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A NEW COUPLING METHOD OF KHALOUTA TRANSFORM AND RESIDUAL POWER SERIES METHOD FOR SOLVING NONLINEAR FRACTIONAL HYPERBOLIC-LIKE EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this paper, we develop a new numerical method called Khalouta residual power series method (KHRPSM) by mixing the Khalouta transform method and the residual power series method for solving nonlinear fractional hyperbolic-like equations with variable coefficients. The KHRPSM resolves nonlinear fractional problems without resorting to He's polynomials and Adomian polynomials. Therefore, the small computational size of this method is the strength of the scheme, which is an advantage compared with various series solution methods. The approximate and exact solutions of a numerical example of the proposed problem are demonstrated by the presented method. The numerical and exact solutions are compared with each other. The obtained results show that KHRPSM is easy to implement and highly effective in constructing approximate analytical solutions to nonlinear fractional problems arising in related fields of science and technology.

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1. Introduction

In recent years, nonlinear fractional partial differential equations (NFPDEs) have been tackled by many researchers because they play an important role in describing many phenomena arising in physics [1], fluid mechanics [2], viscoelasticity [3], chemistry [4], wave propagation [5], biology [6], medicine [7], aerodynamic [8], control theory [9], finance [10], dynamical systems [11], and engineering sciences [12]. The exact solutions of the NFPDEs can help us get familiar with the described process. So, in the past decades, mathematicians have made many efforts in the study of exact solutions of NFPDEs. But for most these equations, no exact solution is known, and in some cases, it is not even clear whether a unique solution exists. Therefore, approximation methods, such as numerical and analytical methods, have been developed. Recently, several numerical and analytical methods have been proposed for the solutions of NFPDEs such as the Adomian decomposition method (ADM) [13], homotopy analysis method (HAM) [14], homotopy perturbation method (HPM) [15], differential transform method (DTM) [16], and the fractional variational iteration method (FVIM) [17].

The novelty of this work lies in the construction of the Khalouta residual power series method which involves the integration of two powerful techniques: the Khalouta transform (KHT), which was first introduced by Ali Khalouta [18], and the residual power series method (RPSM) [19] which is a well-known mathematical technique for solving nonlinear partial differential equations.

KHRPSM provides a simple and fast way to find the coefficients of the recommended series as a solution to the problem. Unlike traditional RPSM, which requires calculating the fractional derivative each time to determine the coefficients for a series, KHRPSM only relies on the concept of the limit at infinity to determine the coefficients.

In this paper we present approximate analytical solutions for a nonlinear fractional hyperbolic-like equation with variable coefficients using KHRPSM.

The hyperbolic-like equation with fractional derivatives is written in operator form as

$$\begin{aligned} \mathfrak{D}_{\theta}^{2\alpha}\mathscr{W}(\mathscr{V},\theta) &= \sum_{i,j=1}^{N}\mathscr{F}_{1ij}(\mathscr{V},\theta,\mathscr{W})\frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m}\mathscr{F}_{2ij}(\mathscr{W}_{x_i},\mathscr{W}_{x_j}) \\ &\quad + \sum_{i=1}^{N}\mathscr{G}_{1i}(\mathscr{V},\theta,\mathscr{W})\frac{\partial^{p}}{\partial x_i^p}\mathscr{G}_{2i}(\mathscr{W}_{v_i}) + \mathscr{H}(\mathscr{V},\theta,\mathscr{W}) + \mathscr{S}(\mathscr{V},\theta), \end{aligned} \tag{1}$$

under the initial conditions

$$\mathscr{W}(\mathscr{V},0) = \mathscr{W}_0(\mathscr{V}), \mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0) = \mathscr{W}_1(\mathscr{V}), \tag{2}$$

where $\mathfrak{D}_{\theta}^{2\alpha}$ is the Caputo fractional derivative operator of order 2α with $1/2 < \alpha \leq 1$, $\mathscr{W} = \{\mathscr{W}(\mathscr{V},\theta), \mathscr{V} = (x_1,x_2,...,x_N) \in \mathbb{R}^N, \theta \geq 0, N \in \mathbb{N}^*\}$, $\mathscr{F}_{1ij}, \mathscr{G}_{1i}$ $i,j \in \{1,2,...,N\}$ are nonlinear functions of $\mathscr{V},\theta$ and $\mathscr{W}$, $\mathscr{F}_{2ij}, \mathscr{G}_{2i}$ $i,j \in \{1,2,...,N\}$ are nonlinear functions of derivatives of $\mathscr{W}$ with respect to x_i and x_j $i,j \in \{1,2,...,N\}$, respectively. Also $\mathscr{H}, \mathscr{S}$ are nonlinear functions and k,m,p are integers.

These types of equations are of considerable significance in various fields of applied sciences, mathematical physics, nonlinear hydrodynamics, engineering physics, biophysics, human movement sciences, astrophysics and plasma physics. These equations describe the evolution of erratic motions of small particles that are immersed in fluids, fluctuations of the intensity of laser light, and velocity distributions of fluid particles in turbulent flows.

The remaining paper is organized in the following manner: The section "Basic definitions and results" gives detailed concepts related to fractional calculus and

Khalouta transform. The section "New formula of multiple fractional Taylor's series" presents a new formula, which will be useful for the KHRPSM. The section "Khalouta residual power series method (KHRPSM)" acquaints the reader with the mathematical formulation of KHRPSM using KHT and RPSM to solve the proposed equations. The section "Numerical experiment" provides a numerical example of the application of the method. Finally, the conclusion of this work is presented in the section entitled "Conclusion".

