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INVERSE LAPLACE TRANSFORM FOR A NEW CLASS OF FUNCTIONS AND ITS APPLICATION TO TIME-FRACTIONAL DIFFUSION-WAVE EQUATIONS

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Abstract. This paper investigates the inverse Laplace transform of a certain class of functions. This inverse Laplace transform is obtained in the form of an infinite series of the three--parameter Mittag-Leffler function. Additionally, we found the sum of an infinite series of Mittag-Leffler functions with three parameters in terms of the Wright function. As an application, we get an exact solution of the time-fractional diffusion-wave equation with the Hilfer-Prabhakar time-fractional derivative using Laplace and Fourier transforms.

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1. Introduction

The Mittag-Leffler function is a vital function that is widely used in fractional calculus. Just as the exponential function that plays an important role in the solution of the integer order differential equations, the Mittag-Leffler function appears in several solutions of problems in fractional order differential and integral equations [1, 2]. These solutions can be rewritten with respect to elementary functions or other special functions, such as the Wright function [1]. There are many generalizations of the Mittag-Leffler function, but in this paper, we are interested in only one of them, called the three-parameter Mittag-Leffler function [3], which is given in the following definition:

Definition 1. [3, 4] The Mittag-Leffler function with three parameters is given by

$$E_{\rho,\mu}^{\gamma}(z)=\sum_{j=0}^{\infty}\frac{\Gamma(\gamma+j)}{\Gamma(\gamma)\Gamma(\rho j+\mu)}\frac{z^j}{j!},\quad Re(\gamma)>0,\ \ Re(\rho)>0,\ \ Re(\mu)>0. \tag{1}$$

When $\gamma = 1$, we obtain the two-parameter Mittag-Leffler function $E_{\rho,\mu}$ and with $\gamma = \mu = 1$ we get the one-parameter Mittag-Leffler function E_ρ. In the case of $\gamma = \mu = \rho = 1$, Eq. (1) reduces to the exponential function.

The Wright and Minardi functions will be used in the following sections, so we give their definitions as follows:

Definition 2 [5] The Wright function $W(z;a,b)$ is defined as

$$W(z;a,b) = \sum_{j=0}^{\infty} \frac{z^j}{j!\,\Gamma(aj+b)}, a > -1. \tag{2}$$

Definition 3 [6] The Mainardi function $M(z;a)$ is defined as

$$M(z;a) = W(-z;-a,1-a) = \sum_{j=0}^{\infty} \frac{(-1)^j z^j}{j!\,\Gamma(-aj+1-a)}, \quad 0 < a < 1. \tag{3}$$

The following section presents the theorems related to the three-parameter Mittag--Leffler and Wright functions. The last section illustrates an application of our theorems.

2. Basic theorems

In this section, we derive two theorems involving the three-parameter Mittag--Leffler function. A subsequent corollary will be discussed after each theorem.

Theorem 1 The inverse Laplace transform of the function $s^{\frac{h}{2}-1}(s^\rho - \omega)^{\frac{\gamma}{2}} \mathrm{e}^{-ys^{\frac{h}{2}}(s^\rho-\omega)^{\frac{\gamma}{2}}}$ is

$$L^{-1}\left\{s^{\frac{h}{2}-1}(s^\rho - \omega)^{\frac{\gamma}{2}} \mathrm{e}^{-ys^{\frac{h}{2}}(s^\rho-\omega)^{\frac{\gamma}{2}}}\right\} = \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\ t^{\frac{(h+\rho\gamma)}{2}(j+1)}} E_{\rho,1-\frac{(h+\rho\gamma)}{2}(j+1)}^{-\frac{\gamma}{2}(j+1)}(\omega t^\rho), \tag{4}$$

where ρ, y, ω are arbitrary positive constants, $0 < h < 2$ and $\gamma \geq 0$.

