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FRACTIONAL OPTIMAL CONTROL APPROACH TO THE DIABETICS MODEL

***Muhammad Nadeem*[1], *Mustafa Habib*[2], *Ayesha Arif*[2], *Muhammad Imran Asjad*[3]**

[1] *School of Mathematics and Statistics, Qujing Normal University, Qujing 655011, China*
[2] *Department of Mathematics, University of Engineering and Technology, Lahore 58400, Pakistan*
[3] *Department of Mathematics, University of Management and Technology, Lahore 54770, Pakistan*
nadeem@mail.qjnu.edu.cn, mustafa@uet.edu.pk, ayeshaarif865@gmail.com
imran.asjad@umt.edu.pk

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Abstract. Diabetes mellitus is one of the most critical diseases, affecting millions of people around the world. This work deals with the fractional optimal control of the dynamics of the population model on diabetes. This framework is based on the fractional order differential problems that describe the population before and after diabetes involving some health problems. We consider the Caputo derivatives for the study of the proposed model. The maximum principle of Pontryagin is utilized to derive the necessary conditions for the optimality of a dynamical system. Using a forward-backward sweep approach with the generalized Euler method accomplishes numerical solutions of formulated issues.

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1. Introduction

Traditional derivatives capture instantaneous rates of change, whereas fractional derivatives, like the Caputo derivative, incorporate memory effects. Fractional derivatives, particularly the Caputo derivative, naturally handle such non-local effects, making them more suitable for modeling systems with long-term memory or distributed effects [1]. Diabetes has become an epidemic and poses a great threat to human health throughout the globe. Along with the impact that diabetes has on human health, its huge and growing socio-economic burden affecting individuals, families and the whole of society should be carefully monitored and investigated. Diabetes is an illness that happens when your blood glucose level is uncommonly high [2, 3]. In the context of diabetic populations, where the disease progression often involves memory-like effects due to the accumulation of damage over time, fractional derivatives can better capture these dynamics and diabetic populations often exhibit non-local effects as well, where the current state of the system depends on past states at distant times. There are different other causes of diabetes, such as genetic mutations, damage to the pancreas, unbalanced hormones, etc. Approximately 1.4 million new cases of diabetes were detected in 2019 [4]. More than 25 percent of them had no idea that they had the illness. Hyperglycemia causes problems such as dental disease, eye difficulties, foot problems, heart disease, renal disease, etc. Everyone can make efforts to reduce their risk of getting certain diabetes-related health issues. Numerous authors have thought about modelling diabetes mellitus mathematically using fractional differential equations with non-integer and integer derivatives [5–7]. Fractal calculus is a straightforward, constructive, and algorithmic way to describing natural phenomena that cannot be achieved using smooth differentiable structures and standard modeling tools such as differential equations [8]. Fractional calculus, often referred to as the calculus of non-integer order integration and differentiation, generalizes the standard differential calculus by extending it to be defined and valid for any real or complex orders. Such a powerful scientific tool had been used within the area of pure mathematics for some time until the researchers witnessed its unique performance, not only in describing, but also for anticipating various natural and artificial phenomena [9–11]. As a generalization of integer-order (classical) differentiation and integration operators, fractional-order operators provide some unique properties for describing non-locality that comes naturally in the dynamics of such complex phenomena. It becomes the main reason that non-integer order calculus is applied tremendously in various areas of sciences and engineering in recent years [12–14]. A wide variety of different mathematical modeling has been constructed to analyze, comprehend and simulate the glucose and insulin dynamics that lead to diabetes. Since the late 1950s, public health professionals have focused on control and elimination of infectious disease organs. Bolie (1961) was a forerunner in this sector [15] where he had developed a basic linear model for glucose and insulin using ordinary differential equations. The International Diabetes Federation (IDF) claims that there is a rising diabetes epidemic that could have catastrophic effects on the planet. Currently, 10.5 % of adults worldwide have diabetes. Jajarmi et al. [16] employed a numerical strategy for the study of fractional modeling and optimal control of diabetes.