2. Basic definitions and results

Definition 1 [20] The Caputo fractional derivative of the function $\mathscr{W}(\mathscr{V},\theta)$ of order $\alpha > 0$ is defined by

$$\mathfrak{D}_\theta^\alpha \mathscr{W}(\mathscr{V},\theta) = \mathfrak{I}_\theta^{n-\alpha} \mathscr{W}^{(n)}(\mathscr{V},\theta),$$

where $n-1 < \alpha \leq n, n \in \mathbb{N}$, $\mathscr{V} \in \mathbb{R}^n, \theta \in \mathbb{R}^+$ and $\mathfrak{I}_\theta^\alpha$ is the fractional Riemann--Liouville integral operator defined as

$$\mathfrak{I}_\theta^\alpha \mathscr{W}(\mathscr{V},\theta) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^\varepsilon (\theta-\varepsilon)^{\alpha-1} \mathscr{W}(\mathscr{V},\varepsilon) d\varepsilon, & \text{if } \alpha > 0, \\ \mathscr{W}(\mathscr{V},\theta), & \text{if } \alpha = 0. \end{cases}$$

Definition 2 [18] The Khalouta transform of a function $\mathscr{W}(\mathscr{V},\theta)$ can be described as

$$\mathbb{KH}[\mathscr{W}(\mathscr{V},\theta)] = \mathscr{K}(\mathscr{V},s,\gamma,\eta) = \frac{s}{\gamma\eta}\int_0^\infty \exp\left(-\frac{s\theta}{\gamma\eta}\right)\mathscr{W}(\mathscr{V},\theta)d\theta,$$

where $s, \gamma, \eta > 0$ are the parameters of the Khalouta transform. □

Theorem 1 *[21] Let $\mathscr{K}_1(\mathscr{V},s,\gamma,\eta)$ and $\mathscr{K}_2(\mathscr{V},s,\gamma,\eta)$ are the Khalouta transfoms of $\mathscr{W}_1(\mathscr{V},\theta)$ and $\mathscr{W}_2,(\mathscr{V},\theta)$ respectively. Then we have*

(1)

$$\mathbb{KH}[a\mathscr{W}_1(\mathscr{V},\theta) + b\mathscr{W}_2(\mathscr{V},\theta)] = a\mathscr{K}_1(\mathscr{V},s,\gamma,\eta) + b\mathscr{K}_2(\mathscr{V},s,\gamma,\eta),$$

where a, b are real numbers.

(2)

$$\mathbb{KH}[\mathfrak{I}_\theta^\alpha \mathscr{W}(\mathscr{V},\theta)] = \left(\frac{\gamma\eta}{s}\right)^\alpha \mathscr{K}(\mathscr{V},s,\gamma,\eta).$$

(3)

$$\mathbb{KH}[\mathfrak{D}_\theta^\alpha \mathscr{W}(\mathscr{V},\theta)] = \left(\frac{s}{\gamma\eta}\right)^\alpha \mathscr{K}(\mathscr{V},s,\gamma,\eta) - \sum_{i=0}^{n-1}\left(\frac{s}{\gamma\eta}\right)^{\alpha-i}\mathscr{W}^{(i)}(\mathscr{V},0).$$

(4)

$$\mathbb{KH}\left[\frac{\theta^{\alpha}}{\Gamma(\alpha+1)}\right] = \left(\frac{\gamma\eta}{s}\right)^{\alpha}, \alpha > -1.$$

(5)

$$\mathbb{KH}\left[\mathfrak{D}_{\theta}^{m\alpha}\mathscr{W}(\mathscr{V},\theta)\right] = \left(\frac{s}{\gamma\eta}\right)^{m\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta) - \sum_{i=0}^{m-1}\left(\frac{s}{\gamma\eta}\right)^{(m-i)\alpha}\mathfrak{D}_{\theta}^{i\alpha}\mathscr{W}(\mathscr{V},0),$$

where $\mathfrak{D}_{\theta}^{m\alpha} = \mathfrak{D}_{\theta}^{\alpha}\mathfrak{D}_{\theta}^{\alpha}...\mathfrak{D}_{\theta}^{\alpha}$*(m−times).* □

3. New formula of multiple fractional Taylor's series

Suppose that the multiple fractional power series representation of the function $\mathscr{W}(\mathscr{V},\theta)$ at $\theta = 0$ has the form [19]

$$\mathscr{W}(\mathscr{V},\theta) = \sum_{m=0}^{\infty}\mathfrak{C}_m(\mathscr{V})\theta^{m\alpha}, n-1 < \alpha \leq n, \mathscr{V} \in \mathbb{R}^n, 0 \leq \theta \leq R,$$

and R is the radius of convergence of the multiple fractional power series.

Theorem 2 *If* $\mathscr{W} \in C(\mathbb{R}\times[0,R))$ *and* $\mathfrak{D}_{\theta}^{m\alpha}\mathscr{W} \in C(\mathbb{R}\times(0,R))$ *for* $m = 0,1,2,...,$ *then the coefficients* $\mathfrak{C}_m(\mathscr{V})$ *will take the form of*

$$\mathfrak{C}_m(\mathscr{V}) = \frac{\mathfrak{D}_{\theta}^{m\alpha}\mathscr{W}(\mathscr{V},0)}{\Gamma(m\alpha+1)},$$

where $\mathfrak{D}_{\theta}^{m\alpha} = \mathfrak{D}_{\theta}^{\alpha}\mathfrak{D}_{\theta}^{\alpha}...\mathfrak{D}_{\theta}^{\alpha}$*(m−times).* □

Lemma 1 *The Khalouta transform of* $\mathscr{W}(\mathscr{V},\theta)$ *given by* $\mathscr{K}(\mathscr{V},s,\gamma,\eta)$*, has multiple fractional Taylor's series representation as*

$$\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \sum_{m=0}^{\infty}\left(\frac{\gamma\eta}{s}\right)^{m\alpha}\Upsilon_m(\mathscr{V}), \tag{3}$$

where $\Upsilon_m(\mathscr{V})$ *represents* m^{th} *coefficient of the new formula of multiple fractional Taylor's series in Khalouta transform.* □

