Journal of Applied Mathematics and Computational Mechanics 2024, 23(2), 66-78
www.amcm.pcz.pl p-ISSN 2299-9965
DOI: 10.17512/jamcm.2024.2.06 e-ISSN 2353-0588

THE METHOD OF DEVELOPING THE COMPLEX STRESS TENSOR BY BASIC STATES FOR THE CONSTRUCTION SOLUTIONS OF SPATIAL BOUNDARY PROBLEMS OF THE LINEAR ELASTICITY THEORY

***Viktor Pabyrivskyi*[1], *Petro Pukach*[1], *Volodymyr Pabyrivskyi*[2]**

[1] *Institute of Applied Mathematics and Fundamental Sciences, Lviv Polytechnic National University, Lviv, Ukraine*
[2] *Institute of Computer Science and Information Technologies, Lviv Polytechnic National University, Lviv, Ukraine*
viktor.v.pabyrivskyi@lpnu.ua, petro.y.pukach@lpnu.ua, volodymyr.v.pabyrivskyi@lpnu.ua

Received: 30 November 2023; Accepted: 10 February 2024

Abstract. The apparatus of holomorphic functions of many complex variables is applied to solving spatial boundary value problems of the linear theory of elasticity. The construction of the solution of the boundary value problem is based on the representation of the displacement vector in the form of J. Dougall through spatial harmonic potentials. The transition from spatial harmonic potentials to holomorphic functions of two complex variables z_1, z_2 was carried out and a boundary value problem for the above functions was formulated. By presenting these holomorphic functions in the form of homogeneous polynomials of order k relative to complex variables z_1, z_2, solutions were constructed by the method of development of the complex tensor of stresses by basic states. The application of this technique is illustrated in the examples of marginal problems, the real components of solutions that correspond to the solutions of Grashof's problem for an elastic beam. Imaginary components of exact analytical solutions are obtained and corresponding structures of external load vectors for elastic beams of complex cross-section are constructed.

MSC 2010: *74B05, 74G05, 74S70*
Keywords: *three-dimensional theory of elasticity, Lamé equation, stress tensor, displacement vector, complex variables, holomorphic function*

1. Introduction

The application of the apparatus of harmonic and biharmonic functions in the presentation of the fundamental solution of equilibrium equations in displacements became the basis of analytical methods for solving problems of the spatial static theory of elasticity. The founders of this method of constructing solutions were Airy, Boussinesq, Papkovich, Neuber and Dougall. Proving the completeness of general solutions, the existence of relationships between them, and the construction of solutions of boundary value problems were discussed by Eubanks and Sternberg, Timoshenko and Goodier, and others [1-6]. In particular, [7, 8] created a universal design scheme for the development of general solutions and assessment of their completeness and non-unity within the framework of the classical theory of elasticity. The issue of optimizing the number of harmonic potentials in the Papkovich--Neuber representation using the variational approach was analyzed in [9].

In the case of a plane problem, Kolosov and Muskhelishvili presented the components of the displacement vector and the stress tensor through two analytic functions of a complex variable and developed a method for constructing solutions for multi-connected domains based on the use of Cauchy-type integrals and conformal mapping. Aleksandrovich [10] considered three-dimensional problems of elasticity theory as a partial class of problems of four-dimensional theory and obtained solutions using the functions of two complex variables. Important results regarding the application of $p-$, (p, q) – analytical functions for solving boundary value problems owned by Polozhiy [11].

The application of functions of complex variables for the construction of solutions to problems of the elasticity theory for inhomogeneous bodies, piezoelectric/ piezomagnetic bimaterials in two-dimensional and three-dimensional settings are covered in works [12-15].

This work aims to expand the set of exact analytical solutions for a confined elastic body of a complex cross-section based on the representation of the fundamental solution of the equilibrium equations in Dougall's form through three spatial harmonic functions [16]. To achieve this aim, the method of setting and constructing solutions of boundary value problems in the spatial theory of elasticity in holomorphic functions of two complex variables is used [17]. On the basis of Dougall's representation, the boundary value problem of the theory of elasticity for complex--valued functions is formulated. On the set of these functions, a subset of holomorphic functions of two variables z_1, z_2 is selected. The obtained results make it possible to present the complex displacement vector and stress tensor through the above-mentioned holomorphic functions and to formulate the basic complex--conjugate problem for harmonic potentials. An algorithm for constructing basic states for a complex stress tensor of order k by presenting holomorphic functions $\Phi_i(z_1, z_2)$ $(i = \overline{1,3})$ in the form of homogeneous polynomials of order k with respect to variables z_1, z_2 has been developed. The structures of the real and imaginary components of the basic state of order k were obtained.

The aim of the work is also to obtain, based on the developed approach, exact analytical solutions, which are imaginary components of complex basic solutions, the real components of which correspond to Grashof's solution. The obtained set of solutions will make it possible to form the structure of the external load for an elastic finite body and, accordingly, to optimize the stress state in the middle of elastic mechanical constructions of a complex cross section.

2. Materials and method

Consider a homogeneous elastic isotropic solid body $K \bigcup \partial K$, which in the initial state is bijectively mapped onto the region $X \bigcup \partial X$ of the Euclidean space. A stationary force load is applied to the surface of the body ∂X.

