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A NOTE ON THE MATHEMATICAL MODEL OF MACHINE VIBRATION

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Abstract. In the article, we investigate the stability of a simple mathematical model describing the vibration of a machine. The model is characterized by a nonlinear second-order differential equation with a delay. We present a detailed stability analysis and examination of the equilibrium of this equation, providing insights into the dynamic behavior of the system. The aim of the article is to investigate the effect of body movement on equilibrium stability and to determine the value of the parameter τ at which body vibration occurs. The research used a methodology for solving the given type of nonlinear differential equations, for example, linearization.

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Keywords: *mathematical model, differential equation, vibration, periodic motion*

1. Introduction

The movement of a body on an uneven surface can induce vibrations, a common issue encountered in machines, vehicles, structures, and buildings. As the speed of machinery increases, the forces exciting these vibrations become more pronounced, potentially leading to significant vibration problems. These excitations can arise from various external sources and manifest as periodic or random motions.

This study focuses on the single degree of freedom model of a vibrating system with viscous damping, as presented in [1]. The equation of motion for this system is considered linear. Previous research has explored similar problems, including the investigation of machine tool vibrations in [2], and other relevant mathematical models are discussed in [3] and the references therein. In the book [4], the author took the opportunity to revise, modify, update and expand the material from publication [1]. This book discusses very comprehensively the analysis of the vibration of dynamic systems and then shows how the techniques and results obtained in vibration analysis may be applied to the study of control system dynamics. Vibrational dynamics is also studied in [5-7]. In the article [8], the authors focus on the dynamic modeling of the entire machine tool. Using the finite element method and virtual simulation, they developed analytical models whose parameters are evaluated experimentally. In the article [9], a nonlinear dynamic model is developed to analyze the wear and vibrations.

Delay differential equations offer a deeper understanding of numerous problems across various fields of natural science and engineering technology. They have also been proposed to model commodity cycles in economics. To write this contribution, we drew inspiration from the model presented in [1] (Fig. 1), which does not include delay. However, we decided to investigate the delay differential equation to gain a deeper insight into the dynamics of machine vibrations.

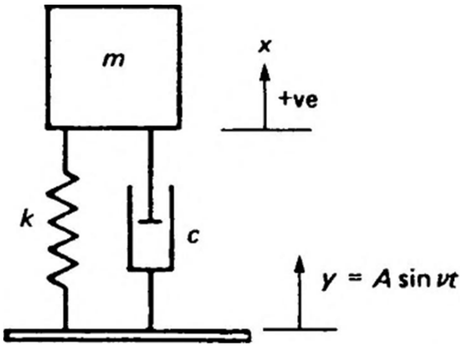

Fig. 1. Single degree of freedom model of a vibrated system with viscous damping

We consider the delay differential equation as follows:

$$mx''(t) = c\left(y'(t) - x'(t-\tau)\right) + k(y(t) - x(t)), \;\; m > 0,$$

$$x''(t) = \frac{c}{m}(y'(t) - x'(t-\tau)) + \frac{k}{m}(y(t) - x(t)).$$

We put $a = c/m$, $b = k/m$, and the equation has a form

$$x''(t) = a\left(y'(t) - x'(t-\tau)\right) + b\left(y(t) - x(t)\right),$$

where a, b, τ are parameters.

The foundation is subjected to vibration $y = \frac{\pi}{2}\sin x$. Then we have the nonlinear delay differential equation

$$x''(t) = a\left(\frac{\pi}{2}x'(t)\cos x(t) - x'(t-\tau)\right) + b\left(\frac{\pi}{2}\sin x(t) - x(t)\right), \;\; t \geq 0. \quad (1)$$

We assume that the parameters $a, b \in (0,\infty)$, $\tau \in [0,\infty)$. It is required to determine the stability of equilibrium and the response x of the body.