Proof

The inversion formula of the Laplace transform is given by [6]

$$L_1 = L^{-1}\left\{s^{\frac{h}{2}-1}(s^\rho - \omega)^{\frac{\gamma}{2}} \mathrm{e}^{-ys^{\frac{h}{2}}(s^\rho-\omega)^{\frac{\gamma}{2}}}\right\} = \frac{1}{2\pi i}\int_{Br} s^{\frac{h}{2}-1}(s^\rho - \omega)^{\frac{\gamma}{2}} \mathrm{e}^{st-ys^{\frac{h}{2}}(s^\rho-\omega)^{\frac{\gamma}{2}}} ds.$$

In the above complex integral, the Bromwich path Br can be deformed into the Hankel path Ha. Assuming that $\delta = st$, we get

$$L_1 = \frac{1}{2\pi i\left(t^{\frac{h}{2}}\right)} \int_{Ha} \delta^{\frac{h}{2}-1}\left(\left(\frac{\delta}{t}\right)^{\rho} - \omega\right)^{\frac{\gamma}{2}} \mathrm{e}^{\delta - y\left(\frac{\delta}{t}\right)^{\frac{h}{2}}\left(\left(\frac{\delta}{t}\right)^{\rho}-\omega\right)^{\frac{\gamma}{2}}} d\delta.$$

Utilizing the Taylor series of the exponential function, we get

$$L_1 = \frac{1}{2\pi i\left(t^{\frac{h}{2}}\right)} \int_{Ha} \delta^{\frac{h}{2}-1}\left(\left(\frac{\delta}{t}\right)^{\rho} - \omega\right)^{\frac{\gamma}{2}} \mathrm{e}^{\delta} \sum_{j=0}^{\infty}\left(\frac{(-1)^j}{j!} y^j \left(\frac{\delta}{t}\right)^{\frac{jh}{2}}\left(\left(\frac{\delta}{t}\right)^{\rho} - \omega\right)^{\frac{j\gamma}{2}}\right) d\delta$$

$$= \frac{1}{2\pi i\left(t^{\frac{h}{2}}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{jh}{2}}} \int_{Ha} \frac{\mathrm{e}^{\delta}\, \delta^{\frac{h}{2}(j+1)+\frac{\rho\gamma}{2}(j+1)-1}}{t^{\frac{\rho\gamma}{2}(j+1)}} \left(1 - \omega\left(\frac{t}{\delta}\right)^{\rho}\right)^{\frac{\gamma}{2}(j+1)} d\delta.$$

Using the Taylor series expansion

$$\left(1 - \omega\left(\frac{t}{\delta}\right)^{\rho}\right)^{\frac{\gamma}{2}(j+1)} = \sum_{i=0}^{\infty} \frac{(-1)^i\, \Gamma\left(\frac{\gamma}{2}(j+1)+1\right)}{\Gamma(i+1)\,\Gamma\left(\frac{\gamma}{2}(j+1)-i+1\right)}\left(\frac{\omega t^{\rho}}{\delta^{\rho}}\right)^i, \quad \left|\omega\left(\frac{t}{\delta}\right)^{\rho}\right| < 1,$$

we get

$$L_1 = \frac{1}{2\pi i\left(t^{\frac{h}{2}}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{jh}{2}+\frac{\rho\gamma}{2}(j+1)}} \sum_{i=0}^{\infty} \frac{(-1)^i\, \Gamma\left(\frac{\gamma}{2}(j+1)+1\right)(\omega t^{\rho})^i}{\Gamma(i+1)\,\Gamma\left(\frac{\gamma}{2}(j+1)-i+1\right)} \int_{Ha} \mathrm{e}^{\delta}\, \delta^{\chi}\, d\delta,$$

where, $\chi = \frac{h}{2}(j+1) + \frac{\rho\gamma}{2}(j+1) - \rho i - 1$.
Using the relation [6], $\frac{1}{\Gamma(a)} = \frac{1}{2\pi i}\int_{Ha} \mathrm{e}^{\delta}\, \delta^{-a}\, d\delta$, we get

$$L_1 = \frac{1}{\left(t^{\frac{h}{2}}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{jh}{2}+\frac{\rho\gamma}{2}(j+1)}} \sum_{i=0}^{\infty} \frac{(-1)^i\, \Gamma\left(\frac{\gamma}{2}(j+1)+1\right)(\omega t^{\rho})^i}{\Gamma(i+1)\,\Gamma\left(\frac{\gamma}{2}(j+1)-i+1\right)\Gamma\left(\rho i + 1 - \frac{(h+\rho\gamma)}{2}(j+1)\right)}.$$