The critical characteristic of fractional order derivatives, known as the memory effect, fractional calculus theory and application, have been extensively utilized to depict dynamical processes in domains of science, engineering, and other fields [17–19]. Agrawal introduced Riemann-Liouville fractional derivative-driven general optimal control problems [20]. The same author developed a reliable numerical framework for the mathematical model and presented related optimal control problems using Caputo derivatives in another study [21]. Numerous scholars have created and studied optimum control problems, where controlled dynamic problems are characterized by R-L derivatives [22–24]. In recent years, authors [25, 26] developed a basic linear model for glucose and insulin using ordinary differential equations. For an optimal control problem with both first-order and non-integer derivatives in the state model, necessary optimality requirements have been derived [27].

The main objective of this research is to investigate fractional mathematical model involving the population evolving from pre-diabetes to diabetes with and without complications by applying fractional optimal control strategies. A new fractional model for diabetes involving the Caputo-derivative is proposed and studied within the non-integer order calculus. The new fractional model has provided better results in terms of simulation and consequently achieved the objective of modelling in comparison to the integer order model. The results are significant in terms of better understanding of disease progression and advantage is gained for the better comprehension of diabetes disease that cannot be achieved by the use of classical approach.

We summarize this study as follows: We define some basic concepts of fractional theory in Section 2. In Section 3, mathematical formulation of the fractional optimal control problem is considered. In Section 4, formulation of a controlled diabetic model is considered, and a necessary condition for optimality of model problem is derived. In Section 5, we present the steps of Forward backward sweep method to obtain the optimal control approximation. Section 6 presents the solution of the problem and experimentation of the results in the form of simulation. Section 7 deals with the concluding remarks.

2. Preliminary concept

This segment provides a quick overview of the fractional optimal control model's mathematical formulation. The Riemann-Liouville derivative and Caputo derivative are two fractional derivatives that are used in engineering and mathematical modelling the most frequently.

Definition 1. The left R-L fractional derivative is given as [25]

$$ {}_{a}^{L}D_{\varsigma}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^{m} \int_{a}^{x} f(t)(x-t)^{(m-1-\alpha)} dt, $$

where $m-1<\alpha<m$. □

Definition 2. The right R-L fractional derivative is given as [25]

$$ {}_{a}^{R}D_{x}^{\alpha} f(x) = -\frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^{m} \int_{a}^{x} (x-t)^{(m-1-\alpha)} f(t) dt. $$

where order α fulfills $m-1 \leq \alpha < m$ where Γ denotes Euler's Gamma function. □

Definition 3. The left Caputo fractional derivative is termed as [25,26]

$$ {}_{a}^{C}D_{x}^{\alpha} g(x) = \frac{1}{\Gamma(m-\alpha)} \int_{a}^{x} g(\tau)(x-\tau)^{(m-1-\alpha)} \left(\frac{d}{dx}\right)^{m} d\tau. $$

Definition 4. The right Caputo fractional derivative is termed as [25,26]

$$ {}_{x}^{C}D_{b}^{\alpha} g(y) = -\frac{1}{\Gamma(m-\alpha)} \int_{x}^{b} g(\tau)(x-\tau)^{(m-1-\alpha)} \left(\frac{d}{dx}\right)^{m} d\tau. $$

Where α is the order of the Caputo derivative. □

Definition 5. The Atangana-Baleanu-Caputo derivative and integral of $z(v,t)$ of order ρ are defined as [26]. For $z \in \mathbb{H}^1(0,T)$ and $0 < \rho < 1$,

$$^{ABC}D^{\rho} z(v,t) := \frac{P(\rho)}{1-\rho} \int_0^t E_{\rho}\left[-\frac{\rho}{1-\rho}(t-\xi)^{\rho}\right] z(v,\xi)d\xi,$$
$$^{ABC}I^{\rho} z(v,t) := \frac{1-\rho}{P(\rho)} z(v,t) + \frac{\rho}{P(\rho)\Gamma(\rho)} \int_0^t (t-\xi)^{\rho-1} z(v,\xi)d\xi,$$

where E_ρ is the Mittag-Leffler function and $P(\rho)$ with $P(\rho)|_{\rho=0,1} = 1$ represents the normalization function. □

