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SOLUTION OF THE HARMONIC OSCILLATOR EQUATION IN CYLINDRICAL COORDINATES WITH FRACTIONAL BOUNDARY CONDITIONS

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Abstract. The boundary value problem consisting of homogeneous second-order ordinary differential equation and the classical and/or fractional boundary conditions is considered. Such an equation can describe the motion of the harmonic oscillator in the one-dimensional cylindrical coordinate. The general solution of this equation includes the Bessel functions of the first and second kinds. The particular solutions of the equation are determined on the basis of various constructions of boundary conditions that, in particular, take into account the left- and right-side fractional derivatives defined in the Riemann-Liouville sense. Also, three illustrative examples of particular solutions on the plots are shown.

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1. Introduction

The ordinary differential equations of the second order [1, 2] are often used to model various physical phenomena. These phenomena include vibrations of harmonic oscillator [3], modeling of stationary heat transfer problems [4, 5], steady-state fluid flow [6], solutions of different Sturm-Liouville problems [7-9], or in electrostatics problems [10]. These equations are often solved as boundary and/or initial value problems. In this work, the focus is only on the boundary value problem. The classical boundary conditions are typically defined as the Dirichlet, Neumann and Robin conditions. With the development of fractional calculus [11, 12], it has became possible to define fractional boundary conditions. The definitions of various fractional derivatives are usually more complex than classical ones, and therefore it is more difficult to obtain exact solutions hence, in such a situation, it is worth considering the use of numerical methods [13, 14]. In my previous paper [15], the fourth-order differential equation with fractional initial-boundary conditions was considered.

In this work, the equation of the harmonic oscillator in 1D axisymmetric cylindrical coordinates is considered. The solution of the equation consists of the sum of the Bessel functions of the first and second kind. Various variants of the classical--fractional boundary conditions are considered. The analysis of example solutions involves examining the influence of the values of the fractional derivative order on the solution. A comparison of the solutions obtained for fractional boundary conditions with classical boundary conditions of the Robin type was also analyzed.

2. Problem and its solution

The second-order differential equation with constant coefficient a in the following form

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{dy(x)}{dx}\right)+a\,y(x)=0 \tag{1}$$

is considered. The general solution of Eq. (1) for $a > 0$ is of the form (i.e. [16])

$$y(x)=C_1\cdot J_0\left(\sqrt{a}\,x\right)+C_2\cdot Y_0\left(\sqrt{a}\,x\right) \tag{2}$$

where J and Y are the Bessel functions of the first and second kind, defined as

$$J_\alpha(x)=\sum_{i=0}^{\infty}\frac{(-1)^i}{i!\Gamma(i+\alpha+1)}\left(\frac{x}{2}\right)^{2i+\alpha} \tag{3}$$

$$Y_n(x)=\lim_{\alpha\to n}\frac{J_\alpha(x)\cos(\alpha\pi)-J_{-\alpha}(x)}{\sin(\alpha\pi)} \tag{4}$$

and C_1 and C_2 are two independent integration constants.

In order to obtain the particular solutions of Eq. (1), the coefficients C_1 and C_2 in Eq. (2) must be determined on the basis of boundary conditions given in the end points x_1 and x_2, where $0 < x_1 < x_2$.

In this paper, Eq. (1) with the following boundary conditions is considered

$$\begin{aligned} x=x_1:\ \Phi_1\left(y(x_1),Dy(x)\big|_{x=x_1},D^{\alpha_1}_{x_2^-}y(x)\Big|_{x=x_1}\right)=0\\ x=x_2:\ \Phi_2\left(y(x_2),Dy(x)\big|_{x=x_2},D^{\alpha_2}_{x_1^+}y(x)\Big|_{x=x_2}\right)=0 \end{aligned} \tag{5}$$

