JAM CM Journal of Applied Mathematics and Computational Mechanics 2024, 23(4), 101-111
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2024.4.09 e-ISSN 2353-0588

CERTAIN CONVERGENCE RESULTS FOR HOMOGENEOUS YOUNG MEASURES WITH DENSITIES

Piotr Puchała

Department of Mathematics, Czestochowa University of Technology
Częstochowa, Poland
piotr.puchala@pcz.pl

Received: 19 August 2024; Accepted: 4 December 2024

Abstract. We consider Young measures associated with elements of sequences of m-oscillating functions. Such Young measures are homogeneous and absolutely continuous with respect to the Lebesgue measure. The total slope of an m-oscillating function is defined, and the basic property of a set of Young measures associated with m-oscillating functions is stated. Next, the relation between weak L^1 convergence of densities and weak convergence, with respect to the total variation norm, of respective Young measures is investigated. The last result unifies and generalizes most examples of Young measures usually presented in the literature.

MSC 2010: *46N10, 74B99*
Keywords: *engineering, oscillating sequences, total slope, Young measures*

1. Introduction

The hidden roots of the concept of a Young measure are in P.G.L. Dirichlet's conviction (shared by many prominent scholars) that bounded from below integral functional attains its infimum. The counterexample provided by Weierstrass in 1870 proved this conviction wrong. The examples presented by O. Bolza and L.C. Young, this time in the 'proper' context of calculus of variations, revealed the nature of 'generalized curves', or 'generalized surfaces' in the multidimensional case, (Young's terminology) of Young measures. Namely, they are weak (or weak*) limits of sequences of rapidly oscillating functions. This follows from the fact that, in general, weak L^1 limits of sequences of bounded functions lie in the second conjugate of L^1. The latter can be identified with the conjugate of L^∞, which is strictly larger with respect to inclusion than L^1.

The first general existence result concerning Young measures, based on the fact that the space of measures is a dual of a relevant function space is presented in [1], the reader may also check [2], where the author investigates Young measures also as disintegrations of respective product measures. An even more general existence result for Young measures together with some advanced applications in engineering can be found in [3]. Detailed treatment of Young measures from various points of view, but concentrated on the mathematical aspects of Young measures are in [4]. The third chapter of [5] is devoted entirely to Young measures and provides a general existence theorem proved in detail.

Young measures appear in the mathematical analysis of certain engineering problems, for example, in the investigation of the infima of the energy functionals of certain shape-memory alloys, like Cu-Al-Ni or Ni-Al, see, for example, paragraph 1.8 in [6], and also [7]. Since the infima of the energy functionals in this case are not attained, the respective minimizing sequences are rapidly oscillating. These oscillations reveal a phenomenon called a microstructure, which can be observed through a microscope. Young measures can be used to analyze the information contained in the microstructure, but obtaining an explicit form of a Young measure in a particular case is difficult. Fortunately, it turns out that probabilistic methods, in particular Monte Carlo simulations, can be of help, see, for example, [8-10], see also [11, 12].

Among other areas in which Young measures play an important role, one can mention nonlinear elasticity and fluid dynamics. An interested reader may check for example [13-15] and the references cited there.

In this article, like in [16], we look at a Young measure as a value of a weakly*-measurable mapping defined on a domain of definition of considered functions. We fix our attention on Young measures associated with bounded functions that are piecewise diffeomorphic. Young measures associated with such functions are homogeneous and absolutely continuous with respect to the Lebesgue measure. We use the fact that for any bounded, Borel, $\mathbb{R}^l$-valued function there exists a Young measure associated with it. This approach makes calculating generalized limits of sequences of rapidly oscillating functions in many practically significant cases easier.