We formulate the static boundary value problem of the linear theory of elasticity for a given isotropic elastic body:

– the equilibrium equation (Lamé equations):

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \otimes (\nabla \cdot \mathbf{u}) = 0 \tag{1}$$

– the boundary condition on the side surface ∂X:

$$\boldsymbol{\sigma}_{\mathbf{n}} \equiv (\mathbf{n} \cdot \hat{\boldsymbol{\sigma}})\big|_{\partial X} = \boldsymbol{\sigma}_{\mathbf{n}}^{+}, \tag{2}$$

– stress tensor $\hat{\boldsymbol{\sigma}}$ is presented through the displacement vector $\mathbf{u}$:

$$\hat{\boldsymbol{\sigma}} = \lambda (\nabla \cdot \mathbf{u}) \hat{\mathbf{I}} + \mu (\nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla). \tag{3}$$

Here $\mathbf{u}$ is the vector of displacements; $\hat{\mathbf{I}}$ is the unit tensor; $\mathbf{n}$ is the vector of the external normal; $\boldsymbol{\sigma}_{\mathbf{n}}$ is the stress vector; $\mathbf{r}$ is the radius vector of an arbitrarily selected point of the body; $\boldsymbol{\sigma}_{\mathbf{n}}^{+}$ is a given vector of surface forces that satisfies the conditions of self-equilibrium of the external load on the lateral surface of the body ∂X; $\nabla \equiv \partial / \partial \mathbf{r}$ is the Hamilton's nabla-operator; $\Delta \equiv \nabla \cdot \nabla$ is the Laplace operator; λ, μ are the elastic Lamé constants; $\otimes$ is the operation of the dyadic product.

2.1. Basic relations and formulation of boundary value problems of the linear spatial theory of elasticity in J. Dougall's harmonic potentials

Consider the presentation of the fundamental solution of Lamé equations (1) $\mathbf{u} = u_i(x_1, x_2, x_3)\mathbf{e_i}$ in Dougall's form [16] through harmonic functions $\varphi_i(x_1, x_2, x_3)$ in the Cartesian coordinate system $\{x_i\}$ ($i = \overline{1,3}$):

$$\begin{gathered} u_1 = x_1 \frac{\partial^2 \varphi_1}{\partial x_3^2} + \frac{\partial \varphi_2}{\partial x_1} + \frac{\partial \varphi_3}{\partial x_2}, \quad u_2 = x_2 \frac{\partial^2 \varphi_1}{\partial x_3^2} + \frac{\partial \varphi_2}{\partial x_2} - \frac{\partial \varphi_3}{\partial x_1}, \\ u_3 = \frac{\partial}{\partial x_3} \left(-x_1 \frac{\partial \varphi_1}{\partial x_1} - x_2 \frac{\partial \varphi_1}{\partial x_2} + 4(\nu - 1)\varphi_1 \right) + \frac{\partial \varphi_2}{\partial x_3}, \end{gathered} \tag{4}$$

where $\mathbf{e_i}$ are basic orts of Cartesian coordinate system ($i = \overline{1,3}$); $\{x_i\}$ are coordinates of an arbitrarily selected material point $x \in X$; ν – Poisson's ratio.

Based on this representation of the displacement vector **u** (4), we find the components of the stress tensor $\hat{\boldsymbol{\sigma}}(x_1,x_2,x_3)$ (3):

$$\sigma_{11}=-\frac{E}{1+\nu}\left(\frac{\partial^2}{\partial x_3^2}\left((2\nu-1)\varphi_1-x_1\frac{\partial\varphi_1}{\partial x_1}\right)-\frac{\partial}{\partial x_1}\left(\frac{\partial\varphi_2}{\partial x_1}+\frac{\partial\varphi_3}{\partial x_2}\right)\right),$$

$$\sigma_{22}=-\frac{E}{1+\nu}\left(\frac{\partial^2}{\partial x_3^2}\left((2\nu-1)\varphi_1-x_2\frac{\partial\varphi_1}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(-\frac{\partial\varphi_2}{\partial x_2}+\frac{\partial\varphi_3}{\partial x_1}\right)\right),$$

$$\sigma_{33}=\frac{E}{1+\nu}\left(\frac{\partial^2}{\partial x_3^2}\left(2(\nu-2)\varphi_1-x_1\frac{\partial\varphi_1}{\partial x_1}-x_2\frac{\partial\varphi_1}{\partial x_2}\right)+\frac{\partial^2\varphi_2}{\partial x_3^2}\right),$$

$$\sigma_{12}=\frac{E}{2(1+\nu)}\left(\frac{\partial^2}{\partial x_3^2}\left(x_2\frac{\partial\varphi_1}{\partial x_1}+x_1\frac{\partial\varphi_1}{\partial x_2}\right)+2\frac{\partial^2\varphi_2}{\partial x_3^2}-\frac{\partial^2\varphi_3}{\partial x_1^2}+\frac{\partial^2\varphi_3}{\partial x_2^2}\right),$$

$$\sigma_{23}=\frac{E}{2(1+\nu)}\left(\frac{\partial}{\partial x_3}\left(-x_1\frac{\partial^2\varphi_1}{\partial x_1\partial x_2}+x_2\left(\frac{\partial^2\varphi_1}{\partial x_3^2}-\frac{\partial^2\varphi_1}{\partial x_2^2}\right)+(4\nu-5)\frac{\partial\varphi_1}{\partial x_2}\right)+2\frac{\partial^2\varphi_2}{\partial x_2\partial x_3}-\frac{\partial^2\varphi_3}{\partial x_1\partial x_3}\right),$$

$$\sigma_{13}=\frac{E}{2(1+\nu)}\left(\frac{\partial}{\partial x_3}\left(-x_2\frac{\partial^2\varphi_1}{\partial x_1\partial x_2}+x_1\left(\frac{\partial^2\varphi_1}{\partial x_3^2}-\frac{\partial^2\varphi_1}{\partial x_1^2}\right)+(4\nu-5)\frac{\partial\varphi_1}{\partial x_1}\right)+2\frac{\partial^2\varphi_2}{\partial x_1\partial x_3}+\frac{\partial^2\varphi_3}{\partial x_2\partial x_3}\right), \tag{5}$$

where $\mu=E/2(1+\nu)$, E – Young's modulus.