3. Fractional optimal control formulation

Optimal control formulation using Caputo fractional derivatives was carried out by Agrawal [28]. The main goal is to develop an optimal control u^*, which reduces the objective functional which involves:

$$J(u) = \int_0^1 F(x,u,t)dt. \tag{1}$$

Considering the fractional dynamics control

$${}_0^C D_t^{\alpha} x(t) = W(x,u,t), \tag{2}$$

and the initial condition is

$$x(0) = x_0, \tag{3}$$

where α is the fractal dimension [29], and $x(t)$ is state variable, $F(x,u,t)$ and $W(x,u,t)$ represent the arbitrary functions. By combining the Lagrange multiplier approach, we derive at the essential conditions for a fractional order controlled problem. Necessary conditions are given as,

$${}_0^C D_t^{\alpha} x(t) = W(x,u,t), \tag{4}$$

$${}_t^C D_1^{\alpha} \lambda = -\frac{\partial F}{\partial x} - \lambda \frac{\partial W}{\partial x}, \tag{5}$$

$$\frac{\partial F}{\partial x} + \lambda \frac{\partial W}{\partial x} = 0, \tag{6}$$

with

$$x(0) = x_0 \quad \text{and} \quad \lambda(1) = 0, \tag{7}$$

where λ is co-state variable referred to as the Lagrange multiplier.

4. Controlled diabetics model with fractional derivatives

In this part, we discuss and obtain the necessary conditions of the fractional optimal control problem. Finding an ideal control that minimizes the number of diabetic patients with and without complications that is, finding an ideal control u^* that minimizes objective functional is the primary objective:

$$J(u) = \int_0^1 F(X,u,t)dt. \tag{8}$$

Considering the fractional dynamics control

$${}_0^C D_t^\alpha X(t) = W(X,u,t), \tag{9}$$

and the initial condition is

$$X(0) = X_0, \tag{10}$$

where

$$X(t) = (E,D,C)^T,$$
$$X(0) = \Big(E(0),D(0),C(0)\Big)^T,$$
$$F(X,u,t) = D(t) + C(t) + Au^2(t),$$

$${}_0^C D_t^\alpha E(t) = I - [\mu + (\beta_3 + \beta_1)(1-u))]E(t), \tag{11}$$

$${}_0^C D_t^\alpha D(t) = \beta_1 E(t)(1-u) - D(t)(\mu + \beta_2(1-u)) + C(t)\gamma, \tag{12}$$

$${}_0^C D_t^\alpha C(t) = \beta_3(1-u)E(t) + \beta_2(1-u)D(t) - C(t)(\delta + \mu + \nu + \gamma). \tag{13}$$

Where $X(t)$ represents a state vector and $u(t)$ represents a control variable. Initial conditions are $E(0) = E_0, D(0) = D_0$ and $C(0) = C_0$, where parameter A represents the weight on the benefits and cost (a balance the size of the terms) at $\alpha = 1$, the proposed and evaluated diabetic controlled dynamical model becomes a classical optimal control problem for uncontrolled diabetes models with and without complications. Also, keep in mind that dynamic constraint equations with initial condition stated above becomes a diabetes disease model when the control functions are set to 0. It is important to point out that there are thorough justifications in the literature for the formulation of essential criteria for optimality of various fractional dynamical systems [30]

$${}_0^C D_t^\alpha X(t) = W(x,u,t), \tag{14}$$

Table 1. Parameters of diabetics

Parameters	Characterization
$E(t)$	Represents the stage before diabetics of a patient
$D(t)$	Represents the diabetics without health problems
$C(t)$	Represents the diabetics with health problems
$u(t)$	The control variable represents the promotion of physical activity and a healthy diet

Table 2. Description of model parameters

Parameters	Definition
$I(t)$	Represents the ration before diabetes
μ	Represents normal death ratio
β_3	Represents probability of evolving diabetes at stage of complication
β_2	Represents probability of diabetic person evolving a complications
β_1	Represents probability of developing diabetes
γ	Represents the ratio rate at the point of health issue
δ	Represents the mortality rate due to complications
ν	Represents the ration of a patient at the point of disabled

$$ {}_{t}^{C}D_{b}^{\alpha}\lambda(t) = \frac{\partial F}{\partial X} + \lambda^{T}\frac{\partial W}{\partial X}, \tag{15} $$

$$ \frac{\partial F}{\partial U} + \lambda^{T}\frac{\partial W}{\partial U} = 0, \tag{16} $$

$$ X(0) = X_0 \quad \text{and} \quad \lambda(b) = 0. $$

here $\lambda(t)$= $(\lambda_1,\lambda_2,\lambda_3)^T$ denotes a variable. In this study, we show that the necessary conditions in Eqs. (14)-(16) are derived on the concepts of the authors in [31, 32]. We use compact form of previously given conditions to produce optimality system for fractional optimal control diabetes model.