The structure of the article is as follows: in the first part of the next section, we recall facts concerning Young measures together with necessary notions from functional analysis that are used in the article. In the second part, we recall the form of Young measures associated respectively with simple functions and with m-oscillating functions. Then we generalize these results, showing and illustrating this with an example that the Young measure associated with a function being a sum of those types of functions is a mixed probability distribution. The third part of the article contains the definition of a total slope of an m-oscillating function and the statement of the compactness property of a set of Young measures associated with m-oscillating functions. The last section contains the main result of the article. We prove a theorem that unifies and generalizes most of the examples of Young measures generated by sequences of piecewise smoothly invertible functions, which are usually presented in the literature. Finally, the Conclusions section closes the main body of the article.

2. Some necessary facts about Young measures

We briefly recall basic facts and set the notation. The reader is referred to the details in [16] and the references cited there.

2.1. Basic definitions and facts

The letter Ω will denote an open subset of $\mathbb{R}^d$ such that $\mu(\Omega) = M > 0$, where μ is the Lebesgue measure on Ω, the letter K will denote a nonempty compact subset of $\mathbb{R}^l$, dy will denote the Lebesgue measure on K, and $\mathscr{U}$ will stand for the set of all Borel measurable functions on Ω with values in K (if $u \in \mathscr{U}$, then $u(x) \in K$).

If Z is a Banach space and Z^* – the space of all continuous linear functionals on Z (shortly: the conjugate of Z), then a real valued mapping $\langle \cdot, \cdot \rangle$ defined on $Z^* \times Z$, that is linear in each variable separately, is called a *dual pair*. We then say that a mapping $g\colon \Omega \to Z^*$ is *weakly*-measurable*, if for any $z \in Z$ the function $x \mapsto \langle g(x), z \rangle$ is measurable.

If (z_n) is a sequence in Z and (f_n) a sequence in Z^*, then we say that (z_n) is *weakly convergent* to $z \in Z$, if for all $f \in Z^*$ there holds $\lim\limits_{n\to\infty} f(z_n) = f(z)$, while (f_n) is *weakly* convergent* to $f \in Z^*$, if for all $z \in Z$ there holds $\lim\limits_{n\to\infty} f_n(z) = f(z)$.

The elements of a set $rca^1(K)$ are probability measures on the set K. They form a subset of a set $rca(K)$ – the set of regular, countably additive scalar measures on K, which endowed with a total variation norm is a Banach space (if $m \in rca(K)$, its total variation norm $|m|(K) = \sup \sum\limits_i |m(K_i)|$, where the supremum is taken over by all partitions of the set K).

A sequence (ρ_n) of bounded measures on a compact set $K \subset \mathbb{R}^l$ converges weakly* to a measure ρ_0, if $\forall \beta \in C(K, \mathbb{R})$, there holds

$$\lim_{n\to\infty} \int_K \beta(k) d\rho_n(k) = \int_K \beta(k) d\rho_0(k).$$

If ρ is a measure on K and for some function $w\colon K \to \mathbb{R}$ integrable with respect to the measure ξ there holds: for any Borel subset A of K we have $\rho(A) = \int_A w(y) d\xi(y)$, then the function w is called a *density* of the measure ρ. In this case ρ is *absolutely continuous* with respect to ξ (shortly: ξ-continuous): $\xi(A) = 0 \Rightarrow \rho(A) = 0$.

Consider mappings $\nu\colon \Omega \ni x \to \nu(x) \in rca(K)$ assigning to the points from the domain of definition of $u \in \mathscr{U}$ the measures on the range of u. We want them to be weakly* measurable and such that the essential supremum of the set $\{\|\nu(x)\|_{\mathrm{rca}(K)} : x \in \Omega\}$ is finite. Then ν is a weakly* measurable mapping if for any $\beta \in C(K)$ the function

$$x \mapsto \int_K \beta(k)(\nu(x))(dk) = \langle \nu(x), \beta \rangle$$

is Borel measurable.

The set of all weakly*-measurable mappings from $L^\infty_{w^*}(\Omega, \mathrm{rca}(K))$, such that their values belong to $rca^1(K)$, is called the set of *Young measures*. It is denoted by $\mathscr{Y}(\Omega, K)$

$$\mathscr{Y}(\Omega,K) := \left\{ \nu = (\nu(x)) \in L^\infty_{w^*}(\Omega, \mathrm{rca}(K)) : \nu_x \in \mathrm{rca}^1(K) \text{ for a.a } x \in \Omega \right\}.$$

We will write ν_x or $(\nu_x)_{x\in\Omega}$ instead of $\nu(x)$.