PROOF Consider the following fractional Taylor series

$$\mathscr{W}(\mathscr{V},\theta) = \Upsilon_0(\mathscr{V}) + \Upsilon_1(\mathscr{V})\frac{\theta^{\alpha}}{\Gamma(\alpha+1)} + \Upsilon_2(\mathscr{V})\frac{\theta^{2\alpha}}{\Gamma(2\alpha+1)} + \Upsilon_3(\mathscr{V})\frac{\theta^{3\alpha}}{\Gamma(3\alpha+1)} + ... \tag{4}$$

Applying the Khalouta transform to both sides of equation (4) and using its linearity property, we get

$$\begin{aligned}\mathbb{KH}[\mathscr{W}(\mathscr{V},\theta)] &= \Upsilon_0(\mathscr{V})+\Upsilon_1(\mathscr{V})\mathbb{KH}\left[\frac{\theta^{\alpha}}{\Gamma(\alpha+1)}\right]+\Upsilon_2(\mathscr{V})\mathbb{KH}\left[\frac{\theta^{2\alpha}}{\Gamma(2\alpha+1)}\right]\\ &\quad +\Upsilon_3(\mathscr{V})\mathbb{KH}\left[\frac{\theta^{3\alpha}}{\Gamma(3\alpha+1)}\right]+\ldots\end{aligned}$$

Using point (4) of Theorem 1, we get

$$\begin{aligned}\mathscr{K}(\mathscr{V},s,\gamma,\eta) &= \Upsilon_0(\mathscr{V})+\Upsilon_1(\mathscr{V})\left(\frac{\gamma\eta}{s}\right)^{\alpha}+\Upsilon_2(\mathscr{V})\left(\frac{\gamma\eta}{s}\right)^{2\alpha}+\Upsilon_3(\mathscr{V})\left(\frac{\gamma\eta}{s}\right)^{3\alpha}+\ldots\\ &= \sum_{m=0}^{\infty}\left(\frac{\gamma\eta}{s}\right)^{m\alpha}\Upsilon_m(\mathscr{V}),\end{aligned}$$

which is a new form of fractional Taylor's series in Khalouta transform form.

Thus, the proof is completed. ■

Lemma 2 *Suppose the function* $\mathbb{KH}[\mathscr{W}(\mathscr{V},\theta)] = \mathscr{K}(\mathscr{V},s,\gamma,\eta)$ *has a multiple fractional power series representation in the new form of the Taylor's series (3). Then we have*

$$\lim_{s\to\infty}\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \Upsilon_0(\mathscr{V}) = \mathscr{W}(\mathscr{V},0).$$

PROOF Taking $\lim\limits_{s\to\infty}$ of equation (5) and performing a simple calculation, we get

$$\lim_{s\to\infty}\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \Upsilon_0(\mathscr{V}) = \mathscr{W}(\mathscr{V},0).$$

Theorem 3 *Suppose that the function* $\mathbb{KH}[\mathscr{W}(\mathscr{V},\theta)] = \mathscr{K}(\mathscr{V},s,\gamma,\eta)$ *has the following multiple fractional power series representation*

$$\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \sum_{m=0}^{\infty}\left(\frac{\gamma\eta}{s}\right)^{m\alpha}\Upsilon_m(\mathscr{V}), \tag{5}$$

then we have

$$\Upsilon_m(\mathscr{V}) = \mathfrak{D}_{\theta}^{m\alpha}\mathscr{W}(\mathscr{V},0),$$

where $\mathfrak{D}_{\theta}^{m\alpha} = \mathfrak{D}_{\theta}^{\alpha}\mathfrak{D}_{\theta}^{\alpha}\ldots\mathfrak{D}_{\theta}^{\alpha}$*(m−times).* □

PROOF Consider that $\mathscr{K}(\mathscr{V},s,\gamma,\eta)$ has multiple fractional power series representation as in equation (5). Then equation (5) becomes

$$\begin{aligned}\mathscr{K}(\mathscr{V},s,\gamma,\eta) &= \Upsilon_0(\mathscr{V})+\left(\frac{\gamma\eta}{s}\right)^{\alpha}\Upsilon_1(\mathscr{V})+\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\Upsilon_2(\mathscr{V})\\ &\quad +\left(\frac{\gamma\eta}{s}\right)^{3\alpha}\Upsilon_3(\mathscr{V})+\ldots\end{aligned} \tag{6}$$

Multiplying equation (6) by $\left(\frac{s}{\gamma\eta}\right)^{\alpha}$, we get

$$\begin{aligned}\Upsilon_1(\mathscr{V}) &= \left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^{\alpha}\Upsilon_0(\mathscr{V})-\left(\frac{\gamma\eta}{s}\right)^{\alpha}\Upsilon_2(\mathscr{V}) \\ &\quad -\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\Upsilon_3(\mathscr{V})-\ldots \end{aligned} \tag{7}$$

Taking $\lim\limits_{s\to\infty}$ on equation (7) and using point (5) of Theorem 1, we get

$$\begin{aligned}\Upsilon_1(\mathscr{V}) &= \lim_{s\to\infty}\left(\left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^{\alpha}\Upsilon_0(\mathscr{V})\right) \\ &= \lim_{s\to\infty}\left(\mathbb{KH}\left[D_t^{\alpha}\mathscr{W}(\mathscr{V},\theta)\right](s,\gamma,\eta)\right). \end{aligned} \tag{8}$$

By Lemma 2, equation (8) becomes

$$\Upsilon_1(\mathscr{V})=\mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0).$$