We write equation (1) as the equivalent first order nonlinear system

$$\begin{aligned} x'(t) &= a\left(\frac{\pi}{2}\sin x(t) - x(t-\tau)\right) + w(t), \\ w'(t) &= b\left(\frac{\pi}{2}\sin x(t) - x(t)\right), \quad t \geq 0. \end{aligned} \tag{2}$$

We obtain the equilibria of the system (2) from the equations

$$\begin{aligned} a\left(\frac{\pi}{2}\sin x - x\right) + w &= 0, \\ b\left(\frac{\pi}{2}\sin x - x\right) &= 0. \end{aligned}$$

We get $E_0 = (0,0)$, $E_1 = \left(\frac{\pi}{2}, 0\right)$, $E_2 = \left(-\frac{\pi}{2}, 0\right)$. We will investigate the stability of $E_1 = \left(\frac{\pi}{2}, 0\right)$. This can be achieved by linearising the system (2) at the equilibrium E_1. The linearised system has the form

$$\begin{aligned} x'(t) &= -ax(t-\tau) + w(t), \\ w'(t) &= -bx(t), \quad t \geq 0. \end{aligned} \tag{3}$$

Since the system (3) is linear, we set

$$x(t) = re^{\lambda t} \text{ and } x(t-\tau) = re^{\lambda t}e^{-\lambda\tau} = x(t)e^{-\lambda\tau}.$$

We obtain

$$\begin{aligned} x'(t) &= -ae^{-\lambda\tau}x(t) + w(t), \\ w'(t) &= -bx(t), \quad t \geq 0. \end{aligned}$$

The corresponding characteristic equation reads

$$\begin{vmatrix} -ae^{-\lambda\tau} - \lambda & 1 \\ -b & -\lambda \end{vmatrix} = 0,$$

$$\lambda^2 + ae^{-\lambda\tau}\lambda + b = 0. \tag{4}$$

The stability will change if λ reaches the imaginary axis of the complex plane.

2. Theoretical results

In this section, we determine the conditions under which the roots of the characteristic equation are purely imaginary and the conditions under which the equilibrium state changes stability.

Lemma 2.1 Equation (4) has the pure imaginary root $\lambda = i\omega$, $\omega > 0$ for

$$\tau = \tau_n\ ,\ \ n = 0, 1, 2, \ldots,$$

where

$$\tau_0 = \frac{\pi}{2\omega}\ ,\ \ \tau_n = \tau_0 + \frac{2\pi n}{\omega}\ ,\ \ n = 1,2, \ldots,$$

$$\omega = \sqrt{\frac{A + \sqrt{A^2 - 4B}}{2}}\ ,\ \ A = a^2 + 2b,\ \ B = b^2.$$

Proof. We substitute $\lambda = i\omega$, $\omega > 0$ into equation (4),

$$-\omega^2 + ia\omega\, e^{-i\tau\omega} + b = 0,$$

$$-\omega^2 + ia\omega\,(\cos\omega\tau - i\sin\omega\tau) + b = 0,$$

$$-\omega^2 + ia\omega\cos\omega\tau + a\omega\sin\omega\tau + b = 0.$$

By separating the real and imaginary parts, we get

$$a\omega\sin\omega\tau = \omega^2 - b,$$

$$a\omega\cos\omega\tau = 0.$$

Eliminating the trigonometric terms from the equations above yields

$$a^2\omega^2\sin^2\omega\tau = (\omega^2 - b)^2,$$

$$a^2\omega^2\cos^2\omega\tau = 0.$$

we obtain

$$a^2\omega^2\,(\sin^2\omega\tau + \cos^2\omega\tau) = (\omega^2 - b)^2,$$

$$a^2\omega^2\ = \omega^4 - 2b\omega^2 + b^2,$$

$$\omega^4 - (a^2 + 2b)\omega^2 + b^2 = 0.$$

We have

$$\omega^4 - A\omega^2 + B = 0,$$

where $A = a^2 + 2b$, $B = b^2$. Then we get

$$\omega^2 = \frac{A \pm \sqrt{A^2 - 4B}}{2}, \quad \omega = \sqrt{\frac{A \pm \sqrt{A^2 - 4B}}{2}}.$$

Since $a > 0$, $\omega > 0$, then

$$\cos \omega\tau = 0, \text{ and } \cos(\omega\tau_n - 2\pi n) = 0, \quad n = 0, 1, 2, \dots .$$

We get

$$\tau_n = \frac{\pi}{2\omega} + \frac{2\pi n}{\omega}, \quad n = 0, 1, 2, \dots .$$

For $n = 0$

$$\tau_0 = \frac{\pi}{2\omega}, \text{ and } \tau_n = \tau_0 + \frac{2\pi n}{\omega}, \quad n = 1, 2, \dots .$$