where $\alpha_1, \alpha_2 \in (0,1)$, the operator D denotes the first-order derivative and the operators $D^{\alpha_1}_{x_2^-}$, $D^{\alpha_2}_{x_1^+}$ are the left- and right-side Riemann-Liouville derivatives defined as [11, 12]

$$D^{\alpha_1}_{x_2^-} y(x) := \frac{-1}{\Gamma(1-\alpha_1)} \frac{d}{dx} \int_x^{x_2} \frac{y(\xi)}{(\xi - x)^{\alpha_1}} d\xi, \text{ for } 0 < \alpha_1 < 1 \tag{6}$$

$$D^{\alpha_2}_{x_1^+} y(x) := \frac{1}{\Gamma(1-\alpha_2)} \frac{d}{dx} \int_{x_1}^{x} \frac{y(\xi)}{(x - \xi)^{\alpha_2}} d\xi, \text{ for } 0 < \alpha_2 < 1 \tag{7}$$

The values of the left- and right-side fractional derivatives of function $y(x)$ (see Eq. (2)) calculated on the endpoints of interval $[x_1, x_2]$ are equal to

$$\begin{aligned} D^{\alpha_1}_{x_2^-} y(x)\Big|_{x=x_1} &= D^{\alpha_1}_{x_2^-} \left\{ C_1 \cdot J_0\left(\sqrt{a}\,x\right) + C_2 \cdot Y_0\left(\sqrt{a}\,x\right) \right\}\Big|_{x=x_1} \\ &= C_1 \cdot D^{\alpha_1}_{x_2^-} J_0\left(\sqrt{a}\,x\right)\Big|_{x=x_1} + C_2 \cdot D^{\alpha_1}_{x_2^-} Y_0\left(\sqrt{a}\,x\right)\Big|_{x=x_1} \end{aligned} \tag{8}$$

$$\begin{aligned} D^{\alpha_2}_{x_1^+} y(x)\Big|_{x=x_2} &= D^{\alpha_2}_{x_1^+} \left\{ C_1 \cdot J_0\left(\sqrt{a}\,x\right) + C_2 \cdot Y_0\left(\sqrt{a}\,x\right) \right\}\Big|_{x=x_2} \\ &= C_1 \cdot D^{\alpha_2}_{x_1^+} J_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2} + C_2 \cdot D^{\alpha_2}_{x_1^+} Y_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2} \end{aligned} \tag{9}$$

The left- and right-sided Riemann-Liouville fractional derivatives for $\alpha \geq 0$ of the above Bessel functions can be calculated using the transformations based on the definitions of the Caputo fractional derivatives. For the left-side Riemann-Liouville fractional derivatives, one obtains

$$\begin{aligned} D^{\alpha_1}_{x_2^-} J_0\left(\sqrt{a}\,x\right)\Big|_{x=x_1} &= \left(\frac{-1}{\Gamma(1-\alpha_1)} \frac{d}{dx} \int_x^{x_2} \frac{J_0\left(\sqrt{a}\,\xi\right)}{(\xi - x)^{\alpha_1}} d\xi \right)\Bigg|_{x=x_1} \\ &= \left(\frac{-1}{\Gamma(1-\alpha_1)} \int_x^{x_2} \frac{\frac{d}{d\xi} J_0\left(\sqrt{a}\,\xi\right)}{(\xi - x)^{\alpha_1}} d\xi + \frac{J_0\left(\sqrt{a}\,x_2\right)}{\Gamma(1-\alpha_1)(x_2 - x)^{\alpha_1}} \right)\Bigg|_{x=x_1} \\ &= \frac{\sqrt{a}}{\Gamma(1-\alpha_1)} \int_{x_1}^{x_2} \frac{J_1\left(\sqrt{a}\,\xi\right)}{(\xi - x_1)^{\alpha_1}} d\xi + \frac{J_0\left(\sqrt{a}\,x_2\right)}{\Gamma(1-\alpha_1)(x_2 - x_1)^{\alpha_1}} \end{aligned} \tag{10}$$

