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ON THE STABILITY ANALYSIS FOR A SEMI-ANALYTICAL SCHEME FOR SOLVING THE FRACTIONAL ORDER BLOOD ETHANOL CONCENTRATION SYSTEM USING LVIM

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Abstract. The purpose of this article is to present the Laplace variational iteration method, which combines the VIM with the Laplace transform approach (LVIM). This combination will result in a better and more quickly convergent sequence since nonlinear fractional differential equations (FDEs) cannot be solved using the Laplace transform. With the use of the fixed point theory, the stability analysis is specifically discussed and examined. The blood ethanol concentration system is solved numerically by using the suggested scheme. This model can be represented by a system of FDEs. The investigation will employ the Caputo-Fabrizio fractional derivative. To provide a more in-depth study of this model, we have taken it in its fractional form so that we can more accurately follow the behavior of the solution in the future and history based on the memory effect of fractional derivatives. We determine the accuracy and efficiency of the provided process by evaluating the absolute errors, and a comparison with the existing published work. The results show that the approach is a useful tool for simulating this model.

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1. Introduction

Numerous authors have remained interested in fractional calculus during the past three decades [1]. Researchers have found that creating novel fractional derivatives with varied singular or non-singular kernels is essential to meet the demand for modeling real-world problems in numerous fields including mechanics, biology, and engineering [2], to name a few. Since most FDEs don't have an exact solution, numerical and approximation approaches must be applied, as stated in [3].

Recently, Caputo and Fabrizio (CF) suggested a novel operator by substituting the regular kernel for the singular kernel in the traditional Caputo derivative [4]. The exponential kernel, which is a non-singular kernel, is employed with the CF operator. It can depict the complete impact of the memory in addition to having two distinct representations for temporal and spatial elements [5]. The space-time fractional diffusion equation has successfully used this novel operator, as shown in [6]. To get an accurate solution for the fractional models stated in CF-operator, there have only been a small number of works done thus far because the CF-operator still new. For more details on the definitions of fractional derivatives and their features, see the citation provided in [7].

We know that the memory property of the fractional derivative means that we can follow the behavior more broadly and accurately. Here, the blood ethanol concentration system (BECS) will be studied and solved in its CF-fractional form. Please see, for example, but not limited to [8] where the authors also considered this model with different forms of the fractional derivative with singular kernel (Caputo) and non-singular kernel (Atangana-Baleanu-Caputo) fractional derivatives. As in many models, such as the basic processes of relaxation, diffusion, oscillations, wave propagation, etc., we can also see that the BECS has been revisited by several authors by introducing fractional-order derivatives in the governing ordinary differential equations (ODEs). This leads to super-slow or intermediate processes that, in mathematical physics, we may refer to as fractional phenomena. On the other side, we know that such models rely on ODEs. Although these models have useful analytical properties when analyzed over a long period, they perform poorly on short-time scales. One reason for this is that they are essentially 'memoryless' which is, as we know, the reservoir of that memory in the presence of the fractional derivative. Finally, we can conclude that including memory effects in modeling the BECS helps us follow its behavior more accurately and closer to reality, and then it can be controlled. For more general and specialized notes related to the model, see also [9, 10].

We employ the VIM [11] approach with the Laplace transform method in the current investigation. The approximate solutions of the fractional blood ethanol concentration system is investigated next using the combination technique, and the Laplace variational iteration [12]. The major benefit of this procedure is that it converges more quickly than the current method does from similar literature [13]. In addition, the LVIM is implemented to solve some problems, such as the non-linear fractional Burger's equation [14], and the Schrödinger equation [15].

2. Preliminaries and concepts

2.1. Some definitions of fractional calculus and Laplace transforms

Definition 1. The Caputo-Fabrizio fractional derivative ${}^{cf}D^{\nu}$ of order $0 \leq \nu \leq 1$ of a function $\varphi(t) \in H^1(a,b)$, is defined by [16]:

$$ {}^{cf}D^{\nu}\varphi(t) = \frac{1}{1-\nu}\int_a^t \varphi'(\tau)e^{\frac{-\nu(t-\tau)}{1-\nu}}d\tau. $$

Definition 2. Let $\varphi(t)$ be a function, then its Laplace transform is defined as:

$$ Ł\{\phi(t)\} = \Phi(s) = \int_0^{\infty} e^{-st}\phi(t)dt, \tag{1} $$

and the Laplace transform of $\varphi(t)$ in the Caputo-Fabrizio sense is given by [5]:

$$ Ł\{{}^{cf}D^{\nu+m}\phi(t)\} = \frac{s^{m+1}Ł\{\phi(t)\} - s^m\phi(0) - s^{m-1}\phi'(0) - ... - \phi^{(m)}(0)}{s+\nu(1-s)}. \tag{2} $$

For $m = 0$, we have:

$$ Ł\{{}^{cf}D^{\nu}\phi(t)\} = \frac{sŁ\{\phi(t)\} - \phi(0)}{s+\nu(1-s)}. \tag{3} $$

2.2. Some stability concepts

Many stability concepts have been developed in the last few decades, among which are Lyapunov stability [17], exponential stability [18], and so on. Then, we use the Banach contraction principle [19] to check the stability of the proposed LVIM. For this, let us recall some main definitions from the fixed point theory.