Since [7], $\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-i+1)} = (-1)^i(-\lambda)_i$, we get

$$L_1 = \frac{1}{\left(t^{\frac{h}{2}}\right)} \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{jh}{2}+\frac{\rho\gamma}{2}(j+1)}} \sum_{i=0}^{\infty} \frac{\left(-\frac{\gamma}{2}(j+1)\right)_i (\omega t^{\rho})^i}{\Gamma(i+1)\,\Gamma\left(\rho i + 1 - \frac{(h+\rho\gamma)}{2}(j+1)\right)}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{(h+\rho\gamma)}{2}(j+1)}} E_{\rho,1-\frac{(h+\rho\gamma)}{2}(j+1)}^{-\frac{\gamma}{2}(j+1)}(\omega t^{\rho}).$$

Corollary 1 [6] The inverse Laplace transform of the function $s^{\frac{h}{2}-1} e^{-ys^{\frac{h}{2}}}$ is given by

$$L^{-1}\left\{s^{\frac{h}{2}-1} \mathrm{e}^{-ys^{\frac{h}{2}}}\right\} = \frac{1}{t^{\frac{h}{2}}} M\left(yt^{-\frac{h}{2}}; \frac{h}{2}\right), \tag{5}$$

where y is an arbitrary positive constant and $0 < h < 2$.

Proof

From Theorem 1, we can put $\gamma = 0$, to get

$$L^{-1}\left\{s^{\frac{h}{2}-1} \mathrm{e}^{-ys^{\frac{h}{2}}}\right\} = \sum_{j=0}^{\infty} \frac{(-1)^j y^j}{j!\, t^{\frac{h}{2}(j+1)}} E_{\rho,1-\frac{h}{2}(j+1)}^{0}(\omega t^{\rho}).$$

From [8], we have $E_{\rho,\mu}^{0}(z) = \frac{1}{\Gamma(\mu)}$. So, we get

$$L^{-1}\left\{s^{\frac{h}{2}-1} \mathrm{e}^{-ys^{\frac{h}{2}}}\right\} = \frac{1}{t^{\frac{h}{2}}} \sum_{j=0}^{\infty} \frac{(-1)^j \left(yt^{-\frac{h}{2}}\right)^j}{j!\, \Gamma\left(-\frac{hj}{2} + 1 - \frac{h}{2}\right)} = \frac{1}{t^{\frac{h}{2}}} M\left(yt^{-\frac{h}{2}}; \frac{h}{2}\right).$$

Theorem 2 For $0 < \mu < 1$, we have

$$\sum_{q=0}^{\infty} \frac{(-1)^q y^q}{q!} t^{-\mu(q+1)} E_{\mu,1-\mu(q+1)}^{-(q+1)}(\omega t^{\mu}) \\ = e^{\omega y}\left(t^{-\mu} W(-yt^{-\mu}; -\mu, 1-\mu) - \omega W(-yt^{-\mu}; -\mu, 1)\right), \tag{6}$$

where y and ω are arbitrary constants.

Proof

$$R.H.S = \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(t^{-\mu}\sum_{i=0}^{\infty}\frac{(-y)^i t^{-\mu i}}{i!\,\Gamma(-\mu i+1-\mu)} - \omega\sum_{i=0}^{\infty}\frac{(-y)^i t^{-\mu i}}{i!\,\Gamma(-\mu i+1)}\right)$$

$$= \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{i=0}^{\infty}\frac{(-y)^i t^{-\mu(i+1)}}{i!\,\Gamma(-\mu(i+1)+1)} - \omega\sum_{i=0}^{\infty}\frac{(-y)^i t^{-\mu i}}{i!\,\Gamma(-\mu i+1)}\right).$$