$$D_{x_2^-}^{\alpha_1} Y_0\left(\sqrt{a}\,x\right)\Big|_{x=x_1} = \frac{\sqrt{a}}{\Gamma(1-\alpha_1)} \int_{x_1}^{x_2} \frac{Y_1\left(\sqrt{a}\,\xi\right)}{(\xi - x_1)^{\alpha_1}} d\xi + \frac{Y_0\left(\sqrt{a}\,x_2\right)}{\Gamma(1-\alpha_1)(x_2 - x_1)^{\alpha_1}} \tag{11}$$

while the right-side Riemann-Liouville derivatives of J_0 and Y_0 are as follows

$$\begin{aligned} D_{x_1^+}^{\alpha_2} J_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2} &= \left(\frac{1}{\Gamma(1-\alpha_2)} \frac{d}{dx} \int_{x_1}^{x} \frac{J_0\left(\sqrt{a}\,\xi\right)}{(x-\xi)^{\alpha_2}} d\xi \right)\Bigg|_{x=x_2} \\ &= \left(\frac{1}{\Gamma(1-\alpha_2)} \int_{x_1}^{x} \frac{\frac{d}{d\xi} J_0\left(\sqrt{a}\,\xi\right)}{(x-\xi)^{\alpha_2}} d\xi + \frac{J_0\left(\sqrt{a}\,x_1\right)}{\Gamma(1-\alpha_2)(x-x_1)^{\alpha_2}} \right)\Bigg|_{x=x_2} \\ &= \frac{-\sqrt{a}}{\Gamma(1-\alpha_2)} \int_{x_1}^{x_2} \frac{J_1\left(\sqrt{a}\,\xi\right)}{(x_2-\xi)^{\alpha_2}} d\xi + \frac{J_0\left(\sqrt{a}\,x_1\right)}{\Gamma(1-\alpha_2)(x_2-x_1)^{\alpha_2}} \end{aligned} \tag{12}$$

$$D_{x_1^+}^{\alpha_2} Y_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2} = \frac{-\sqrt{a}}{\Gamma(1-\alpha_2)} \int_{x_1}^{x_2} \frac{Y_1\left(\sqrt{a}\,\xi\right)}{(x_2-\xi)^{\alpha_2}} d\xi + \frac{Y_0\left(\sqrt{a}\,x_1\right)}{\Gamma(1-\alpha_2)(x_2-x_1)^{\alpha_2}} \tag{13}$$

3. Illustrative examples

This section outlines the methods for constructing specific solutions to the differential equation in question. The general form of the boundary conditions, as defined in Eq. (5), is examined in detail for three distinct combinations, each applied to the domain $[x_1,x_2]$, on both sides.

Example 1 In this example, the following fractional boundary conditions in the following forms are taken into account

$$\begin{aligned} x = x_1 &: \; D_{x_2^-}^{\alpha_1} y(x)\Big|_{x=x_1} = B_1 \\ x = x_2 &: \; D_{x_1^+}^{\alpha_2} y(x)\Big|_{x=x_2} = B_2 \end{aligned} \tag{14}$$

After substitution of the general solution (2) into Eqs. (14), the following system of linear equations is obtained

$$\begin{aligned} C_1 D^{\alpha_1}_{x_2^-} J_0\left(\sqrt{a}x\right)\Big|_{x=x_1} + C_2 D^{\alpha_1}_{x_2^-} Y_0\left(\sqrt{a}x\right)\Big|_{x=x_1} = B_1 \\ C_1 D^{\alpha_2}_{x_1^+} J_0\left(\sqrt{a}x\right)\Big|_{x=x_2} + C_2 D^{\alpha_2}_{x_1^+} Y_0\left(\sqrt{a}x\right)\Big|_{x=x_2} = B_2 \end{aligned} \tag{15}$$