One of the corollaries of the basic Theorem 3.6 in [5] (see also Theorem 2 in [16]) states that with any $u \in \mathscr{U}$ we can associate a Young measure $\nu^u \in \mathscr{Y}(\Omega, K)$. If the Young measure $(\nu^u_x)_{x\in\Omega}$ does not depend on the parameter $x \in \Omega$, it is called *homogeneous*. More precisely, according to the Convention 3.1, the Theorem 3.6 in [5] and the Theorem 3.1 in [17], we will use the following definition of the homogeneous Young measure.

Definition 1 (i) We say that a mapping $\nu \in \mathscr{Y}(\Omega, K)$ is a *homogeneous Young measure* if it is constant on Ω;

(ii) let ν^u be a Young measure associated with a Borel function $u\colon \Omega \to K$. We say that $\nu^u \in \mathscr{Y}(\Omega, K)$ is a homogeneous Young measure if it is constant on Ω and is an image of the measure $\frac{1}{M} d\mu$ under u, i.e. $\nu^u = \frac{1}{M} d\mu \circ u^{-1}$. □

2.2. Young measures associated with particular classes of functions

We divide the set Ω into n subsets $\Omega_1, \Omega_2, \dots, \Omega_n$, forming an open partition of this set. We assume that the Lebesgue measure of the set Ω_i is positive and is equal to m_i, $i \in I := \{1, 2, \dots, n\}$. Then, obviously, $\sum_{i=1}^{n} m_i = M$.

If a function $u\colon \Omega \to K$ is piecewise constant and takes value p_i on the set Ω_i, $i = 1, 2, \dots, n$, then a Young measure ν^u associated with this function is homogeneous and is of the form

$$\nu^u = \frac{1}{M} \sum_{i=1}^{n} m_i \delta_{p_i}. \tag{1}$$

Let us now consider functions $u_i\colon \Omega_i \to K \subset \mathbb{R}^d$ with inverses u_i^{-1} that are continuously differentiable on $u(\Omega_i)$ and let $K_i := \overline{u_i(\Omega_i)}$ be compact. We denote the Jacobian of u_i^{-1} by $J_{u_i^{-1}}$. Denoting by $\mathbf{1}_A$ the characteristic function of a set A, let us call a function u an *m-oscillating function* if it is of the form

$$u(x) = \sum_i u_i(x) \mathbf{1}_{\Omega_i}(x). \tag{2}$$

The letter 'm' in the above definition refers to the fact that in the one dimensional case the functions u_i and u_i^{-1} are strictly monotonic ones. A Young measure ν^u associated with an m-oscillating function is a homogeneous Young measure that is absolutely continuous with respect to the Lebesgue measure dy on the set K. Its density g is given by the formula

$$g(y) = \frac{1}{M} \sum_{\{i: y \in K_i\}} |J_{u_i^{-1}}(y)|. \tag{3}$$

Using the facts that a finite sum of measurable maps is measurable and that the absolutely continuous measure of a countable set is zero, we can 'unify' the above two cases as follows. Let the sets I_1, I_2 form a partition of the set I of indices. Let the function $u\colon \mathbb{R}^d \supset \Omega \to K \subset \mathbb{R}^d$ be of the form $u = u^c + d^d$, where

- u^c is piecewise constant on a set $\bigcup_{i \in I_1} \Omega_i$ and takes respectively the value p_i on the set Ω_i, $i \in I_1$;
- u^d is piecewise invertible on a set $\bigcup_{i \in I_2} \Omega_i$, such that the inverse of a function u_i^d is continuously differentiable on a set $K_i := \overline{u_i^d(\Omega_i)}$, $i \in I_2$.