Assuming $i+1=r$, we get

$$R.H.S = \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{r=1}^{\infty}\frac{(-y)^{r-1} t^{-\mu r}}{(r-1)!\,\Gamma(-\mu r+1)} - \omega\sum_{i=0}^{\infty}\frac{(-y)^i t^{-\mu i}}{i!\,\Gamma(-\mu i+1)}\right)$$

$$= \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{r=1}^{\infty}\frac{(-y)^{r-1} t^{-\mu r}}{(r-1)!\,\Gamma(-\mu r+1)} - \omega\sum_{r=1}^{\infty}\frac{(-y)^r t^{-\mu r}}{r!\,\Gamma(-\mu r+1)} - \omega\right)$$

$$= \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{r=1}^{\infty}\frac{(-y)^{r-1} t^{-\mu r}(r+\omega y)}{(r)!\,\Gamma(-\mu r+1)} - \omega\right)$$

$$= \left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{r=0}^{\infty}\frac{(-y)^{r-1} t^{-\mu r}(r+\omega y)}{(r)!\,\Gamma(-\mu r+1)}\right)$$

$$L.H.S = \sum_{q=0}^{\infty}\frac{(-1)^q y^q}{q!}\,t^{-\mu(q+1)}\,E_{\mu,1-\mu(q+1)}^{-(q+1)}(\omega t^{\mu}).$$

It is well known that [8]

$$E_{\alpha,\beta}^{-n}(t) = \sum_{p=0}^{n}\frac{(-1)^p\binom{n}{p}}{\Gamma(\alpha p+\beta)}t^p\,, \;\; n\in N.$$

So, we have

$$L.H.S=\sum_{q=0}^{\infty}\left(\frac{(-1)^q y^q t^{-\mu(q+1)}}{q!}\sum_{p=0}^{q+1}\frac{(-1)^p\binom{q+1}{p}(\omega t^{\mu})^p}{\Gamma\left(\mu p+1-\mu(q+1)\right)}\right)=$$

$$=\sum_{q=0}^{\infty}\sum_{p=0}^{q+1}\frac{(-1)^{q-p}(q+1)y^q\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}$$

$$=\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\frac{(-1)^{q-p}(q+1)y^q\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}-\frac{y^q\omega^{q+1}}{q!}\right)$$

$$=\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\frac{(-1)^{q-p}(q-p+p+1)y^p y^{q-p}\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}\right)-\omega\sum_{q=0}^{\infty}\frac{y^q\omega^q}{q!}$$

$$=\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\frac{(-1)^{q-p}(q-p)y^p y^{q-p}\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}\right)$$

$$+\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\frac{(-1)^{q-p}\,p\,y^p y^{q-p}\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}\right)$$

$$+\sum_{q=0}^{\infty}\left(\sum_{p=0}^{q}\frac{(-1)^{q-p}\,y^p y^{q-p}\omega^p t^{-\mu(q-p+1)}}{p!\,(q-p+1)!\,\Gamma(-\mu(q-p+1)+1)}\right)-\omega\sum_{q=0}^{\infty}\frac{y^q\omega^q}{q!}$$

$$=\left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{p(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)$$

$$+\left(\sum_{q=0}^{\infty}\frac{(q)(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)$$

$$+\left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)-\omega\sum_{q=0}^{\infty}\frac{y^q\omega^q}{q!}$$

$$=\left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{p(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)$$

$$+\left(\omega y\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)$$

$$+\left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}\right)-\omega\sum_{q=0}^{\infty}\frac{y^q\omega^q}{q!}$$

$$=\left(\sum_{q=0}^{\infty}\frac{(\omega y)^q}{q!}\right)\left(\sum_{p=0}^{\infty}\frac{(\omega y+p+1)(-1)^p y^p t^{-\mu(p+1)}}{(p+1)!\,\Gamma(-\mu(p+1)+1)}-\omega\right).$$

Let $r = p + 1$, to get

$$L.H.S = \left(\sum_{q=0}^{\infty} \frac{(\omega y)^q}{q!}\right)\left(\sum_{r=1}^{\infty} \frac{(r+\omega y)(-1)^{r-1} y^{r-1} t^{-\mu r}}{r!\ \Gamma(-\mu r + 1)} - \omega\right)$$

$$= \left(\sum_{q=0}^{\infty} \frac{(\omega y)^q}{q!}\right)\left(\sum_{r=0}^{\infty} \frac{(r+\omega y)(-y)^{r-1} t^{-\mu r}}{r!\ \Gamma(-\mu r + 1)}\right) = R.H.S$$

Corollary 2 For the arbitrary constants y and ω, we have

$$\sum_{l=0}^{\infty} \frac{(-1)^l y^l}{l!} t^{-\frac{1}{2}(l+1)} E_{\frac{1}{2},1-\frac{1}{2}(l+1)}^{-(l+1)}\left(\omega t^{\frac{1}{2}}\right) = e^{\omega y}\left(\frac{1}{\sqrt{\pi t}} e^{-\frac{y^2}{4t}} - \omega\, erfc\left(\frac{y}{2\sqrt{t}}\right)\right), \quad (7)$$

where $erfc(z)$ is the complementary error function [9].