that can be written in the matrix form as

$$\mathbf{A} \cdot \mathbf{C} = \mathbf{B} \tag{16}$$

where

$$\mathbf{A} = \begin{bmatrix} D^{\alpha_1}_{x_2^-} J_0\left(\sqrt{a}x\right)\Big|_{x=x_1^+} & D^{\alpha_1}_{x_2^-} Y_0\left(\sqrt{a}x\right)\Big|_{x=x_1^+} \\ D^{\alpha_2}_{x_1^+} J_0\left(\sqrt{a}x\right)\Big|_{x=x_2^-} & D^{\alpha_2}_{x_1^+} Y_0\left(\sqrt{a}x\right)\Big|_{x=x_2^-} \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{17}$$

from which the constants C_1 and C_2 are determined.

Figure 1 shows the plots of example particular solutions (2) with the above-mentioned boundary conditions.

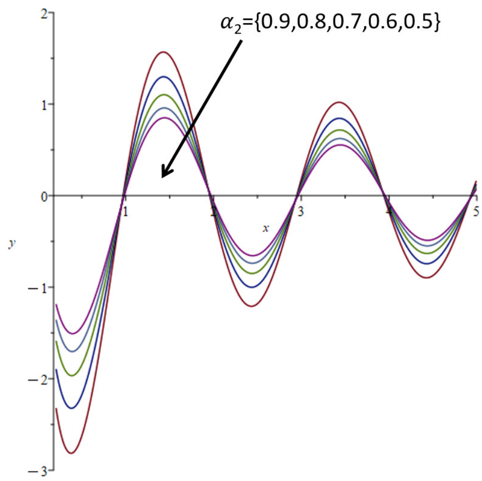

Fig. 1. The solutions of Eq. (1) for $x_1 = 0.2$, $x_2 = 5$, $a = 10$ and boundary conditions $D^{0.6}_{5^-} y(x)\Big|_{x=0.2} = 0.5,\ D^{\alpha_2}_{0.2^+} y(x)\Big|_{x=5} = 1.3$ for $\alpha_2 \in \{0.5, 0.6, 0.7, 0.8, 0.9\}$

Example 2 Here, two cases of the following boundary conditions are considered

$$\begin{aligned} x = x_1 &: \ y(x_1) = B_1 \\ x = x_2 &: \ D^{\alpha_2}_{x_1^+} y(x)\Big|_{x=x_2} = B_2 \end{aligned} \tag{18}$$

and

$$\begin{aligned} x = x_1 &: \ y(x_1) = B_1 \\ x = x_2 &: \ (1-\alpha_2)y(x_2) + \alpha_2 Dy(x_2) = B_2 \end{aligned} \tag{19}$$

In a similar manner as in the first example, the systems of equations are created, where the matrices **A** take the forms

$$\mathbf{A} = \begin{bmatrix} J_0\left(\sqrt{a}\,x_1\right) & Y_0\left(\sqrt{a}\,x_1\right) \\ D^{\alpha_2}_{x_1^+} J_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2^-} & D^{\alpha_2}_{x_1^+} Y_0\left(\sqrt{a}\,x\right)\Big|_{x=x_2^-} \end{bmatrix} \tag{20}$$

and

$$\mathbf{A} = \begin{bmatrix} J_0\left(\sqrt{a}\,x_1\right) & Y_0\left(\sqrt{a}\,x_1\right) \\ (1-\alpha_2)J_0\left(\sqrt{a}\,x_2\right) - \alpha_2\sqrt{a}J_1\left(\sqrt{a}\,x_2\right) & (1-\alpha_2)Y_0\left(\sqrt{a}\,x_2\right) - \alpha_2\sqrt{a}Y_1\left(\sqrt{a}\,x_2\right) \end{bmatrix} \tag{21}$$

respectively. The matrices **B** are the same.

In Figures 2 and 3, the plots of particular solutions (2) with the above sets of boundary conditions are presented.