Definition 3. Let $(\Omega, |.|)$ be a metric space. A mapping $A : \Omega \to \Omega$ is called a contraction mapping, if for all $\varphi_1, \varphi_2 \in \Omega$, and $0 < \gamma < 1$ we have:

$$ |A\varphi_1 - A\varphi_2| \leq \gamma|\varphi_1 - \varphi_2|. \tag{4} $$

This means that every pair of points $\varphi_1, \varphi_2 \in \Omega$ has images that are closer than the points φ_1, φ_2 or, i.e.

$$ \frac{|A\varphi_1 - A\varphi_2|}{|\varphi_1 - \varphi_2|}, $$

does not exceed γ [19].

Further, let us recall Picard's existence and uniqueness theorem for differential equations. So we consider the following first-order initial value problem:

$$ \dot{u}(t) = A(t;u), \qquad u(t_0) = u_0, \tag{5} $$

with t_0 and u_0 two given real numbers. Let A be a continuous mapping on the rectangle,

$$ \bar{R} = \{(t;u) : \ |t-t_0| \leq a, \ |u-u_0| \leq b\}. $$

Thus, A is bounded on $\bar{R}$. Then, for all $(t;u) \in \bar{R}$, we can write $|A(t;u)| \leq c$.

Assume that about its second argument, A meets the Lipschitz condition on $\bar{R}$. Consequently, a Lipschitz constant exists γ such that, for all $(t;u),(t;v) \in \bar{R}$,

$$|A(t;u) - A(t;v)| \leq \gamma |u - v|. \tag{6}$$

Then the initial value problem (5) has a unique solution in the interval $(t_0 - \sigma, t_0 + \sigma)$ under these circumstances, where $\sigma < \{a, \frac{b}{c}, \frac{1}{\gamma}\}$ (see [19]).

2.3. The proposed fractional blood ethanol concentration system

Calculating the amounts of alcohol in a person's blood $\Psi(t)$ and stomach $\Phi(t)$ is the main goal of this subsection. The primary source of the actual data used in the current research study was an experimental study conducted in [9]. Based on the fractional derivative kinetic reaction ($0 < \nu \leq 1$), the model takes the following form:

$$D^{\nu}\Phi(t) = -\beta^{\nu}\Phi(t), \tag{7}$$

$$D^{\nu}\Psi(t) = \beta^{\nu}\Phi(t) - \mu^{\nu}\Psi(t), \tag{8}$$

$$\Phi(0) = \bar{\Phi}_0, \qquad \Psi(0) = 0, \tag{9}$$

where $\Phi(t)$ is the concentration of alcohol in the stomach [mg/L], $\Psi(t)$ is the concentration of alcohol in the blood [mg/L], β is the rate law constant 1 [min^{-1}], μ is the rate law constant 2 [min^{-1}] at any time t [min], and Φ_0 is the initial alcohol concentration in the stomach [mg/L] [9].
The exact solution of this system is defined as [10]:

$$\begin{aligned} \Phi(t) &= \Phi_0 E_{\nu}(-\beta^{\nu} t^{\nu}), \\ \Psi(t) &= \Phi_0 \beta^{\nu} \sum_{r=0}^{\infty} \sum_{q=0}^{\infty} \frac{(-\beta^{\nu})^r (-\mu^{\nu})^q}{\Gamma(r\nu + q\nu + \nu + 1)} t^{r\nu + q\nu + \nu}. \end{aligned} \tag{10}$$

With this model (7)-(9) in its fractional form, we can more precisely characterize the effect of the solution in the future and in history based on the memory effect of fractional derivatives. As it is also known, although the mathematical models with integer derivatives play an important role and have significance in understanding the dynamics of such systems, they have some limitations, such as these systems do not have memory or non-local effects; therefore, these models in the usual form are sometimes inappropriate, and it is necessary to convert several models into FDEs form. As a general case, FDEs are frequently used in the study of anomalous phenomena in nature and in the theory of complex systems as well as when taking into account the properties of the curve over a large extent [9, 10].