The boundary value problem of the linear theory of elasticity (1)-(3) is reformulated as a boundary value problem on harmonic functions $\varphi_i(x_1,x_2,x_3)$:

$$\Delta(\varphi_1,\varphi_2,\varphi_3)=0, \tag{6}$$

that satisfy the corresponding boundary conditions:

$$\boldsymbol{\sigma}_{\mathbf{n}}\equiv(\mathbf{n}\cdot\hat{\boldsymbol{\sigma}})\big|_{\partial X}=(n_i\sigma_{ij}\mathbf{e}_{\mathbf{j}})\big|_{\partial X}=\sigma_{n_i}\mathbf{e}_{\mathbf{j}}\ \ (i,j=\overline{1,3}). \tag{7}$$

2.2. Construction of the solutions to the problem of the elasticity theory by the method of development of the stress tensor according to the basic states

On the basis of the methodology proposed in the work [17], complex variables z_1, z_2, z_3 are introduced in accordance with real variables x_1, x_2, x_3 in the Cartesian coordinate system, as follows: $z_1 = x_1 + ix_2$, $z_2 = x_2 + ix_3$, $z_3 = x_3 + ix_1$.

The introduced complex variables and harmonic potentials $\varphi_i(x_1,x_2,x_3)$, $\varphi_i^*(x_1,x_2,x_3)$, from the representation of displacement vectors $\mathbf{u}(x_1,x_2,x_3)$ and $\mathbf{u}^*(x_1,x_2,x_3)$ (4) make it possible to form functions from complex variables z_1, z_2, z_3 as follows: $F_i(z_1,z_2,z_3)=\varphi_i(x_1,x_2,x_3)+i\varphi_i^*(x_1,x_2,x_3)$, $(i=\overline{1,3})$.

In the continued methods, on the set of complex-valued functions $F_i(z_1,z_2,z_3)$, a subset of complex-conjugate problems is distinguished, which are described by holomorphic functions $\Phi_i(z_1,z_2)$, $(i=\overline{1,3})$, from two complex variables z_1, z_2, for which the following conditions are fulfilled: $F_i(z_1,z_2,z_3)/\partial z_3=0$, which impose bonds on harmonic potentials $\varphi_i(x_1,x_2,x_3)$, $\varphi_i^*(x_1,x_2,x_3)$ and are a generalization of the Cauchy-Riemann conditions:

$$\frac{\partial\varphi_i}{\partial x_1}-\frac{\partial\varphi_i}{\partial x_3}=\frac{\partial\varphi_i^*}{\partial x_2},\quad \frac{\partial\varphi_i}{\partial x_2}=\frac{\partial\varphi_i^*}{\partial x_3}-\frac{\partial\varphi_i^*}{\partial x_1}.$$

Thus, the boundary value problem on harmonic functions $\varphi_i(x_1,x_2,x_3)$ (6)-(7) is reformulated as a boundary value problem on holomorphic functions $\Phi_i(z_1,z_2)$ of two complex variables z_1, z_2 that satisfy the Laplace equation:

$$\Delta(\Phi_1,\Phi_2,\Phi_3)=0 \tag{8}$$

and the corresponding boundary conditions:

$$\mathbf{P}_n\equiv\left(\mathbf{n}\cdot\hat{\mathbf{P}}\right)\Big|_{\partial X}=\left(n_iP_{ij}\mathbf{e}_j\right)\Big|_{\partial X}=P_{n_1}^+\mathbf{e}_1+P_{n_2}^+\mathbf{e}_2+P_{n_3}^+\mathbf{e}_3=\mathbf{P}_n^+. \tag{9}$$

where $\hat{\mathbf{P}}(z_1,z_2,z_3)=\hat{\boldsymbol{\sigma}}(x_1,x_2,x_3)+i\hat{\boldsymbol{\sigma}}^*(x_1,x_2,x_3)$ is the complex stress tensor; $\Delta=\partial^2/\partial z_1\partial z_2$ – the Laplace operator.

Let us present the complex vector of displacements $\mathbf{w}(z_1,z_2,z_3)=\omega_i\mathbf{e_i}$ through holomorphic functions $\Phi_i(z_1,z_2)$ $(i=\overline{1,3})$:

$$\begin{aligned}
\omega_1&=u_1+iu_1^*=-\frac{1+i}{2}(z_1-iz_2-z_3)\frac{\partial^2\Phi_1}{\partial z_2^2}+\left(\frac{\partial\Phi_2}{\partial z_2}+i\frac{\partial\Phi_2}{\partial z_1}\right)+\frac{\partial\Phi_3}{\partial z_1},\\
\omega_2&=u_2+iu_2^*=-\frac{1+i}{2}(z_2-iz_3-z_1)\frac{\partial^2\Phi_1}{\partial z_2^2}-\frac{\partial\Phi_2}{\partial z_1}+\left(\frac{\partial\Phi_3}{\partial z_2}+i\frac{\partial\Phi_3}{\partial z_1}\right),\\
\omega_3&=u_3+iu_3^*=\frac{1-i}{2}(z_2-iz_3-z_1)\frac{\partial^2\Phi_1}{\partial z_2^2}+4(\nu-1)i\frac{\partial\Phi_1}{\partial z_2}+i\frac{\partial\Phi_3}{\partial z_2}.
\end{aligned} \tag{10}$$