$$ {}_{C}^{0}D_{t}^{\alpha}E(t) = I - [\mu + (\beta_3 + \beta_1)(1-u))]E(t), \tag{17} $$

$$ {}_{C}^{0}D_{t}^{\alpha}D(t) = \beta_1 E(t)(1-u) - D(t)(\mu + \beta_2(1-u)) + C(t)\gamma, \tag{18} $$

$$ {}_{C}^{0}D_{t}^{\alpha}C(t) = \beta_3(1-u)E(t) + \beta_2(1-u)D(t) - C(t)(\delta + \mu + \nu + \gamma), \tag{19} $$

$$ {}_{t}^{C}D_{b}^{\alpha}\lambda_1 = (\lambda_1 - \lambda_2)(1-u^*)\beta_1 + (\lambda_1 - \lambda_3)(1-u^*)\beta_3 + \mu\lambda_1, \tag{20} $$

$$ {}_{t}^{C}D_{b}^{\alpha}\lambda_2 = -1+\beta_2(1-u^*)(\lambda_2-\lambda_3)+\mu\lambda_2, \tag{21}$$

$$ {}_{t}^{C}D_{b}^{\alpha}\lambda_3 = -1+(\lambda_3-\lambda_2)\gamma+\lambda_3(\mu+\nu+\delta), \tag{22}$$

Furthermore, optimal control is termed as

$$u^* = \frac{1}{2A}[\beta_1 E(\lambda_2-\lambda_1)+\beta_3 E(\lambda_3-\lambda_1)+\beta_2 D(\lambda_3-\lambda_2)] \tag{23}$$

$E(0)=E_0, D(0)=D_0$ and $C(0)=C_0$ $\lambda_1(t_f)=0, \lambda_2(t_f)=0, \lambda_3(t_f)=0$

5. Forward backward sweep method

The method to solve the optimality system comprising equation (17) to equation (23) is the Forward Backward Sweep Method (FBSM). The name of the method suggests how an algorithm solves state equations forward in time and adjoint equation backward in time. A rough outline of the Algorithm is given below:
Step 1: Making an initial guess for the control function (23).
Step 2: Using the initial condition of the state equation and initial guess for the control, the state equations (17)-(19) are solved forward in time.
Step 3: Using the transversality condition and values for control and state equations, solve the adjoint equations (20)-(22) backward in time.
Step 4: Update u by entering the new value of state equation and adjoint equations.
Step 5: Check convergence. If the values of variables in this iteration and the last iteration are negligibly close, output the current values as solutions. If the values are not close, return to step 2. For Steps 2 and 3, the generalized Euler method is used. Using a step size h and ODE $x(t) = f(t,x(t))$, the approximation of $x(t+h)$ given $x(t)$ is

$$x(t+h) \approx x(t)+\frac{h^{\alpha}}{\Gamma(\alpha+1)}f(t,x(t)), \quad 0<\alpha<1 \tag{24}$$

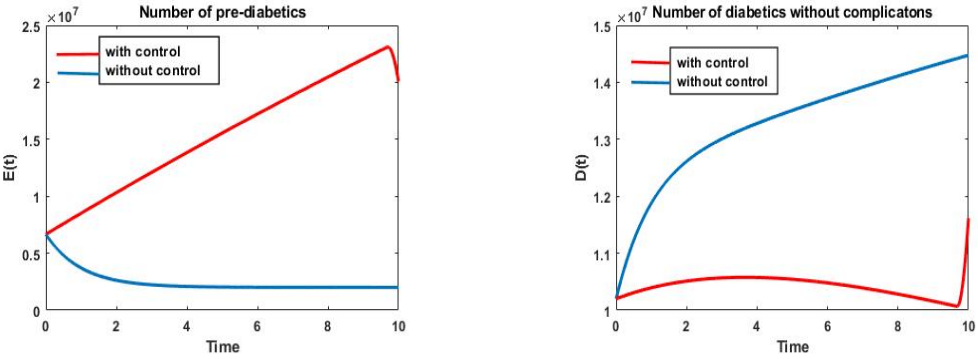
For the execution of step-5, the sufficient requirement is $\| x-oldx \| = \sum_{i-1}^{N+1}$ to be small, where $N+1$ is the length of vector. Further, the relative error, for the state vector, x, is given below. Note that the k represents the iteration step, not the k-th element of x.