Lemma 2.2 Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of characteristic equation (4) for which holds

$$\alpha(\tau_n) = 0, \quad \omega(\tau_n) = \omega, \quad n = 0, 1, 2, \dots .$$

If

$$2\omega > a, \tag{5}$$

then

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda = i\omega, \tau = \tau_n} > 0.$$

Proof. We insert $\lambda(\tau)$ into equation (4)

$$\lambda^2(\tau) + ae^{-\lambda(\tau)\tau}\lambda(\tau) + b = 0,$$

and by differentiation of the equation with respect to the parameter τ, we get

$$2\lambda\frac{d\lambda}{d\tau} + ae^{-\lambda\tau}\left(-\frac{d\lambda}{d\tau}\tau - \lambda\right)\lambda + ae^{-\lambda\tau}\frac{d\lambda}{d\tau} = 0,$$

$$2\lambda\frac{d\lambda}{d\tau} - a\lambda\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau} - a\lambda^2 e^{-\lambda\tau} + ae^{-\lambda\tau}\frac{d\lambda}{d\tau} = 0,$$

$$\frac{d\lambda}{d\tau}\left(2\lambda - a\lambda\tau e^{-\lambda\tau} + ae^{-\lambda\tau}\right) = a\lambda^2 e^{-\lambda\tau},$$

$$\frac{d\lambda}{d\tau} = \frac{a\lambda^2 e^{-\lambda\tau}}{2\lambda + a(1-\lambda\tau)e^{-\lambda\tau}},$$

$$\frac{d\lambda}{d\tau} = \frac{a\lambda^2}{2\lambda e^{\lambda\tau} + a(1-\lambda\tau)},$$

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda e^{\lambda\tau} + a(1-\lambda\tau)}{a\lambda^2}.$$

For $\lambda = i\omega$, $\tau = \tau_n$, we obtain

$$\begin{aligned}\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega,\tau=\tau_n} &= \frac{2i\omega e^{i\omega\tau_n} + a(1-i\omega\tau_n)}{-a\omega^2} \\ &= \frac{a(i\omega\tau_n - 1) - 2i\omega e^{i\omega\tau_n}}{a\omega^2} \\ &= \frac{a(i\omega\tau_n - 1) - 2i\omega(\cos\omega\tau_n + i\sin\omega\tau_n)}{a\omega^2}.\end{aligned}$$

Separating the real part, we get

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega,\tau=\tau_n} = \frac{2\omega - a}{a\omega^2}.$$

With regard to condition (5), we obtain

$$Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\Bigg|_{\lambda=i\omega,\tau=\tau_n} > 0.$$

The next theorem follows from Lemma 2.1 and Lemma 2.2.

Theorem. Suppose that condition (5) holds and $a > 0$, $b > 0$. Then equilibrium $E_1 = \left(\frac{\pi}{2}, 0\right)$ is asymptotically stable for $\tau \in [0, \tau_0)$ and unstable for $\tau > \tau_0$.

3. Results of numerical simulation

In this section, we illustrate the obtained results on a specific example. The following example and corresponding graphic representation was processed using the program MATHEMATICA.

Example. Let $a = 0.5$, $b = 0.25$. Then the system (2) has the form

$$\begin{aligned} x'(t) &= 0.5\left(\frac{\pi}{2}\sin x(t) - x(t-\tau)\right) + w(t), \\ w'(t) &= 0.25\left(\frac{\pi}{2}\sin x(t) - x(t)\right),\ t \geq 0. \end{aligned} \tag{6}$$

The constant $A = \frac{3}{4}$ and $B = \frac{1}{16}$. Then

$$\omega = \sqrt{\frac{\frac{3}{4}+\sqrt{\frac{5}{16}}}{2}} = \sqrt{\frac{3+\sqrt{5}}{8}} = \frac{1}{2}\sqrt{\frac{3+\sqrt{5}}{2}}.$$

For the parameter τ_0, we get

$$\tau_0 = \frac{\pi}{2\omega} = \pi\sqrt{\frac{2}{3+\sqrt{5}}} = 0.618034\,\pi.$$

Figures 2 and 3 show the solution of the system (6) for the $\tau = 0.5\pi \in (0, \tau_0)$. The solution converges to $E_1 = \left(\frac{\pi}{2}, 0\right)$.