Proof

It is known that [10]

$$W\left(-t; -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} e^{-\frac{t^2}{4}},\ \ W\left(-t; -\frac{1}{2}, 1\right) = \mathrm{erfc}\left(\frac{t}{2}\right).$$

Using the above relations and putting $\mu = \frac{1}{2}$ in Theorem 2, we complete the proof.

3. Illustrating application

In this section, we use the Laplace and Fourier transforms with the help of the previous theorems to get an exact solution of the following time-fractional diffusion--wave equation with Hilfer-Prabhakar derivative [10]

$$^{C}D_{\rho,\omega,0^+}^{\gamma,\mu} u(x,t) = a\, u_{xx}, \quad (8)$$

associated with the conditions

$$u(x,0) = P\delta(x),\ \ 0 < \mu \leqslant 2, \quad (9)$$

$$u_t(x,0) = 0,\ \ 1 < \mu \leqslant 2, \quad (10)$$

where $^{C}D_{\rho,\omega,0^+}^{\gamma,\mu} u(x,t)$ is the Hilfer-Prabhakar derivative of order μ that will be defined in Appendix A associated with some of its properties. The time-fractional diffusion-wave equations can be used to model many physical and engineering phenomena such as in electrodynamics [11], wave propagation in viscoelastic media [12], and anomalous diffusion in porous media with fractal structure [13].

Now, we can get the exact solution of Eq. (8) with the conditions (9) and (10) using the Laplace and Fourier transforms as follows:

By using Lemma 1 and then applying the Laplace transform to Eq. (8), we obtain

$$\begin{aligned} s^{\mu}(1-\omega s^{-\rho})^{\gamma}\,\tilde{u}(x,s) - s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}\,u(x,0) \\ - s^{\mu-2}(1-\ \omega s^{-\rho})^{\gamma}\left.\frac{\partial u(x,t)}{\partial t}\right|_{t=0} = a\frac{\partial^2}{\partial x^2}\tilde{u}(x,s). \end{aligned} \tag{11}$$

Then, using the initial conditions (9) and (10), we get

$$s^{\mu}(1-\omega s^{-\rho})^{\gamma}\,\tilde{u}(x,s) - P\delta(x)\,s^{\mu-1}(1-\omega s^{-\rho})^{\gamma} = a\frac{\partial^2}{\partial x^2}\tilde{u}(x,s). \tag{12}$$

Now, applying the Fourier transform to Eq. (12) gives

$$\tilde{u}^*(r,s) = \frac{Ps^{\mu-1}(1-\omega s^{-\rho})^{\gamma}}{s^{\mu}(1-\omega s^{-\rho})^{\gamma} + ar^2}. \tag{13}$$

After that, using Eq. (A6) and then taking the inverse Fourier transform to Eq. (13), we get

$$\begin{aligned} \tilde{u}(x,s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Ps^{\mu-1}(1-\omega s^{-\rho})^{\gamma}}{s^{\mu}(1-\omega s^{-\rho})^{\gamma}+ar^2}\mathrm{e}^{-irx}\,\mathrm{d}r \\ &= \frac{P}{2\sqrt{a}}s^{\frac{1}{2}(\mu-\rho\gamma)-1}(s^{\rho}-\omega)^{\frac{\gamma}{2}}\mathrm{e}^{\frac{-|x|}{\sqrt{a}}s^{\frac{1}{2}(\mu-\rho\gamma)}(s^{\rho}-\omega)^{\frac{\gamma}{2}}}. \end{aligned} \tag{14}$$