The components of the stress tensor $\hat{\mathbf{P}}(z_1,z_2,z_3)=\lambda(\nabla\cdot\mathbf{w})\hat{\mathbf{I}}+\mu(\nabla\otimes\mathbf{w}+\mathbf{w}\otimes\nabla)$ take the form:

$$P_{11} = -(\lambda + \mu)\frac{\partial^2 \Phi_1}{\partial z_2^2} + 2\mu i \frac{\partial^2 \Phi_2}{\partial z_1^2} + 2\mu \frac{\partial^2 \Phi_3}{\partial z_1^2},$$

$$P_{22} = -2(\lambda + \mu)\frac{\partial^2 \Phi_1}{\partial z_2^2} - \mu(1+i)(z_2 - iz_3 - z_1)\frac{\partial^3 \Phi_1}{\partial z_2^3} - 2\mu i \frac{\partial^2 \Phi_2}{\partial z_1^2} + 2\mu\left(\frac{\partial^2 \Phi_3}{\partial z_2^2} - \frac{\partial^2 \Phi_3}{\partial z_1^2}\right),$$

$$P_{33} = -2\lambda \frac{\partial^2 \Phi_1}{\partial z_2^2} + \mu(1+i)(z_2 - iz_3 - z_1)\frac{\partial^3 \Phi_1}{\partial z_2^3} - 2\mu \frac{\partial^2 \Phi_3}{\partial z_2^2},$$

$$P_{12} = -\mu \frac{(1+i)}{2}(z_1 - iz_2 - z_3)\frac{\partial^3 \Phi_1}{\partial z_2^3} + \mu\left(\frac{\partial^2 \Phi_2}{\partial z_2^2} - 2\frac{\partial^2 \Phi_2}{\partial z_1^2}\right) + 2\mu i \frac{\partial^2 \Phi_3}{\partial z_1^2},$$

$$P_{23} = \mu(1-i)(z_2 - iz_3 - z_1)\frac{\partial^3 \Phi_1}{\partial z_2^3} + \mu(4\nu - 5)i\frac{\partial^2 \Phi_1}{\partial z_2^2} + 2\mu i \frac{\partial^2 \Phi_3}{\partial z_2^2},$$

$$P_{13} = \mu \frac{(1-i)}{2}(z_1 - iz_2 - z_3)\frac{\partial^3 \Phi_1}{\partial z_2^3} + \mu i \frac{\partial^2 \Phi_2}{\partial z_2^2}. \qquad (11)$$

Thus, the problem of constructing a complex displacement vector $\mathbf{w}(z_1, z_2, z_3)$ (10) and a stress tensor $\hat{\mathbf{P}}(z_1, z_2, z_3)$ (11), which are represented by holomorphic functions $\Phi_i(z_1, z_2)$ of two complex variables z_1, z_2 and satisfy the boundary conditions (9), will be interpreted as *the basic complex-conjugate boundary value problem* of the theory of elasticity.

To deduce the structure solutions of the above-mentioned boundary-value problems, we present the holomorphic functions $\Phi_i(z_1, z_2)$ in the form of polynomials of order n in degrees of complex variables z_1, z_2:

$$\Phi_i^{(n)}(z_1, z_2) = \sum_{k=0}^{n} Q_i^{(k)}(z_1, z_2), \ (i = \overline{1,3}) \qquad (12)$$

where

$$Q_m^{(k)}(z_1, z_2) = \sum_{j=0}^{k} a_m^{((k-j)j)} z_1^{k-j} z_2^j, \ (j, k = \overline{0, n}) \qquad (13)$$

are homogeneous polynomials of degree k in complex variables z_1, z_2 that satisfy the Laplace equation: $\partial^2 Q_m^{(k)}(z_1, z_2) \big/ \partial z_1 \partial z_2 = 0$,

$$a_m^{((k-j)j)} = \alpha_m^{((k-j)j)} + i\beta_m^{((k-j)j)}; \ \alpha_m^{((k-j)j)}, \beta_m^{((k-j)j)}, \ (m = \overline{1,3})$$

are real numbers.

If we use representation (13) for homogeneous polynomials $Q_m^{(k)}$, then relations (12) acquire the structure of "k-ary" forms (unary, binary, ternary, etc.) with respect to independent complex variables z_1, z_2, which we will use to construct solutions of the original boundary value problem:

$$Q_m^{(k)}\left(z_1,z_2\right)=a_m^{(k0)}z_1^k+a_m^{(0k)}z_2^k,\ (m=\overline{1,3}),(k=\overline{0,n}). \tag{14}$$

To construct a complex stress tensor $\hat{\mathbf{P}}^{(n)}$ of order n, we present holomorphic functions $\Phi_m\left(z_1,z_2\right)$ in the form of homogeneous polynomials $Q_m^{(n)}\left(z_1,z_2\right)$ of degree $n+2$:

$$\Phi_m^{(n+2)}\left(z_1,z_2\right)=a_m^{((n+2)0)}z_1^{(n+2)}+a_m^{(0(n+2))}z_2^{(n+2)},\ (m=\overline{1,3}). \tag{15}$$