Similarly, we multiply equation (6) by $\left(\frac{s}{\gamma\eta}\right)^{2\alpha}$ and we get

$$\begin{aligned}\Upsilon_2(\mathscr{V}) &= \left(\frac{s}{\gamma\eta}\right)^{2\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^{2\alpha}\Upsilon_0(\mathscr{V})-\left(\frac{s}{\gamma\eta}\right)^{\alpha}\Upsilon_1(\mathscr{V}) \\ &\quad -\left(\frac{\gamma\eta}{s}\right)^{\alpha}\Upsilon_3(\mathscr{V})+\ldots \end{aligned} \tag{9}$$

Taking $\lim\limits_{s\to\infty}$ on equation (9) and using point (5) of Theorem 1, we get

$$\begin{aligned}\Upsilon_2(\mathscr{V}) &= \lim_{s\to\infty}\left(\begin{array}{c}\left(\frac{s}{\gamma\eta}\right)^{2\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^{2\alpha}\Upsilon_0(\mathscr{V}) \\ -\left(\frac{s}{\gamma\eta}\right)^{\alpha}\Upsilon_1(\mathscr{V})\end{array}\right) \\ &= \lim_{s\to\infty}\left(\mathbb{KH}\left[\mathfrak{D}_{\theta}^{2\alpha}\mathscr{W}(\mathscr{V},\theta)\right](s,\gamma,\eta)\right). \end{aligned} \tag{10}$$

By Lemma 2, equation (10) becomes

$$\Upsilon_2(\mathscr{V})=\mathfrak{D}_{\theta}^{2\alpha}\mathscr{W}(\mathscr{V},0).$$

To complete the proof, we use the principle of mathematical induction method. Suppose $\Upsilon_{m-1}(\mathscr{V})=\mathfrak{D}_{\theta}^{(m-1)\alpha}\mathscr{W}(\mathscr{V},0)$. Multiplying equation (6) by $\left(\frac{s}{\gamma\eta}\right)^{m\alpha}$

and using point (5) of Theorem 1 and Lemma 2, we get

$$\begin{aligned}\Upsilon_m(\mathscr{V}) &= \lim_{s\to\infty}\left(\begin{array}{c}\left(\frac{s}{\gamma\eta}\right)^{m\alpha}\mathscr{K}(\mathscr{V},s,\gamma,\eta)-\left(\frac{s}{\gamma\eta}\right)^{m\alpha}\Upsilon_0(\mathscr{V})-\\ \left(\frac{s}{\gamma\eta}\right)^{(m-1)\alpha}\mathfrak{D}_\theta^{\alpha}\mathscr{W}(\mathscr{V},0)-\left(\frac{s}{\gamma\eta}\right)^{(m-2)\alpha}\mathfrak{D}_\theta^{2\alpha}\mathscr{W}(\mathscr{V},0)\\ -\ldots-\left(\frac{s}{\gamma\eta}\right)^{\alpha}\mathfrak{D}_\theta^{(m-1)\alpha}\mathscr{W}(\mathscr{V},0)\end{array}\right)\\ &= \lim_{s\to\infty}\left(\mathbb{KH}\left[\mathfrak{D}_\theta^{m\alpha}\mathscr{W}(\mathscr{V},\theta)\right](s,\gamma,\eta)\right)\\ &= \mathfrak{D}_\theta^{m\alpha}\mathscr{W}(\mathscr{V},\theta)\end{aligned}$$

■

Remark 1 The inverse Khalouta transform of the series extension in Theorem 3 has the form of the following multiple fractional power series

$$\mathscr{W}(\mathscr{V},\theta)=\sum_{m=0}^{\infty}\frac{\mathfrak{D}_\theta^{m\alpha}\mathscr{W}(\mathscr{V},0)}{\Gamma(m\alpha+1)}\theta^{m\alpha}, 0<\alpha\leq 1, \theta\geq 0.$$

In the following theorem, we explain and determine the convergence conditions of the new form of multiple fractional Taylor's formula.

Theorem 4 *Let $\mathscr{W}(\mathscr{V},\theta)$ be a piecewise continuous function defined on $\mathbb{R}^n\times\mathbb{R}^+$ and of exponential order and let $\mathbb{KH}[\mathscr{W}(\mathscr{V},\theta)]=\mathscr{K}(\mathscr{V},s,\gamma,\eta)$ be represented as the new form of multiple fractional Taylor's formula explained in Theorem 3. If $\left|\mathbb{KH}\left[\mathfrak{D}_\theta^{(k+1)\alpha}\mathscr{W}(\mathscr{V},\theta)\right]\right|\leq\mathfrak{T}(\mathscr{V})$ on $\mathbb{R}^n\times(0,d]$ with $0<\alpha\leq 1$, then the remainder $R_k(\mathscr{V},s,\gamma,\eta)$ of the newform of multiple fractional Taylor's formula satisfies the following inequality*

$$|R_k(\mathscr{V},s,\gamma,\eta)|\leq\left(\frac{\gamma\eta}{s}\right)^{(k+1)\alpha}\mathfrak{T}(\mathscr{V}).$$

4. Khalouta residual power series method (KHRPSM)

Theorem 5 *Consider the following nonlinear fractional hyperbolic-like equations with variable coefficients (1) under the initial conditions (2). Then the Khalouta fractional power series solution of the proposed equation is described as an infinite series expansion which rapidly converges to the exact solution as follows*

$$\mathscr{W}(\mathscr{V},\theta)=\lim_{k\to\infty}\mathscr{W}_k(\mathscr{V},\theta),$$

where $\mathscr{W}_k(\mathscr{V},\theta)$ is the $k^{th}-$approximate solution given by

$$\mathscr{W}_k(\mathscr{V},\theta)=\sum_{i=0}^{k}\mathscr{W}_i(\mathscr{V})\frac{\theta^{i\alpha}}{\Gamma(i\alpha+1)}.$$