Using Theorem 1 by putting $h=\mu-\rho\gamma, y=\frac{|x|}{\sqrt{a}}$, the inverse Laplace transform of Eq. (14) becomes

$$u(x,t) = \frac{P}{2\sqrt{a}}\sum_{k=0}^{\infty}\left(\frac{1}{k!}\left(\frac{-|x|}{\sqrt{a}}\right)^{k} t^{-\frac{\mu}{2}(k+1)}E_{\rho,1-\frac{\mu}{2}(k+1)}^{-\frac{\gamma}{2}(k+1)}(\omega t^{\rho})\right), \tag{15}$$

which is a general solution to Eq. (8)-Eq. (10). We can get a special solution for Eq. (8)-Eq. (10) in the form of the Wright function by using Theorem 2 and setting $\gamma=2$ and $\rho=\frac{\mu}{2}$ in solution (15). In this case, we get

$$u(x,t) = \frac{P}{2\sqrt{a}}\mathrm{e}^{\frac{\omega|x|}{\sqrt{a}}}\left(t^{-\frac{\mu}{2}}W\left(\frac{-|x|}{\sqrt{a}}t^{-\frac{\mu}{2}};\frac{-\mu}{2},1-\frac{\mu}{2}\right) - \omega W\left(\frac{-|x|}{\sqrt{a}}t^{-\frac{\mu}{2}};\frac{-\mu}{2},1\right)\right). \tag{16}$$

4. Conclusions

In this paper, an inverse Laplace transform in the form of an infinite series of the three-parameter Mittag-Leffler functions is derived for a new class of functions, which is given in Theorem 1. This theorem generalizes the work done in [6] as mentioned and reproved in Corollary 1. The inverse Laplace transform obtained in Theorem 1 enabled us to obtain a new closed-form solution (15) of the time--fractional diffusion-wave equation (8)-(10) in the form an infinite series of the three-parameter Mittag-Leffler functions. In Theorem 2, we have obtained the sum of an infinite series of Mittag-Leffler functions with three parameters in terms of the Wright function. The results obtained in Theorem 2 enabled us to get the exact solution (16) of the time-fractional diffusion-wave equation (8)-(10) when $\gamma = 2$ and $\rho = \frac{\mu}{2}$.

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Appendix A

The Hilfer-Prabhakar derivative at $0 < \mu \leq 1$ is given by [3]

$$\begin{aligned} {}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t) &= \boldsymbol{E}^{-\gamma}_{\rho,1-\mu,\omega,0^+}f'(t) \\ &= \int_0^t (t-y)^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega(t-y)^{\rho})f'(y)dy \\ &= f'(t) * t^{-\mu}E^{-\gamma}_{\rho,1-\mu}(\omega t^{\rho}). \end{aligned} \tag{A1}$$

With the help of Eq. (10), the Laplace transform of the operator (A1) is obtained as [3]

$$L\left\{{}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t)\right\} = s^{\mu-1}(1-\omega s^{-\rho})^{\gamma}\big(s\,L\{f(t)\} - f(0)\big). \tag{A2}$$

The Hilfer-Prabhakar derivative at $0 < \mu \leq 2$, is given by [3]

$$\begin{aligned} {}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t) &= \boldsymbol{E}^{-\gamma}_{\rho,2-\mu,\omega,0^+}f''(t) \\ &= \int_0^t (t-y)^{1-\mu}E^{-\gamma}_{\rho,2-\mu}(\omega(t-y)^{\rho})f''(y)dy \\ &= f''(t) * t^{1-\mu}E^{-\gamma}_{\rho,2-\mu}(\omega t^{\rho}). \end{aligned} \tag{A3}$$

With the help of Eq. (10), the Laplace transform of the operator (A3) is obtained as

$$L\left\{{}^{C}D^{\gamma,\mu}_{\rho,\omega,0^+}f(t)\right\} = s^{\mu-2}(1-\omega s^{-\rho})^{\gamma}(s^2\,L\{f(t)\} - sf(0) - f'(0)). \tag{A4}$$

The Fourier transform of a function $\varphi(x)$ is defined by [5]

$$\mathcal{F}\{\varphi(x)\} = \tilde{\varphi}(r) = \int_{-\infty}^{\infty} \varphi(x)e^{irx}dx, \tag{A5}$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}\{\tilde{\varphi}(r)\} = \varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\varphi}(r)e^{-irx}dr. \tag{A6}$$