Then the components of the complex stress tensor $\hat{\mathbf{P}}\left(z_1,z_2,z_3\right)$ of order n are given as follows:

$$\begin{gathered}
P_{11}^{(n)}=2(n+1)(n+2)\Big(\mu\left(ia_2^{((n+2)0)}+a_3^{((n+2)0)}\right)z_1^n-(\lambda+\mu)a_1^{(0(n+2))}z_2^n\Big),\\
P_{22}^{(n)}=2(n+1)(n+2)\Big(-\mu\left(ia_2^{((n+2)0)}+a_3^{((n+2)0)}\right)z_1^n+\\
+\left(-(\lambda+\mu)a_1^{(0(n+2))}+\mu\left(a_3^{(0(n+2))}-(1+i)na_1^{(0(n+2))}\right)\right)z_2^n+\\
+\mu(1+i)na_1^{(0(n+2))}z_1z_2^{n-1}+\mu(i-1)na_1^{(0(n+2))}z_2^{n-1}z_3\Big),\\
P_{33}^{(n)}=2(n+1)(n+2)\Big(-(\lambda+\mu)a_1^{((n+2)0)}+\mu\left((1+i)na_1^{(0(n+2))}-a_3^{((n+2)0)}\right)z_2^n-\\
-\mu(1+i)na_1^{(0(n+2))}z_1z_2^{n-1}-\mu(i-1)na_1^{(0(n+2))}z_2^{n-1}z_3\Big),\\
P_{12}^{(n)}=\mu(n+1)(n+2)\Big(2\left(-a_2^{((n+2)0)}+ia_3^{((n+2)0)}\right)z_1^n+\left(a_2^{(0(n+2))}+(i-1)na_1^{(0(n+2))}\right)z_2^n-\\
-(1+i)na_1^{(0(n+2))}z_1z_2^{n-1}+(1+i)na_1^{(0(n+2))}z_2^{n-1}z_3\Big),\\
P_{23}^{(n)}=2\mu(n+1)(n+2)\Big(\left((2\nu-5/2)ia_1^{(0(n+2))}-(i-1)na_1^{(0(n+2))}+ia_3^{(0(n+2))}\right)z_2^n+\\
-(1-i)na_1^{(0(n+2))}z_1z_2^{n-1}-(1+i)na_1^{(0(n+2))}z_2^{n-1}z_3\Big),\\
P_{13}^{(n)}=\mu(n+1)(n+2)\Big(\left(-(1+i)na_1^{(0(n+2))}+ia_2^{(0(n+2))}\right)z_2^n+\\
+(1-i)na_1^{(0(n+2))}z_1z_2^{n-1}-(1-i)na_1^{(0(n+2))}z_2^{n-1}z_3\Big).
\end{gathered} \tag{16}$$

To form the appropriate structure of the boundary conditions on the side surface of the body ∂X with the normal **n**, for each basic stress state $\hat{\mathbf{P}}^{(k)}(z_1, z_2, z_3)$ of order k, there is the following expression of the stress vector:

$$\mathbf{P}_{\mathbf{n}}^{(\mathbf{k})} \equiv \left(\mathbf{n} \cdot \hat{\mathbf{P}}^{(k)}\right)\Big|_{\partial X} = \left(n_i P_{ij}^{(k)} \mathbf{e}_{\mathbf{j}}\right)\Big|_{\partial X}. \quad (17)$$

Next, the solution of the boundary value problem is constructed for the complex stress tensor $\hat{\mathbf{P}}^{(k)}$ of order k (*the base state of order* k), which is generated by homogeneous polynomials:

$$\Phi_m^{(k+2)}(z_1, z_2) \equiv Q_m^{(k+2)}(z_1, z_2) = a_m^{((k+2)0)} z_1^{(k+2)} + a_m^{(0(k+2))} z_2^{(k+2)}, \ (m = \overline{1,3}). \quad (18)$$

will be interpreted as a solution of order k of the basic boundary value problem.

3. Example of the solutions of the complex-conjugate boundary problems for an elastic beam

The application of the method of holomorphic functions of two complex variables will be illustrated by examples of constructing complex-conjugate solutions for an elastic finite beam.

An elastic beam with a length of l is considered, the closed contour of its cross--section is described by equation $f(x_1, x_2) = (\pm x_2/b)^{1/\zeta} + \left(x_1^2/a^2 - 1\right) = 0$, $(0 < \zeta \le 1)$, $(a, b = const)$, which is bent by a force P applied to its edge and parallel to one of the main axes of the cross-section (in particular, x_2).

Let's take the origin at the center of gravity of the fixed end of the beam. Axis x_3 coincides with the middle line of the beam, and axes x_1 and x_2 coincide with the main axes of the cross section. Assume that the normal stress in a particular section at a distance x_3 from the fixed end is distributed in the same way as it is in the case of pure bending: $\sigma_{33} = -P(l - x_3)\, x_2 / I$, where I is the moment of inertia of the cross section of the beam.

Consider the real component of the solution, which corresponds to the solution known from the literature [18] for a given elastic beam, and is presented as a superposition of the basic states of the stress tensor of the zeroth, first, and second order:

$$\hat{\mathbf{P}} = \hat{\mathbf{P}}^{(0)} + \hat{\mathbf{P}}^{(1)} + \hat{\mathbf{P}}^{(2)},$$

where

$$\sigma_{11}^{(0)} = \sigma_{22}^{(0)} = \sigma_{33}^{(0)} = \sigma_{12}^{(0)} = \sigma_{13}^{(0)} = 0,\ \sigma_{11}^{(2)} = \sigma_{22}^{(2)} = \sigma_{12}^{(2)} = 0,\ \sigma_{11}^{(1)} = \sigma_{22}^{(1)} = \sigma_{23}^{(1)} = \sigma_{12}^{(1)} = \sigma_{13}^{(1)} = 0,$$
$$\operatorname{Re} P_{33} = \sigma_{33}^{(1)} + \sigma_{33}^{(2)} = -\frac{Pl}{I}x_2 + \frac{Pl}{I}x_2 x_3,\ \operatorname{Re} P_{13} = \sigma_{13}^{(2)} = -\frac{P\nu}{(1+\nu)I}x_1 x_2,$$
$$\operatorname{Re} P_{23} = \sigma_{23}^{(0)} + \sigma_{23}^{(2)} = \frac{P\nu}{2(1+\nu)I}\left(b^2 - x_2^2\right). \tag{19}$$