Then the Young measure ν^u associated with u is homogeneous and it is a mixed probability distribution of the form

$$\nu^u = \frac{1}{M}\Big(\sum_{\{i\colon\, y \in K_i \setminus p_i\}} |J_{(u_i^d)^{-1}}(y)|dy + \sum_{i \in I_1} m_i \delta_{p_i} \Big).$$

Taking now the function u as a function generating a sequence of oscillating functions, we obtain a sequence (u_n). To each element of u_n there is associated the same Young measure of the same form as above.

Consider, for a one dimensional illustration of the above, a function u given by the formula

$$\begin{aligned} u(x) &:= \sqrt{x} \cdot \mathbf{1}_{(0,9)}(x) + \mathbf{1}_{[9,12)\cup[19,24)}(x) + \\ &\quad + (x-12) \cdot \mathbf{1}_{[12,14)}(x) + 4 \cdot \mathbf{1}_{[14,17)}(x) + 3 \cdot \mathbf{1}_{[17,19)}(x) + (\tfrac{1}{3}x - 8) \cdot \mathbf{1}_{[24,30)}(x). \end{aligned}$$

Here, we have $\Omega = (0,30)$ and $K = [0,3] \cup \{4\}$ and

$$\nu^u = \frac{1}{30} \cdot \begin{cases} (2y+4)dy & \text{for } y \in [0,2] \setminus \{1\} \\ 8\delta_1 & \text{for } y = 1 \\ 2ydy & \text{for } y \in (2,3) \\ 2\delta_3 & \text{for } y = 3 \\ 3\delta_4 & \text{for } y = 4 \end{cases} .$$

In particular, we have

$$\nu^u(K) = \frac{1}{30}\Big(\int_0^2 (2y+4)dy + \int_2^3 2ydy + 8 + 2 + 3 \Big) = 1,$$

which shows that ν^u is a probability measure on K.

3. Total slope of an m-oscillating function

The equation (3) shows that the density of an m-oscillating function u depends only (apart from the Lebesgue measure of Ω) on the 'steepness' of the graph of u. This suggests introducing the following notion.

Definition 2 Let an m-oscillating function u be given by the equation (2). The total slope Jt_u of f is defined by

$$Jt_u(y) := \sum_{\{i: y \in K_i\}} |J_{u_i^{-1}}(y)|.$$

Example 1 Consider the following examples from Section 3.1.b in [5].

(i) a sequence (u_n) of m-oscillating nonperiodic functions, where for each $n \in \mathbb{N}$ we have

$$u_n(x) := \begin{cases} \big(x(n+k-1)-k+1\big)\frac{n+k}{n}, & x \in \left(\frac{k-1}{n+k-1}, \frac{k}{n+k}\right),\ k \in \mathbb{N} \text{ odd}, \\ \big(k-x(n+k)\big)\frac{n+k-1}{n}, & x \in \left[\frac{k-1}{n+k-1}, \frac{k}{n+k}\right),\ k \in \mathbb{N} \text{ even}. \end{cases}$$

Then, the sequence (Jt_{u_n}) of the respective total slopes is constant with each element equal to 1. This means that the Young measure associated with each u_n (and therefore the Young measure generated by the sequence (u_n)) is a homogeneous one, and it is absolutely continuous with respect to the Lebesgue measure on $[0,1]$ with a density that is equal to 1 a.e;

(ii) let $u_n(t) = \sin(2\pi nt)$, $t \in (0,1)$, $n \in \mathbb{N}$. Then

$$\forall n \in \mathbb{N} \quad Jt_{u_n}(y) = \frac{1}{\pi\sqrt{1-y^2}},$$

and thus the Young measure associated with each u_n is homogeneous and absolutely continuous with respect to the Lebesgue measure. Its density is equal to Jt_{u_n}. □

Recall that the subset of a normed space is *relatively weakly compact* if its closure in a weak topology of this space is a compact set in this topology.

The next result follows from the two facts: the first is, that a Young measure associated with an *m*-oscillating function is homogeneous and has a density (see the equation (3)), while the second fact is a Theorem 1.64 in [4].