PROOF To prove our result, we first define the following nonlinear operator

$$
\begin{aligned}
\mathscr{N}(\mathscr{W}(\mathscr{V},\theta)) &= \sum_{i,j=1}^{N} \mathscr{F}_{1ij}(\mathscr{V},\theta,\mathscr{W}) \frac{\partial^{k+m}}{\partial x_i^k \partial x_j^m} \mathscr{F}_{2ij}(\mathscr{W}_{x_i},\mathscr{W}_{x_j}) \\
&\quad + \sum_{i=1}^{N} \mathscr{G}_{1i}(\mathscr{V},\theta,\mathscr{W}) \frac{\partial^p}{\partial x_i^p} \mathscr{G}_{2i}(\mathscr{W}_{x_i}) + \mathscr{H}(\mathscr{V},\theta,\mathscr{W}) + \mathscr{S}(\mathscr{V},\theta).
\end{aligned}
$$

Thus, equation (1) can be written in the form

$$
\mathfrak{D}_{\theta}^{2\alpha}\mathscr{W}(\mathscr{V},\theta) = \mathscr{N}(\mathscr{W}(\mathscr{V},\theta)). \tag{11}
$$

Operating the Khalouta transform on both sides of equation (11) and using point (*1*) of Theorem 1, we get

$$
\mathbb{KH}\left[\mathfrak{D}_{\theta}^{2\alpha}\mathscr{W}(\mathscr{V},\theta)\right] = \mathbb{KH}\left[\mathscr{N}(\mathscr{W}(\mathscr{V},\theta))\right]. \tag{12}
$$

According to Theorem 1 and the initial conditions in equation (2), then equation (12) becomes

$$
\begin{aligned}
\mathscr{K}(\mathscr{V},s,\gamma,\eta) &= \mathscr{W}(\mathscr{V},0) + \left(\frac{\gamma\eta}{s}\right)^{\alpha} \mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0) \\
&\quad + \left(\frac{\gamma\eta}{s}\right)^{2\alpha} \mathbb{KH}\left[\mathscr{N}\left(\mathbb{KH}^{-1}\left[\mathscr{K}(\mathscr{V},s,\gamma,\eta)\right]\right)\right],
\end{aligned} \tag{13}
$$

where $\mathbb{KH}^{-1}\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \mathscr{W}(\mathscr{V},\theta)$.

Based on Theorem 3, we assume that the approximate solution of the Khalouta equation (13) has the following Khalouta fractional expansion

$$
\mathscr{K}(\mathscr{V},s,\gamma,\eta) = \sum_{m=0}^{\infty} \left(\frac{\gamma\eta}{s}\right)^{m\alpha} \Upsilon_m(\mathscr{V}).
$$

Following $\lim_{s\to\infty} \mathscr{K}(\mathscr{V},s,\gamma,\eta) = \Upsilon_0(\mathscr{V}) = \mathscr{W}(\mathscr{V},0)$ and using Theorem 3, we have

$$
\Upsilon_1(\mathscr{V}) = \mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0),
$$

and the $k^{th}-$ Khalouta series solution take the following form

$$
\begin{aligned}
\mathscr{K}_k(x,s,\gamma,\eta) &= \sum_{m=0}^{k} \left(\frac{\gamma\eta}{s}\right)^{m\alpha} \Upsilon_m(\mathscr{V}) \\
&= \Upsilon_0(\mathscr{V}) + \left(\frac{\gamma\eta}{s}\right)^{\alpha} \Upsilon_1(\mathscr{V}) + \sum_{m=2}^{k} \left(\frac{\gamma\eta}{s}\right)^{m\alpha} \Upsilon_m(\mathscr{V}).
\end{aligned}
$$

Now, we define separately the Khalouta fractional residual function of equation (13) and the $k^{th}-$ Khalouta fractional residual function, as

$$\begin{aligned}\mathbb{KH}\mathscr{R}es(\mathscr{V},s,\gamma,\eta) &= \mathscr{K}(\mathscr{V},s,\gamma,\eta)-\mathscr{W}(\mathscr{V},0)-\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0)\\ &\quad -\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\mathbb{KH}\left[\mathscr{N}\left(\mathbb{KH}^{-1}\left[\mathscr{K}(\mathscr{V},s,\gamma,\eta)\right]\right)\right],\end{aligned}$$

and

$$\begin{aligned}\mathbb{KH}\mathscr{R}es_k(\mathscr{V},s,\gamma,\eta) &= \mathscr{K}_k(\mathscr{V},s,\gamma,\eta)-\mathscr{W}(\mathscr{V},0)-\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0)\\ &\quad -\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\mathbb{KH}\left[\mathscr{N}\left(\mathbb{KH}^{-1}\left[\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)\right]\right)\right]. \quad (14)\end{aligned}$$

Substituting the series form of $\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)$ in equation (14) and multiplying both sides by $\left(\frac{s}{\gamma\eta}\right)^{k\alpha}$ as follows

$$\begin{aligned}&\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\mathbb{KH}\left[\mathscr{R}es_k(\mathscr{V},s,\gamma,\eta)\right]\\ =\ &\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\left(\begin{array}{c}\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)-\mathscr{W}(\mathscr{V},0)-\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0)\\ -\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\mathbb{KH}\left[\mathscr{N}\left(\mathbb{KH}^{-1}\left[\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)\right]\right)\right]\end{array}\right). \quad (15)\end{aligned}$$

Taking $\lim\limits_{s\to\infty}$ at both sides of equation (15)

$$\begin{aligned}&\lim_{s\to\infty}\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\mathbb{KH}\left[\mathscr{R}es_k(\mathscr{V},s,\gamma,\eta)\right]\\ =\ &\lim_{s\to\infty}\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\left(\begin{array}{c}\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)-\mathscr{W}(\mathscr{V},0)\\ -\left(\frac{\gamma\eta}{s}\right)^{\alpha}\mathfrak{D}_{\theta}^{\alpha}\mathscr{W}(\mathscr{V},0)\\ -\left(\frac{\gamma\eta}{s}\right)^{2\alpha}\mathbb{KH}\left[\mathscr{N}\left(\mathbb{KH}^{-1}\left[\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)\right]\right)\right]\end{array}\right).\end{aligned}$$