Using relations (16) ($n = 0$) and (19), we obtain a system of equations for finding the coefficients of holomorphic functions $\Phi_m^{(2)}(z_1, z_2)$ for the real part of the stress tensor of order zero $\operatorname{Re}\hat{\mathbf{P}}^{(0)}$, and after analyzing them, we obtain:

$$\alpha_1^{(02)} = \alpha_2^{(02)} = \alpha_3^{(02)} = \alpha_2^{(20)} = 0,\ \beta_1^{(02)} = \beta_2^{(02)} = \beta_3^{(20)},\ \alpha_3^{(20)} = \beta_2^{(20)},$$
$$\beta_3^{(02)} = \frac{Pb^2}{2\mu(1+\nu)I}. \tag{20}$$

This result, based on the relation (16) ($n = 0$), makes it possible to obtain the imaginary part of the stress tensor $\operatorname{Im}\hat{P}^{(0)}$:

$$\sigma_{11}^{(0)(*)} = \sigma_{12}^{(0)(*)} = \sigma_{13}^{(0)(*)} = \sigma_{23}^{(0)(*)} = 0,\ \sigma_{22}^{(0)(*)} = -\frac{Pb^2}{2(1+\nu)I},\ \sigma_{33}^{(0)(*)} = \frac{Pb^2}{2(1+\nu)I}. \tag{21}$$

Let's proceed to the analysis of the real part of the stress tensor $\operatorname{Re}\hat{\mathbf{P}}^{(1)}$ of the first order. Using the relation (16) ($n = 1$), we obtain a system of equations for finding the coefficients of holomorphic functions $\Phi_m^{(3)}(z_1, z_2)$:

$$\alpha_1^{(03)} = \alpha_2^{(03)} = \alpha_3^{(03)} = \alpha_1^{(30)} = 0,\ \beta_1^{(03)} = \beta_2^{(03)} = \beta_3^{(03)} = 0,$$
$$\alpha_2^{(30)} = -\beta_3^{(30)},\ \alpha_3^{(30)} = \beta_2^{(30)},\ \alpha_3^{(03)} = \frac{Pl}{\mu I}. \tag{22}$$

and, accordingly, the expressions for the imaginary part of the stress tensor $\operatorname{Im}\hat{\mathbf{P}}^{(1)}$:

$$\sigma_{11}^{(1)(*)} = \sigma_{12}^{(1)(*)} = \sigma_{13}^{(1)(*)} = 0,$$
$$\sigma_{22}^{(1)(*)} = 12\frac{Pl}{I}x_3,\ \sigma_{33}^{(1)(*)} = -12\frac{Pl}{I}x_3,\ \sigma_{23}^{(1)(*)} = 12\frac{Pl}{I}x_2. \tag{23}$$

Similarly, after analyzing the real part of the second-order stress tensor $\operatorname{Re}\hat{\mathbf{P}}^{(2)}$ based on relation (16) ($n = 2$), we will obtain a system of equations for finding the coefficients of holomorphic functions $\Phi_m^{(4)}(z_1, z_2)$.

$$\alpha_1^{(04)}=\alpha_2^{(04)}=\alpha_3^{(04)}=\alpha_1^{(40)}=0,\ \beta_2^{(04)}=\beta_3^{(04)}=\beta_1^{(40)}=0,$$
$$\alpha_2^{(40)}=-\beta_3^{(40)},\ \alpha_3^{(40)}=\beta_2^{(40)},\ \beta_1^{(04)}=-\frac{P\nu}{24\mu(1+\nu)I}. \tag{24}$$

As a result, we will obtain expressions for the imaginary part of the stress tensor $\operatorname{Im}\hat{\mathbf{P}}^{(2)}$.

$$\sigma_{11}^{(2)(*)}=-\frac{(\lambda+\mu)}{\mu}\frac{P\nu}{(1+\nu)I}\left(x_2^2-x_3^2\right),\ \sigma_{22}^{(2)(*)}=\frac{P\nu}{(1+\nu)I}\left(\frac{(\lambda+3\mu)}{\mu}x_2^2-\frac{(\lambda+\mu)}{\mu}x_3^2\right),$$
$$\sigma_{33}^{(2)(*)}=\frac{P\nu}{(1+\nu)I}\left(\frac{(\lambda-2\mu)}{\mu}x_2^2-\frac{\lambda}{\mu}x_3^2\right),\ \sigma_{12}^{(2)(*)}=\frac{P\nu}{(1+\nu)I}x_1x_2,$$
$$\sigma_{23}^{(2)(*)}=(4\nu-7)\frac{P\nu}{(1+\nu)I}x_1x_2,\ \sigma_{13}^{(2)(*)}=-\frac{P\nu}{(1+\nu)I}x_1x_3. \tag{25}$$

The obtained results (21), (23), (25) make it possible to form the general imaginary part of the stress tensor $\operatorname{Im}\hat{\mathbf{P}}$:

$$\operatorname{Im}P_{11}=\sigma_{11}^{(0)(*)}+\sigma_{11}^{(1)(*)}+\sigma_{11}^{(2)(*)}=\frac{P\lambda}{IE}\left(x_2^2-x_3^2\right),$$
$$\operatorname{Im}P_{22}=\sigma_{22}^{(0)(*)}+\sigma_{22}^{(1)(*)}+\sigma_{22}^{(2)(*)}=\frac{P}{I}\left[-\frac{b^2}{2(1+\nu)}+12lx_3+\frac{\lambda}{E}\left((3-4\nu)x_2^2-x_3^2\right)\right],$$
$$\operatorname{Im}P_{33}=\sigma_{33}^{(0)(*)}+\sigma_{33}^{(1)(*)}+\sigma_{33}^{(2)(*)}=\frac{P}{I}\left[\frac{b^2}{2(1+\nu)}-12lx_3+2\frac{\lambda}{E}\left((-1+3\nu)x_2^2-\nu x_3^2\right)\right],$$
$$\operatorname{Im}P_{12}=\sigma_{12}^{(0)(*)}+\sigma_{12}^{(1)(*)}+\sigma_{12}^{(2)(*)}=\frac{P\nu}{(1+\nu)I}x_1x_2,$$
$$\operatorname{Im}P_{13}=\sigma_{13}^{(0)(*)}+\sigma_{13}^{(1)(*)}+\sigma_{13}^{(2)(*)}=-\frac{P\nu}{(1+\nu)I}x_1x_3,$$
$$\operatorname{Im}P_{23}=\sigma_{23}^{(0)(*)}+\sigma_{23}^{(1)(*)}+\sigma_{23}^{(2)(*)}=\frac{P}{I}\left[12lx_2+\frac{(4\nu-7)}{(1+\nu)}x_2x_3\right]. \tag{26}$$

Let us present the vectors of external loads for a given elastic beam, which correspond to the imaginary part of the solution $\operatorname{Im}\mathbf{P_n}$.

On the surface ∂X_3: $x_3=0,\ \mathbf{n}_3=\mathbf{e}_3=(0,0,1)$:

$$\operatorname{Im}\mathbf{P}_{\mathbf{n_3}} \equiv \left(\mathbf{n}_3\cdot \operatorname{Im}\hat{\mathbf{P}}\right)\Big|_{x_3=0} = \frac{P}{I}\left[\left(12lx_2\right)\mathbf{e_2} + \left(\frac{b^2}{2(1+\nu)} + 2\left(\frac{\nu(-1+3\nu)}{(1+\nu)(1-2\nu)}x_2^2\right)\right)\mathbf{e}_3\right]. \tag{27}$$

On the surface ∂X_{-3}: $x_{-3}=l$, $\mathbf{n}_{-3}=-\mathbf{e}_3=(0,0,-1)$:

$$\operatorname{Im}\mathbf{P}_{\mathbf{n_{-3}}} \equiv \left(\mathbf{n}_{-3}\cdot \operatorname{Im}\hat{\mathbf{P}}\right)\Big|_{x_3=l} = -\frac{P}{I}\left[\left(\frac{\nu}{(1+\nu)}lx_1\right)\mathbf{e}_1 + \left(\frac{5+16\nu}{(1+\nu)}lx_2\right)\mathbf{e_2} + \right.$$
$$\left. + \left(\left(\frac{b^2}{2(1+\nu)} - 2l^2\frac{6-6\nu-13\nu^2}{(1+\nu)(1-2\nu)}\right) + 2\left(\frac{\nu(-1+3\nu)}{(1+\nu)(1-2\nu)}x_2^2\right)\right)\mathbf{e}_3\right]. \tag{28}$$

On the surface ∂X_2: $f(x_1,x_2)=(\pm x_2/b)^{1/\zeta}+\left(x_1^2/a^2-1\right)=0$, $\mathbf{n_2}=n_{2i}\,\mathbf{e}_i=\vec{\nabla}f\big/\left|\vec{\nabla}f\right|$,

$$n_{21}=\frac{2\zeta b^{1/\zeta}x_1}{\sqrt{4\zeta^2 b^{2/\zeta}x_1^2+a^4x_2^{2\left(1/\zeta-1\right)}}},\quad n_{22}=\frac{a^2(\pm x_2)^{\left(1/\zeta-1\right)}}{\sqrt{4\zeta^2 b^{2/\zeta}x_1^2+a^4x_2^{2\left(1/\zeta-1\right)}}}.$$

As a result, we get the following structure of the external load vector on the surface ∂X_2:

$$\operatorname{Im}\mathbf{P}_{\mathbf{n}_2}=\left(n_{2i}\operatorname{Im}P_{ij}\,\mathbf{e_j}\right)\Big|_{\partial X_2}=\frac{P}{I}\frac{\lambda}{E}\frac{b^{1/\zeta}}{\sqrt{4\zeta^2 b^{2/\zeta}x_1^2+a^4x_2^{2\left(1/\zeta-1\right)}}}\left(2\zeta x_1\left(x_2^2-x_3^2\right)+\right.$$
$$\left.+c\left(a^2-x_1^2\right)\left(\left(3-4\nu\right)(\pm x_2)+(\pm x_2)^{(-1)}\left(\frac{E}{\lambda}\left(-\frac{b^2}{2(1+\nu)}+12lx_3\right)-x_3^2\right)\right)\mathbf{e}_1\right)+$$
$$+\left(2\zeta(1-2\nu)x_1^2x_2+(\pm x_2)^{(-1)}\left(a^2-x_1^2\right)\left(\frac{E}{\lambda}\left(-\frac{b^2}{2(1+\nu)}+12lx_3\right)+(3-4\nu)x_2^2-x_3^2\right)\right)\mathbf{e}_2+$$
$$+\left(-2\zeta(1-2\nu)x_1^2x_3+\frac{E}{\lambda}\left(a^2-x_1^2\right)\left(12l+\frac{(4\nu-7)}{(1+\nu)}x_3\right)\right)\mathbf{e}_3. \tag{29}$$

where $\lambda=E\nu\big/(1+\nu)(1-2\nu)$.

4. Conclusion

In this work, the apparatus of holomorphic functions of two complex variables is used to find a set of exact analytical solutions to spatial boundary value problems of the linear theory of elasticity.