$$\frac{\| x^{(k)}-x^{(k+1)} \|}{\| x^{(k)} \|}, \tag{25}$$

where δ is the accepted tolerance. The relative error is then solved so that there is no division because it is possible that $\| x^{(k)} \|$. When this is done, the relative error becomes $\| x^{(k)} \| - \| x^{(k)}-x^{(k+1)} \| \geq 0$. When this is true for all three vectors being tested, the algorithm stops and the current control is the optimal control approximation.

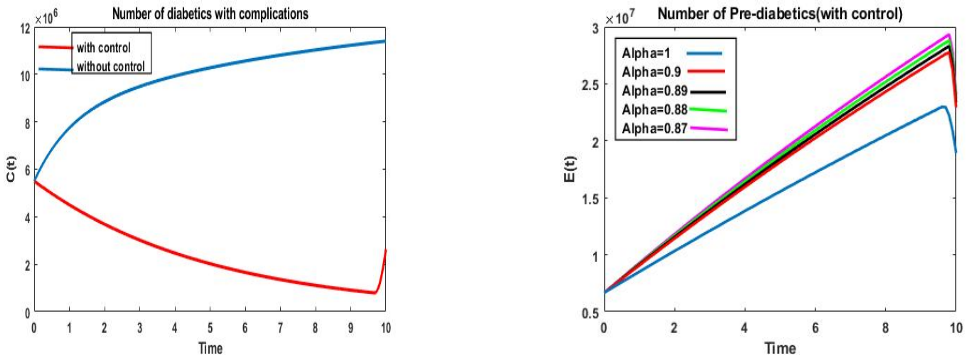
6. Numerical solution

In this segment, we discuss on our obtained results so that we solve the optimality system of a biological model and use a FBSM based on the generalized Euler method. The Forward-backward approach using generalized Euler method is used in this work to numerically compute the optimality system's solutions. One of numerical approaches for fractional order models is the generalised euler method, which has recently been used in a TB model along with the forward-backward algorithm. The numerical simulations are performed using the initial conditions and model parameter values listed below: $E(0) = 6660000$, $D(0) = 10200000$, $C(0) = 5500000$, $\beta_3 = 0.5$, $\nu = 0.05$, $\beta_1 = 0.5$, $\beta_2 = 0.1$, $I = 2000000$, $\gamma = 0.08$, $\delta = 0.05$, $\mu = 0.02$, $A = 3550000$, $h = 0.01$, $T = nh = 10R$ years. It is clear from the plots generated that the number of pre diabetics and the population without complications is increasing, and there is a significant reduction in the population with complications of non-integer orders with time-dependent controls (when $\alpha = 0.87, 0.88, 0.89, 0.90, 1$). It can be seen by the graphs that the number of pre-diabetics are increasing by applying control. Similarly, the ratio of people without complications and population with complications is decreasing by applying the optimal control technique. Figure 1a shows the dynamics of prediabetic population with the inclusion of zero versus non zero control variable. Figure 1b represents the dynamics of the diabetic population having no complications arising from disease with the inclusion of a zero versus non zero control variable. Figure 2a demonstrates the dynamics of a diabetic population with complications arising from disease with the inclusion of the zero versus non zero control variable. Figure 2b reveals the dynamics of the prediabetic population using fractional optimal control with different values of α. Figure 3a shows the dynamics of the diabetic population having complications using fractional optimal control with different values of α. Figure 3b demonstrates the number of diabetics with complications of fractional optimal control with different values of α and Figure 4a shows the comparison of pre-diabetics with different values of α without control. We showed the comparison of diabetics without complications with different values of α without control in Figure 4b, whereas the comparison of diabetics with complications with different values of α without control in Figure 5. The accuracy of a fractional derivative model compared to an integer order derivative model in predicting the dynamics of a pre-diabetic population can be observed from the simulation put into Figure 1. The graphs of the pre-diabetic population with fractional order derivative is showing the trend of population in more ascending order as compared to the graph of integer order derivative.