Theorem 1 *Consider the set $\mathscr{A}$ of m-oscillating functions defined on a nonempty, bounded open set $\Omega \subset \mathbb{R}^d$ of a positive Lebesgue measure, having values in a compact set $K \subset \mathbb{R}^d$. Then the set of the Young measures associated with the elements of $\mathscr{A}$ is relatively weakly compact.* □

4. Weak sequential convergence of functions and measures

We begin with recalling classical theorems concerning weak sequential convergence of functions and measures that will be needed in the sequel. The expression 'weak convergence of the sequence of measures' will be meant as the 'weak convergence of the sequence of measures as elements of the Banach space $\mathrm{rca}(K)$' (with the total variation norm). The term 'convergence' is always understood as 'sequential convergence'.

Let $(X, \mathscr{A}, \rho)$ be a measure space, and consider a sequence (u_n) of scalar functions defined on X and integrable with respect to the measure ρ (that is, $\forall n \in \mathbb{N}\; u_n \in L^1_\rho(X)$) and a function $u \in L^1_\rho(X)$. Recall that (u_n) converges weakly to u if $\forall g \in L^\infty_\rho(X)$ there holds

$$\lim_{n\to\infty} \int_X u_n g d\rho = \int_X u g d\rho.$$

The following theorem characterizes weak L^1 convergence of functions and weak convergence of measures. We refer the reader to [4, 18].

Theorem 2 *(a) (J. Dieudonné, 1957) Let X be a locally compact Hausdorff space and $(X, \mathscr{A}, \rho)$ – a measure space with ρ regular. A sequence $(u_n) \subset L^1_\rho(X)$ converges weakly to some $u \in L^1_\rho(X)$ if and only if $\forall A \in \mathscr{A}$ the limit*

$$\lim_{n\to\infty} \int_A u_n d\rho$$

exists and is finite;

(b) let X be a locally compact Hausdorff space, and denote by $\mathscr{B}(X)$ the σ-algebra of Borel subsets of X. A sequence (ρ_n) of scalar measures on $\mathscr{B}(X)$ converges weakly to some scalar measure ρ on $\mathscr{B}(X)$ if and only if $\forall A \in \mathscr{B}(X)$ the limit

$$\lim_{n\to\infty} \rho_n(A)$$

exists and is finite;

(c) (Vitali-Hahn-Saks theorem) let $(X, \mathscr{A}, \mu)$ be a measure space with μ nonnegative finite, and let (ρ_n) be a sequence of μ-continuous scalar measures on $\mathscr{A}$. If for any $A \in \mathscr{A}$ the limit $\lim_{n\to\infty} \rho_n(A)$ exists, then the formula:

$$\forall A \in \mathscr{A} \quad \rho(A) := \lim_{n\to\infty} \rho_n(A)$$

defines a μ-continuous scalar measure on $\mathscr{A}$. □

Corollary 1 *(a) Let (ρ_n) be a sequence of measures having respective densities f_n, $n \in \mathbb{N}$. Then the sequence (f_n) is weakly convergent in $L^1(X)$ to some function h if and only if the sequence (ρ_n) is weakly convergent to some measure η;*

(b) assume additionally, that $X \subset \mathbb{R}^l$ is compact, and let (ρ_n) be a sequence of homogeneous Young measures having respective densities f_n. Then the sequence (f_n) is weakly convergent in $L^1(X)$ to some function h if and only if the sequence (ρ_n) is weakly convergent to some measure η. □

Lemma 1 *Let (ρ_n) be a sequence of homogeneous Young measures on the set K, having respective densities f_n with respect to the Lebesgue measure dy on K. If the sequence (f_n) is weakly convergent in $L^1(K)$ to certain function h, then the sequence (ρ_n) is weakly convergent in $rca(K)$ to a measure η. This measure is a homogeneous Young measure with density h.* □