The method proposed in [17], which is based on the fundamental solution in the form of J. Dougall, made it possible to represent the complex displacement vector $\mathbf{w}(z_1, z_2, z_3)$ and the complex stress tensor $\hat{\mathbf{P}}(z_1, z_2, z_3)$ through holomorphic functions of two complex variables z_1, z_2 and to formulate the basic complex--conjugate problems of the spatial theory of elasticity. Using the representation of holomorphic functions $\Phi_i(z_1, z_2)$ $(i = \overline{1,3})$ in the form of corresponding homogeneous polynomials $Q_m^{(n)}(z_1, z_2)$ $(m = \overline{1,3})$ of degree $n+2$, the structure of complex basic solutions $\hat{\mathbf{P}}^{(n)}$ of order n was obtained. This approach made it possible to obtain natural connections between harmonic potentials of complex-conjugate solutions, specify the structure of the real and imaginary parts of the basic states $\hat{\mathbf{P}}^{(k)}$ and, accordingly, form a subset of exact analytical solutions. An example of a boundary value problem illustrates the application of the technique, the real components of whose solutions correspond to the solution of the Grashof problem for a beam of complex cross-section. Imaginary components of the solutions were obtained and the structure of the corresponding external loads was specified on each of the lateral surfaces for the above-mentioned problems.

The examples given in the work can be effectively used to describe the stress-deformed state of structural elements of industrial equipment in order to calculate and optimize the parameters of their reliable functioning.

References

[1] Lurie, S.A., Volkov-Bogorodskiy, D.B., & Belov, P.A. (2023). On general representations of Papkovich-Neuber solutions in gradient elasticity. *Lobachevskii Journal of Mathematics*, *44*, 2336-2351. DOI: 10.1134/S199508022306032X.

[2] Solyaev, Y.O., & Korolenko, V.A. (2023). Application of Papkovich-Neuber general solution for crack problems in strain gradient elasticity. *Lobachevskii Journal of Mathematics*, *44*, 2469-2479. DOI: 10.1134/S1995080223060434.

[3] Solyaev, Y.O. (2022). *Complete general solutions for equilibrium equations of isotropic strain gradient elasticity*. DOI: 10.48550/arXiv.2207.08863.

[4] Kong, D., Cui, Z., Pan, Y., & Zhang, K. (2012). On the Papkovich-Neuber formulation for stocks flows driven by translating/rotating prolate spheroid at angles. *International Journal of Pure and Applied Mathematics*, *75*(4), 455-483.

[5] Ike, C.C., Onah, H.N., & Nwoji, C.U. (2017). Bessel functions for axisymmetric elasticity problems of the elastic half space soil: a potential function method. *Nigerian Journal of Technology*, *36*(3), 773-781. DOI: 10.4314/njt.v36i3.16.

[6] Palaniappan, D. (2011). A general solution of equations of equilibrium in linear elasticity, *Applied Mathematical Modelling*, *35*, 5494-5499. DOI: 10.1016/j.apm.2011.01.041.

[7] Wang, M.Z., Xu, B.X, & Gao, C.F. (2008). Recent general solutions in linear elasticity and their applications. *Applied Mechanics Reviews*, *61*(3). DOI: 10.1115/1.2909607.

[8] Revenko, V.P. (2009). Solving the three-dimensional equations of the linear theory of elasticity. *International Journal of Applied Mechanics*, *45*(7), 730-741. DOI: 10.1007/s10778-009-0225-4.

[9] Pabyrivskyi, V., Pukach, P., Pabyrivska, N., & Beregova, G. (2022). Concretization of the mathematical models of linear elasticity theory by representation of general solution in terms of harmonic potentials in Papkovich-Neuber form. *Proceedings of the Romanian Academy Series A – Mathematics, Physics, Technical Sciences, Information Science*, *23*(2), 113-121.

[10] Aleksandrovich, A.I. (1977). Applying the theory of functions of two complex variables to the theory of elasticity. *Doklady Akademii Nauk SSSR*, *232*(3), 542-544.

[11] Polozhiy, G.N. (1973). *Theory and Application of p-analytical and (p, q)-analytical Functions*. Kyiv: Naukova Dumka.

[12] Sharafutdinov, G.Z. (2018). The application of complex potentials to the problems of plane stress state for an inhomogeneous material. *Mechanics of Solids*, *53*, 146-154. DOI: 10.3103/S0025654418030135.

[13] Onopriienko, O., Loboda, V., Sheveleva, A., & Lapusta, Y. (2018). An interface crack with mixed electro-magnetic conditions at it faces in a piezoelectric/piezomagnetic bimaterial under anti-plane mechanical and in-plane electric loadings. *Acta Mechanica et Automatica*, *12*(4), 301-310. DOI: 10.2478/ama-2018-0046.

[14] Matysiak, S.J., Kulchytsky-Zhygailo, R., & Petrowski, D.M. (2018). Stress distribution in an elastic layer resting on a Winkler foundation with an emptiness. *Bulletin of the Polish Academy of Sciences: Technical Sciences*, *66*(53), 721-727. DOI: 10.24425/125339.

[15] Chigirinsky, V., & Naumenko, O. (2019). Studying the stressed state of elastic medium using the argument functions of a complex variable. *Eastern-European Journal of Enterprise Technologies*, *5/7*(101), 27-35. DOI: 10.15587/1729-4061.2019.177514.

[16] Dougall, J. (1914). An analytical theory of the equilibrium of an isotropic elastic rod of circular section. *Transaction of Royal Society, Edinburg*, *49*(17), 895-978.

[17] Pabyrivskyi, V., & Burak, Y. (2023). *Method of Holomorphic Functions of Two Complex Variables for Spatial Problems of the Elasticity Theory*. Lviv: Spolom.

[18] Timoshenko, S.P., & Goodier, J.N. (1970). *Theory of Elasticity.* 3rd ed., New York: McGraw Hill.