Therefore, we solve the following system iteratively in order to obtain the unknown coefficients $\Upsilon_k(\mathscr{V})$

$$\lim_{s\to\infty}\left(\frac{s}{\gamma\eta}\right)^{k\alpha}\mathbb{KH}\mathscr{R}es_k(\mathscr{V},s,\gamma,\eta)=0, k=2,3,4,...$$

Next, we collect the obtained results of $\Upsilon_m(\mathscr{V})$, and substitute them into the series expansion (14) to find the form of the the $k^{th}-$ Khalouta series solutions $\mathscr{K}_k(\mathscr{V},s,\gamma,\eta)$.

We apply the inverse Khalouta transform on the form $(\mathscr{V}, s, \gamma, \eta)$ to obtain the $k^{th}-$approximate solutions of the original equation, as follows

$$\mathscr{W}_k(\mathscr{V}, \theta) = \sum_{i=0}^{k} \mathscr{W}_i(\mathscr{V}) \frac{\theta^{i\alpha}}{\Gamma(i\alpha+1)},$$

where $\mathscr{W}_i(\mathscr{V}) = \mathfrak{D}_\theta^{i\alpha} \mathscr{W}(\mathscr{V}, 0)$.

Finally, the Khalouta fractional power series solution of equations (1) and (2), is given by

$$\begin{aligned} \mathscr{W}(\mathscr{V}, \theta) &= \lim_{k\to\infty} \mathscr{W}_k(\mathscr{V}, \theta) \\ &= \lim_{k\to\infty} \sum_{i=0}^{k} \mathscr{W}_i(\mathscr{V}) \frac{\theta^{i\alpha}}{\Gamma(i\alpha+1)} \\ &= \sum_{i=0}^{\infty} \mathscr{W}_i(\mathscr{V}) \frac{\theta^{i\alpha}}{\Gamma(i\alpha+1)}. \end{aligned}$$

Thus, the proof is completed. ■

5. Numerical experiment

Example 1 Let us consider the two dimensional nonlinear fractional hyperbolic-like equation with variable coefficients [22]

$$\mathfrak{D}_\theta^{2\alpha} \mathscr{W}(x,y,\theta) = \frac{\partial^2}{\partial x \partial y}(\mathscr{W}_{xx}\mathscr{W}_{yy}) - \frac{\partial^2}{\partial x \partial y}(xy\mathscr{W}_x\mathscr{W}_y) - \mathscr{W}, \tag{16}$$

under the initial conditions

$$\mathscr{W}(x,y,0) = e^{xy}, \mathfrak{D}_\theta^{\alpha} \mathscr{W}(x,y,0) = e^{xy}, \tag{17}$$

$\mathscr{W} = \left\{ \mathscr{W}(x,y,\theta), (x,y,\theta) \in \mathbb{R}^2 \times \mathbb{R}^+ \right\}$ and $1/2 < \alpha \leq 1$.

By applying the same steps in KHRPSM as described in Section 3, then the Khalouta fractional power series solution is given as

$$\mathscr{K}(x,y,s,\gamma,\eta) = \sum_{m=0}^{\infty} \left(\frac{\gamma\eta}{s}\right)^{m\alpha} \Upsilon_m(x,y),$$

and the $k^{th}-$ Khalouta series solution, is given by

$$\mathscr{K}_k(x,y,s,\gamma,\eta) = e^{xy} + \left(\frac{\gamma\eta}{s}\right)^{\alpha} e^{xy} + \sum_{m=2}^{k} \left(\frac{\gamma\eta}{s}\right)^{m\alpha} \Upsilon_m(x,y). \tag{18}$$

and the coefficients $\Upsilon_m(x,y), m \geq 2$, are as follows

$$\begin{aligned}
\Upsilon_2(x,y) &= -e^{xy}, \\
\Upsilon_3(x,y) &= -e^{xy}, \\
\Upsilon_4(x,y) &= e^{xy}, \\
\Upsilon_5(x,y) &= e^{xy}, \\
&\vdots \\
\Upsilon_{2k}(x,y) &= (-1)^k e^{xy}, \\
\Upsilon_{2k+1}(x,y) &= (-1)^k e^{xy}.
\end{aligned}$$

Now, putting the values of $\Upsilon_m(x,y), m = 2,3,4,...,2k,k2+1$ into equation (18), we obtain

$$\begin{aligned}
\mathscr{K}_k(x,y,s,\gamma,\eta) &= e^{xy} + \left(\frac{\gamma\eta}{s}\right)^{\alpha} e^{xy} - \left(\frac{\gamma\eta}{s}\right)^{2\alpha} e^{xy} - \left(\frac{\gamma\eta}{s}\right)^{3\alpha} e^{xy} + \left(\frac{\gamma\eta}{s}\right)^{4\alpha} e^{xy} \\
&+ \left(\frac{\gamma\eta}{s}\right)^{5\alpha} e^{xy} + ... + (-1)^k \left(\frac{\gamma\eta}{s}\right)^{2k\alpha} e^{xy} \\
&+ (-1)^k \left(\frac{\gamma\eta}{s}\right)^{(2k+1)\alpha} e^{xy}. \qquad (19)
\end{aligned}$$