PROOF By the parts (a) and (b) of the Theorem 2 for any $A \in \mathscr{B}(K)$, the limit $\lim\limits_{n\to\infty} \rho_n(A) = \lim\limits_{n\to\infty} \int\limits_A f_n(y)dy$ exists and is finite. This means that the sequence (ρ_n) is weakly convergent to some measure η, which by the Vitali-Hahn-Saks theorem has a density with respect to the Lebesgue measure on K. Denote this density with the letter r. Choose and fix $A \in \mathscr{B}(K)$ and consider an inequality

$$\Big|\int\limits_A r(y)dy - \int\limits_A h(y)dy\Big| \le \Big|\int\limits_A r(y)dy - \eta(A)\Big| + \big|\eta(A) - \rho_n(A)\big| + \\ + \Big|\rho_n(A) - \int\limits_A f_n(y)dy\Big| + \Big|\int\limits_A f_n(y)dy - \int\limits_A h(y)dy\Big|.$$

The first and third terms on the right-hand-side of the above inequality vanish while the second and fourth ones tend to zero as $n \to \infty$. Thus $h = r$ dy-almost everywhere on K.

Now from the inequality

$$\Big|1 - \int\limits_K h(y)dy\Big| \le \Big|1 - \int\limits_K f_n(y)dy\Big| + \Big|\int\limits_K f_n(y)dy - \int\limits_K h(y)dy\Big|$$

it follows that η is a probability measure on K. Since a weak limit (whenever it exists) is unique, then a mapping $\Omega \ni x \to \eta \in rca(K)$ is constant, so it is weakly* measurable. Thus, η is a homogeneous Young measure on K and has the density h. ■

Consider again the examples from Section 3.1.b in [5] of the sequences of m-oscillating functions and the Young measures they generate. As it is already known, the Young measures associated with each element of the particular sequence form a constant, hence trivially (weakly) convergent, sequence. It follows from the fact that the sequence of the total slopes of these m-oscillating functions is constant. It is a special case of a more general situation described in the next result.

Theorem 3 *Consider a sequence (u_n) of m-oscillating functions. Assume further that the sequence (Jt_{u_n}) of respective total slopes is monotonic almost everywhere with respect to the Lebesgue measure dy on K. Then the sequence (ρ_n) of Young measures associated with the functions u_n is weakly convergent to the homogeneous Young measure ρ. Moreover, the measure ρ has a density that is equal to the weak L^1 limit of the sequence of densities of the Young measures associated with the functions u_n.* □

PROOF An element u_n of the sequence (u_n) is of the form

$$u_n = \sum_{i=1}^{l(n)} u_i^n(x)\mathbf{1}_{\Omega_i^n}(x),$$

where $\{\Omega_1^n, \dots, \Omega_{l(n)}^n\}$ is an open partition of the set Ω determined by u_n. This means in particular that the set $K_i^n := \overline{u_i^n(\Omega_i^n)}$ is compact and $\bigcup_{i=1}^{l(n)} K_i^n = K$.

The total slope of this function is given by the formula

$$Jt_{u_n}(y) = \sum_{\{i: y \in K_i^n\}} |J_{(u_i^n)^{-1}}(y)|$$

and thus the Young measure ρ_n associated with the function u_n, homogeneous and absolutely continuous with respect to the Lebesgue measure dy on K, has a density equal to $\frac{1}{M} Jt_{u_n}$.

We can assume that the sequence (Jt_{u_n}) is nondecreasing. Then for any $m, n \in \mathbb{N}$, $m \leq n$ and any $A \in \mathscr{B}(K)$ there holds an inequality

$$\int_A Jt_{u_m} dy \leq \int_A Jt_{u_n} dy$$

which means that for any fixed $A \in \mathscr{B}(K)$ the sequence $\left(\int_A Jt_{u_n} dy \right)$ is convergent. The result now follows from Theorem 2 and Lemma 1. ■

5. Conclusions

Homogeneous Young measures are first examples of Young measures. Although they are the simplest in the whole set of Young measures associated with bounded functions, they are important because they appear, for example, in optimization, when searching the minimizers of the multiwell problems, see, for instance [7], [5] or [3] and the references therein. The majority of the specific examples of Young measures that can be found in the literature are homogeneous ones, see, for example, [6], examples 4.8, 4.9; [7], paragraphs 3.2, 3.3, 4.6, 6.2; [5] section 3.1.b; [4], example 3.44; [3], paragraphs 2.3, 4.5, 5.4.