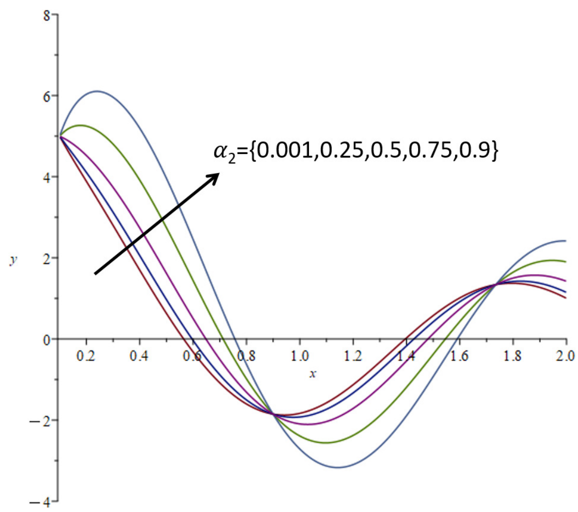

Fig. 2. The solutions of Eq. (1) for $x_1 = 0.1$, $x_2 = 2$, $a = 14$ and boundary conditions $y(0.1) = 5$, $D^{\alpha_2}_{0.1^+} y(x)\Big|_{x=2} = 1$ for $\alpha_2 \in \{0.001, 0.25, 0.5, 0.75, 0.9\}$

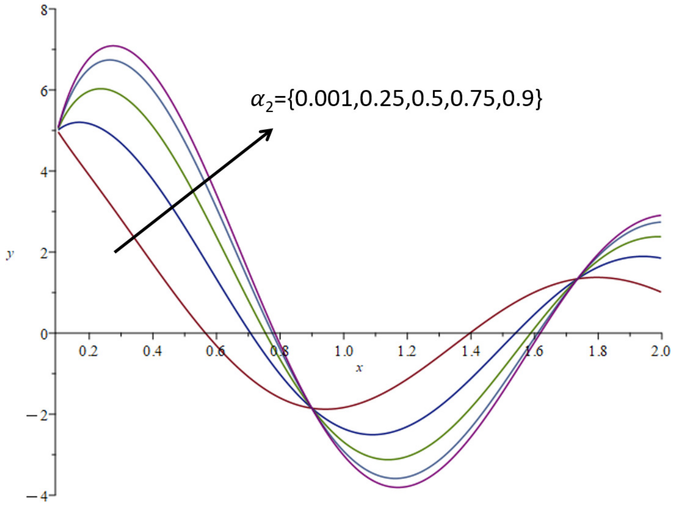

Fig. 3. The solutions of Eq. (1) for $x_1 = 0.1$, $x_2 = 2$, $a = 14$ and boundary conditions $y(0.1)=5,\ (1-\alpha_2)y(0.1)+\alpha_2 Dy(0.1)=1$ for $\alpha_2 \in \{0.001, 0.25, 0.5, 0.75, 0.9\}$

The calculations and plots are created in the Maple software.

4. Conclusions

Solutions of the considered second-order differential equation with fractional boundary conditions seem to be an interesting alternative to solutions with classical boundary conditions. The use of the fractional-order derivatives in boundary conditions gives wider possibilities of applying these solutions, among others, in problems of fractional mechanics. Based on the results presented in the second example, it can be seen that the solutions with fractional boundary conditions are similar (though different) to those with the classical Robin condition. The paper has been presented the construction of three different variants of boundary conditions given at both endpoints of the considered interval. Based on the described constructions, one can create, in an analogous way, other forms of boundary conditions described by any linear functional dependencies taking into account both the value of function, its first derivative and/or fractional derivatives.

Future plans include conducting research on the feasibility of applying the proposed boundary conditions to a broader range of linear differential equations, specifically those of the second or higher order. One also expects wide possibilities of using the fractional derivatives in the constructions of boundary conditions to solve systems of linear differential equations.

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