Applying the inverse Khalouta transform on both sides of equation (19), we obtain

$$\begin{aligned}
\mathscr{W}_k(x,y,\theta) &= e^{xy} + \frac{\theta^{\alpha}}{\Gamma(\alpha+1)} e^{xy} - \frac{\theta^{2\alpha}}{\Gamma(2\alpha+1)} e^{xy} - \frac{\theta^{3\alpha}}{\Gamma(3\alpha+1)} e^{xy} + \frac{\theta^{4\alpha}}{\Gamma(4\alpha+1)} e^{xy} \\
&+ ... + (-1)^k \frac{\theta^{2k\alpha}}{\Gamma(2k\alpha+1)} e^{xy} + (-1)^k \frac{\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} e^{xy} \\
&= \left(1 - \frac{\theta^{2\alpha}}{\Gamma(2\alpha+1)} + \cdot \frac{\theta^{4\alpha}}{\Gamma(4\alpha+1)} + ... + (-1)^k \frac{\theta^{2k\alpha}}{\Gamma(2k\alpha+1)} e^{xy}\right) e^{xy} \\
&+ \left(\frac{\theta^{\alpha}}{\Gamma(\alpha+1)} - \frac{\theta^{3\alpha}}{\Gamma(3\alpha+1)} + ... + (-1)^k \frac{\theta^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}\right) e^{xy} \\
&= \left(\sum_{i=0}^{k} (-1)^i \frac{\theta^{2i\alpha}}{\Gamma(2i\alpha+1)} + \sum_{i=0}^{k} (-1)^k \frac{\theta^{(2i+1)\alpha}}{\Gamma((2i+1)\alpha+1)}\right) e^{xy}. \qquad (20)
\end{aligned}$$

As $k \to \infty$, equation (20), can be expressed as the following

$$\begin{aligned}
\mathscr{W}(x,y,\theta) &= \left(\sum_{i=0}^{\infty} (-1)^i \frac{\theta^{2i\alpha}}{\Gamma(2i\alpha+1)} + \sum_{i=0}^{\infty} (-1)^k \frac{\theta^{(2i+1)\alpha}}{\Gamma((2i+1)\alpha+1)}\right) e^{xy} \\
&= (\cos(\theta^{\alpha},\alpha) + \sin(\theta^{\alpha},\alpha)) e^{xy}. \qquad (21)
\end{aligned}$$

Now, if we substitute $\alpha = 1$ in equation (21), we get the exact solution

$$\mathscr{W}(x,\theta) = (\cos(\theta) + \sin(\theta))\, e^{xy}.$$

It is the same result that was obtained in [22].

Figures 1 and 2 show the 2D and 3D plots of KHRPSM in solving equations (16)-(17). Table 1 shows the comparison of the 6^{th}-order approximate solution and the exact solution and their associated absolute errors for (16)-(17) for different values of α. Numerical simulations show that the current technique's solutions are in good agreement with the exact results. The numerical solutions show that only a few terms are sufficient for obtaining an approximate result, which is efficient, accurate, and reliable.

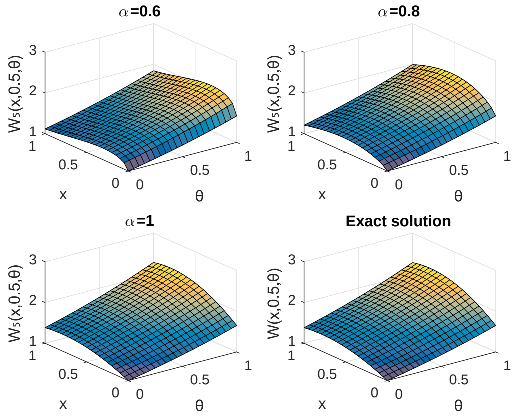

Fig. 1. 3D plots of the 6^{th}-order approximate solution obtained by KHRPSM and exact solution at $y = 0.5$

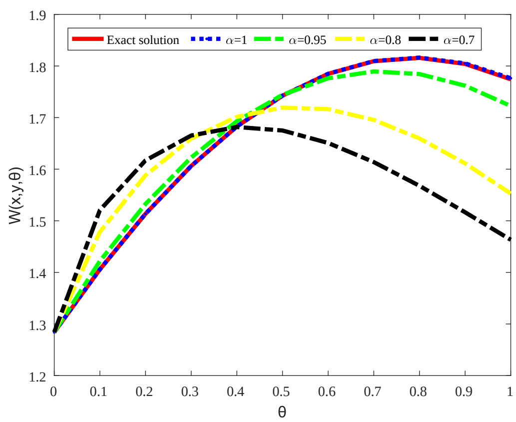

Fig. 2. 2D plots of the 6^{th}-order approximate solution obtained by KHRPSM and exact solution at $x = y = 0.5$

Table 1. Numerical values of the 6^{th}-order approximate solution obtained by KHRPSM and exact solution at $x = y = 0.5$

t	$\alpha = 0.7$	$\alpha = 0.8$	$\alpha = 0.9$	$\alpha = 1$		Absolute error
	$\mathscr{W}_{KHRPSM}$	$\mathscr{W}_{KHRPSM}$	$\mathscr{W}_{KHRPSM}$	$\mathscr{W}_{KHRPSM}$	$\mathscr{W}_{exact}$	$\lvert\mathscr{W}_{exact} - \mathscr{W}_{KHRPSM}\rvert$
0.1	1.5207	1.4784	1.4394	1.4058	1.4058	1.8085×10^{-9}
0.3	1.6652	1.6594	1.6375	1.6061	1.6061	1.3536×10^{-6}
0.5	1.6750	1.7193	1.7411	1.7425	1.7424	2.9725×10^{-5}
0.7	1.6137	1.6956	1.7634	1.8095	1.8093	6.7065×10^{-2}
0.9	1.5164	1.6112	1.714	1.805	1.8040	1.0547×10^{-3}

□

6. Conclusion

This work presents a new coupling method called the Khalouta residual power series method (KHRPSM) which has great potential in constructing approximate solutions and even exact solutions for nonlinear fractional hyperbolic-like equations with variable coefficients. To understand the analytical procedure of the above method, a numerical example is presented for the analytical result of the proposed problem. The effectiveness of KHRPSM has been proven through graphical and numerical results. We can observe from these graphs and tables that the approximate results obtained by KHRPSM are in perfect agreement with their respective exact solutions. The advantage of KHRPSM is that it significantly reduces the numerical calculations required to construct the solutions for this class of equations compared to existing methods, e.g., the differential transform method (DTM), homotopy perturbation method (HPM), and the Adomian decomposition method (ADM). Thus, we can conclude that our new technique is easy to apply, accurate, adaptable, and effective depending on the obtained results. Our future goal is to apply the KHRPSM to other types of NFPDEs appearing in other scientific fields.