Usually, calculating an explicit form of generalized limit of a function sequence, which in our case is a measure-valued mapping, is difficult and requires advanced methods of functional analysis. In [16] and this article, we propose a method that generalizes and simplifies, at least to some extent, calculating those limits in some particular cases that can be met in applications.

When dealing with sequence of fast oscillating functions, it often seems useful not to consider the generalized limit of this sequence, but to consider the relevant sequence of Young measures. The elements of this sequence are the Young measures associated with the respective elements of the function sequence of interest. Having this sequence obtained, one can investigate the existence and the form of the weak (or weak*) limits of this sequence of measures.

References

[1] Ball, J.M. (1989). A version of the fundamental theorem for Young measures. *PDEs and Continuum Models of Phase Transitions*, Lecture Notes in Physics, (344), Springer, 207-215.

[2] Valadier, M. (1990). Young measures. *Methods of Nonconvex Analysis*. Lecture Notes in Mathematics, (1446), Springer, 152-188.

[3] Pedregal, P. (2000). *Variational Methods in Nonlinear Elasticity*. Society for Industrial and Applied Mathematics.

[4] Florescu, L.C., & Godet-Thobie, Ch. (2012). *Young Measures and Compactness in Measure Spaces*. Walter de Gruyter GmbH & Co. KG.

[5] Roubíček, T. (2020). *Relaxation in Optimization Theory and Variational Calculus*, second edition. Walter de Gruyter.

[6] Rindler, P. (2018). *Calculus of Variations*. Springer International Publishing AG, part of Springer Nature.

[7] Müller, S. (1999). Variational Models for Microstructure and Phase Transitions. *Calculus of variations and geometric evolution problems*, Lecture Notes in Mathematics, (1713), Springer, 85-210.

[8] Grzybowski, A.Z., & Puchała, P. (2017). Monte Carlo simulation in the evaluation of the Young functional values. *IEEE 14th International Scientific Conference on Informatic*, 221-226, DOI: 10.1109/INFORMATICS.2017.8327250.

[9] Grzybowski, A.Z., & Puchała, P. (2018). On Young functionals related to certain class of rapidly oscillating sequences. *IAENG Int. J. Appl. Math.*, *48*(4), 381-386.

[10] Grzybowski, A.Z., & Puchała, P. (2019). Classical Young Measures Generated by Oscillating Sequences with Uniform Representation. *Transactions on Engineering Technologies. WCECS 2017*, Springer, 1-11.

[11] Jisha, C.R. (2022). Q-functional applications. *Mathematica Applicanda*, *50*(2), 217-233.

[12] Jisha, C.R. (2022). Q-measure-valued solution of a hyperbolic partial differential equation. *PDEs Appl. Math.*, *6*, 1-9.

[13] Emmrich, E., & Puhst, D. (2015). Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics. *Nonlinearity*, *28*, 285-307.

[14] Fjordholm, U. S., Mishra, S., & Tadmor. E. (2016). On the computation of measure-valued solutions. *Acta Numerica*, *25*, 567-679.

[15] Gallenmüller, D., & Wiedemann, E. (2021). On the selection of measure-valued solutions for the isentropic Euler system. *J. Diff. Eq.*, *271*, 979-1006.

[16] Puchała, P. (2023). Certain convergence results for homogeneous singular Young measures. *J. Appl. Math. Comput. Mech.*, *22*(4), 44-52.

[17] Puchała, P. (2017). A simple characterization of homogeneous Young measures and weak L^1 convergence of their densities. *Optimization*, *66*(2), 197-203.

[18] Benedetto, J.J., & Czaja, W. (2009). *Integration and Modern Analysis*. Boston: Birkhäuser.