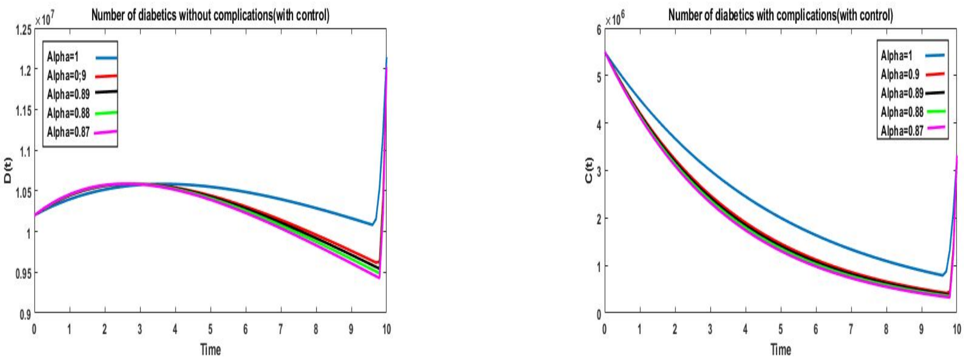
(a) (b)

Fig. 1. Number of pre-diabetics with and without control (a), Number of diabetics without complications with and without control (b)



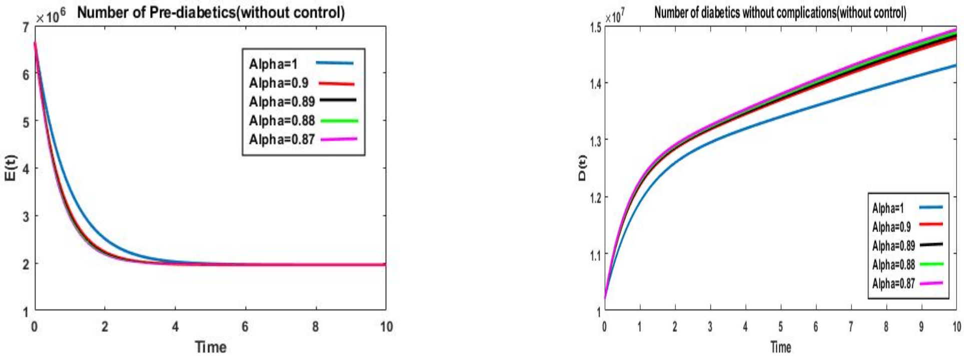
(a) (b)

Fig. 2. Number of diabetics with complications with and without control (a), Number of pre-diabetics of fractional optimal control with different values of α (b)



(a) (b)

Fig. 3. Number of diabetics without complications of fractional optimal control with different values of α (a), Number of diabetics with complications of fractional optimal control with different values of α (b)



(a) (b)

Fig. 4. Comparison of pre-diabetics with different values of α without control (a), Comparison of diabetics without complications with different values of α without control (b)

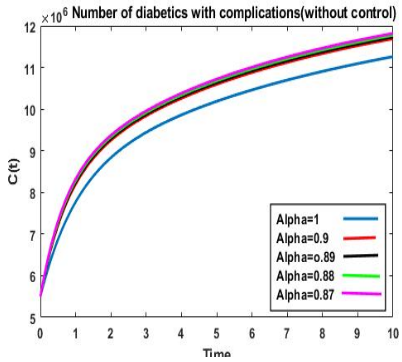


Fig. 5. Comparison of diabetics with complications with different values of α without control

7. Conclusion

In the present research, a mathematical model described by fractional derivatives has been considered and presents a population infectious of pre-diabetics where diabetics are considered with and without health problems. The simulations that occur show that when control is applied, the diabetic population with and without complication is reduced. We have substantially more accurate numerical solutions to the optimal control issue in the sense of Caputo derivative than the integer orders. The extended Euler method is generally simpler and computationally less expensive compared to more advanced numerical techniques. This can be advantageous when dealing with large-scale models or when computational resources are limited. However, in optimality problems where high accuracy is required, the trade-off between computational efficiency and solution accuracy needs to be carefully evaluated. Further, in optimality problems related to diabetic population models, where precise predictions are often crucial for informing medical decisions and treatment strategies, the accuracy of the numerical solution is paramount.

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