References

[1] Akinyemi, L., & Iyiola, O. (2020). Exact and approximate solutions of time-fractional models arising from physics via Shehu transform. *Mathematical Method in Applied Sciences*, *43*(12), 7442-7464.

[2] Alhasan, A.S.H., Saranya, S., & Al-Mdallal, Q.M. (2024). Fractional derivative modeling of heat transfer and fluid flow around a contracting permeable infinite cylinder: Computational study. *Partial Differential Equations in Applied Mathematics*, *11*, 100794, DOI: 10.1016/j.padiff.2024 .100794.

[3] Shen, L.J. (2020). Fractional derivative models for viscoelastic materials at finite deformations. *International Journal of Solids and Structures*, *190*, 226-237.

[4] Ganie, A.H., Mofarreh, F., Alharthi, N.S., & Khan, A. (2025). Novel computations of the time-fractional chemical Schnakenberg mathematical model via non-singular kernel operators. *Boundary Value Problems*, *2025*(2), DOI: 10.1186/s13661-024-01979-4.

[5] Al-Qurashi, M., Rashid, S., Sultana, S. Jarad, F., & Alsharif, A.M. (2022). Fractional-order partial differential equations describing propagation of shallow water waves depending on

power and Mittag-Leffler memory. *AIMS Mathematics*, *7*(7), 12587-12619. DOI: 10.3934/math .2022697.

[6] Khalouta, A. (2024). Existence, uniqueness and convergence solution of nonlinear Caputo--Fabrizio fractional biological population model. *Sahand Communications in Mathematical Analysis*, *21*(3), 165-196, DOI: 10.22130/scma.2023.2005194.1360.

[7] Ibrahim, R.W., Jalab, H.A., Karim, F.K., Alabdulkreem, E., & Ayub, M.N. (2022). A medical image enhancement based on generalized class of fractional partial differential equations. *Quantitative Imaging in Medicine and Surgery*, *12*(1), 172-183. DOI: 10.21037/qims-21-15.

[8] Abro, K.A. (2022). Numerical study and chaotic oscillations for aerodynamic model of wind turbine via fractal and fractional differential operators. *Numerical Methods for Partial Differential Equations*, *38*(5), 1180-1194.

[9] Saber, S. (2024). Control of chaos in the Burke-Shaw system of fractal-fractional order in the sense of Caputo-Fabrizio. *Journal of Applied Mathematics and Computational Mechanics*, *23*(1), 83-96, DOI: 10.17512/jamcm.2024.1.07.

[10] Chauhan, R.P., Kumar, S., Alkahtani, B.S.T., & Alzaid, S.S. (2024). A study on fractional order financial model by using Caputo–Fabrizio derivative. *Results in Physics*, *57*, 107335.

[11] Qin, Y., Khan, A., Ali, I., Al Qurashi, M., Khan, H., Shah, R., & Baleanu, D. (2020). An efficient analytical approach for the solution of certain fractional-order dynamical systems. *Energies*, *13*, 1-14.

[12] Jannelli, A. (2020). Numerical solutions of fractional differential equations arising in engineering sciences. *Mathematics*, *8*(2), 215, DOI: 10.3390/math8020215.

[13] Shukur, A.M. (2015). Adomian decomposition method for certain space-time fractional partial differential equations. *IOSR Journal of Mathematics*, *11*(1), 55-65.

[14] Yin, X.B., Kumar, S., & Kumar, D. (2015). A modified homotopy analysis method for solution of fractional wave equations. *Advances in Mechanical Engineering*, *7*(12), 1-8.

[15] Hamdi-Cherif, M., Belghaba, K., & Ziane, D. (2016). Homotopy perturbation method for solving the fractional Fisher's equation. *International Journal of Analysis and Applications*, *10*(1), 9-16.

[16] Arikoglu, A., & Ozkol, I. (2007). Solution of fractional differential equations by using differential transform method. *Chaos, Solitons & Fractals*, *34*(5), 1473-1481.

[17] Singh, B.K., & Kumar, P. (2017). Fractional variational iteration method for solving fractional partial differential equations with proportional delay. *International Journal of Differential Equations*, 2017, DOI: 10.1155/2017/5206380.

[18] Khalouta, A. (2023). A new exponential type kernel integral transform: Khalouta transform and its applications. *Mathematica Montisnigri*, *57*, 5-23, DOI: 10.20948/mathmontis-2023-57-1.

[19] El-Ajou, A., Abu Arqub, O., Al Zhour, Z., & Momani, Sh. (2013). New results on fractional power series: theories and applications. *Entropy*, *15*(12), 5305-5323.

[20] Kilbas, A.A., Srivastava, H.M., & Trujillo, J.J. (20036). *Theory and Application of Fractional Differential equations*. Elsevier, North-Holland.

[21] Khalouta, A. (2024). Khalouta transform via different fractional derivative operators. *Estn. Samar. Gos. Tekhn. Univ., Ser. Fiz.-Mat. Nauki*, *28*(2), 407-425, DOI: 10.14498/vsgtu2082.

[22] Khalouta, A. (2022). A novel iterative method to solve nonlinear wave-like equations of fractional order with variable coefficients. *Revista Colombiana de Matematicas*, *56*(1), 13-34.