusion

The locally defined operators were studied mainly in the context of operators acting between the spaces of real-valued functions defined on the closed subsets of $\mathbb{R}^n$. Now, we describe operators of such a type acting between the spaces of generalized Hölder functions taking values in Hilbert normed spaces or defined on compact metric spaces. More precisely, we prove that very local operator $K$ mapping the space of Hölder functions $H_\phi(X,\mathbb{R})$ defined on compact metric spaces $(X,\rho_X)$ and taking values in $\mathbb{R}$ into another Hölder space $H_\psi(X,Z)$ of functions taking values in a real normed space $(Z,|\cdot|_Z)$ is a Nemytskij superposition operator. We notice that this result remains valid on replacing $\mathbb{R}$ by Hilbert space $(Y,|\cdot|_Y)$ in the case where $\phi(t) = t^\alpha$ and $\alpha \in (0,\frac{1}{2}]$ or on replacing $(X,\rho_X)$ by Hilbert space $\left(X',|\cdot|_{X'}\right)$ and $\mathbb{R}$ by Hilbert space $(Y,|\cdot|_Y)$ in the case where $\phi(t) = t^\alpha$and $\alpha \in (0,1]$. Moreover, we observe that the pair $\left(H_\phi(X,\mathbb{R}), H_\psi(X,Z)\right)$ has the uniform Matkowski property. This extends the main results of [1] and [2] for $k = 0$.

## References

[1] Wróbel, M. (2011). Locally defined operators in the Hölder spaces. *Nonlinear Anal.*, *TMA 74*, 317-323.

[2] Wróbel, M. (2020). Locally defined operators in the space of C$^{k,\omega}$-functions. *Math. Slovaca*, *70*(3), 745-752.

[3] Brunner, H. (2017). *Volterra Integral Equations. An Introduction to the Theory and Applications*. Cambridge University Press.

[4] Runst, T., & Sickel, W. (2001). *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations*. Berlin, New York: De Gruyter.

[5] Matkowski, J. (2011). Uniformly bounded composition operators between general Lipschitz functions normed spaces. *Topol. Methods Nonlinear Anal.*, *38*(2), 395-406.

[6] Banaś, J., & Nalepa, R. (2016). On a measure of noncompactness in the space of functions with tempered increments. *J. Math. Anal. Appl.*, *435*(2), 1634-1651.

[7] McShane, E.J. (1934). Extension of range of functions. *Bull. Amer. Math. Soc.*, *40*, 837-842.

[8] Minty, G.J. (1970). On the extension of Lipschitz, Lipschitz-Hölder continuous and monotone functions. *Bull. Amer. Math. Soc.*, *76*(2), 334-339.

[9] Matkowski, J. (2009). Uniformly continuous superposition operators in the Banach space of Hölder functions. *J. Math. Anal. Appl.*, *359*(1), 56-61.

[10] Appell, J., Banaś, J., & Merentes, N. (2014). *Bounded Variation and Around*. Berlin, Boston: De Gruyter.