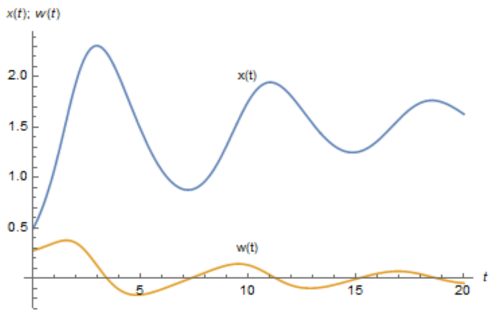

Fig. 2. Components of the solution of the system (6)

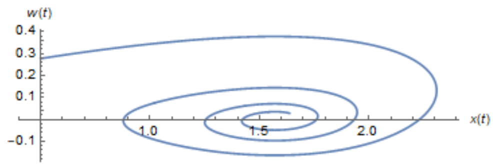

Fig. 3. The solution of the system (6) in the phase plane

Figures 4 and 5 show the periodic solution of the system (6) for the $\tau = \tau_0 =$ $= 0.618034\pi$. There are vibrations of the body.

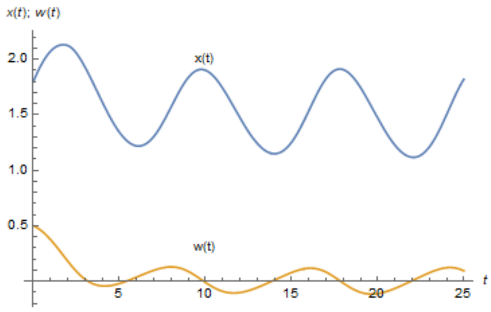

Fig. 4. Periodic components of the solution of the system (6)

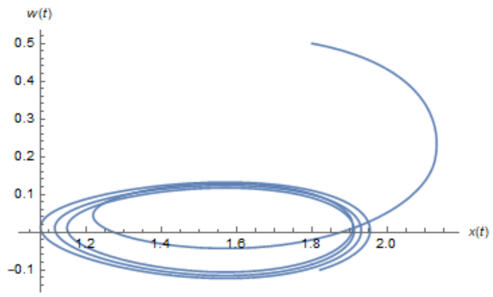

Fig. 5. Periodic solution of the system (6) in the phase plane

Figures 6 and 7 show the loss of the stability of E_1. The $\tau = \pi$.

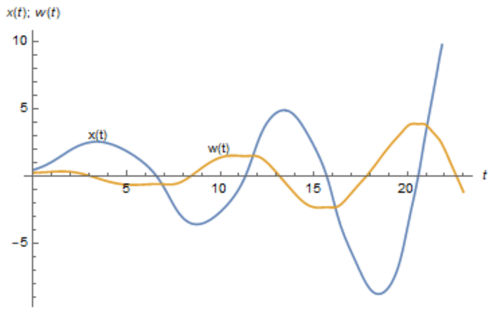

Fig. 6. Components of the unstable solution of the system (6)

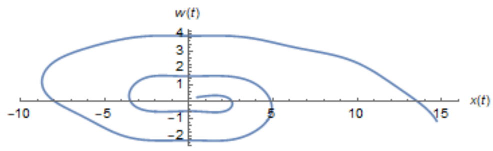

Fig. 7. Unstable solution of the system (6) in the phase plane

Conclusion

In the paper, we examine the value of the parameter τ at which the vibrations of the body arise. The vibration occurs when the parameter τ exceeds $\tau_0 > 0$, at which point the equilibrium E_1 loses its stability. Simulation on a specific example confirms the obtained theoretical results. For $0 \leq \tau < \tau_0$, the equilibrium E_1 is stable (see Figs. 2 and 3). If $\tau = \tau_0$, the system (6) has a periodic solution, indicating the presence of vibrations (see Figs. 4 and 5). If $\tau > \tau_0$, the equilibrium E_1 is unstable (see Figs. 6 and 7). We assume that the foundation is subjected to vibration $y = \frac{\pi}{2} \sin x$. It would be interesting to observe how the vibrational properties of the system change with a different vibrational function. In addition, we believe that the model is also suitable for solving other vibration problems.

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