3. Basis of the Laplace VIM

This section is focused on presenting the basis of the Laplace VIM. For this, recall the general FDE:

$$D^{\nu}u(t)+L(u(t))+N(u(t))=g(t), \qquad \text{subject to} \qquad u(0)=u_0, \tag{11}$$

where $L(u(t)), N(u(t))$, and $g(t)$ are linear, nonlinear and known functions, respectively. Additionally, we consider D^{ν} in the CF-sense.

Applying the VIM in Equation (11) we get:

$$u_{n+1}(t)=u_n(t)+\lambda[D^{\nu}u(t)+L(u(t))+N(u(t))-g(t)].$$

Additionally, applying the Laplace transform, the variable t becomes a new one s, such that:

$$u_{n+1}(s)=u_n(s)+\lambda Ł[D^{\nu}u_n(t)+L(\tilde{u}_n(t))+N(\tilde{u}_n(t))-g(t)], \tag{12}$$

where $\tilde{u}_n(t)$ etc. are restricted values, which means $\delta\tilde{u}_n(t)=0$. Using relation Equation (3) we get:

$$Ł\{D^{\nu}u_n(t)\}=\frac{s\,u_n(s)-u_n(0)}{s+\nu(1-s)}, \qquad Ł\{\delta D^{\nu}u_n(t)\}=\frac{s\,\delta u_n(s)-\delta u_n(0)}{s+\nu(1-s)},$$

where $\delta u_n(0)=0$. Then, we obtain:

$$Ł\{\delta D^{\nu}u_n(t)\}=\frac{s\,\delta u_n(s)}{s+\nu(1-s)}.$$

From the optimization conditions,

$$\frac{\delta\tilde{u}_{n+1}(s)}{\delta\tilde{u}_n(s)}=0, \quad \delta\tilde{u}_n=0,$$

we get

$$1+\lambda\left[\frac{s\,\delta\tilde{u}_n(s)}{s+\nu(1-s)\delta\tilde{u}_n(s)}\right]=0.$$

The above equation gives $\lambda=-\dfrac{s+\nu(1-s)}{s}$.
Replacing Equation (12), we obtain:

$$u_{n+1}(s)=u_n(s)-\Big(\frac{s+\nu(1-s)}{s}\Big)Ł[D^{\nu}u_n(t)+L(u_n(t))+N(u_n(t))-g(t)].$$

Using the inverse Laplace transform we get:

$$u_{n+1}(t)=u_n(t)-Ł^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big)Ł[D^{\nu}u_n+L(u_n)+N(u_n)-g(t)]\Big]. \tag{13}$$

Replacing $n = 0, 1, 2, ...,$ we get the following successive approximations, $u_1(t)$, $u_2(t), ...,$ which converge to the exact solution, i.e.,

$$u(t) = \lim_{n\to\infty} u_n(t).$$

4. Stability analysis of the LVIM scheme

Theorem 1. The Laplace variational iteration scheme (13) is unconditionally stable.

PROOF Considering the above discussion in Section 2, and the scheme (13) to check the stability of LVIM as follows:

$$u_{n+1}(t) = A(u_{n+1}) = u_n(t) - Ł^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big) Ł[D^{\nu}u_n(t) + L(u_n(t)) + N(u_n(t)) - g(t)]\Big].$$

Then we have:

$$|A(u_{n+1}) - A(u_{m+1})| \leq |u_n - u_m| - Ł^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big) Ł\Big[D^{\nu}(u_n - u_m) + L(u_n - u_m) + N(u_n - u_m)\Big]\Big],$$

which implies,

$$\frac{|A(u_{n+1}) - A(u_{m+1})|}{|u_n - u_m|} \leq 1 - Ł^{-1}\Big[\Big(\frac{s+\nu(1-s)}{s}\Big) Ł\Big[D^{\nu}(u_n - u_m) + L(|u_n - u_m|) + N(|(u_n - u_m)|)\Big]\Big] = \gamma.$$

Then, we obtain:

$$\frac{|A(u_{n+1}) - A(u_{m+1})|}{|u_n - u_m|} \leq \gamma,$$

which means $|A(u_{n+1}) - A(u_{m+1})| \leq \gamma|u_n - u_m|$. Then, the proposed scheme is unconditionally stable in view of Equation (6). ■

5. Implementation of Laplace VIM

How the LVIM works is explained in [20], and how different types of differential equations may be solved with it. It was demonstrated in the source that VIM has been applied in a good form for solving FDEs. Following the discussion in [21], in this section we expand the application of the LVIM to solve the BECS [21]:

The correction functional for the system (7)-(8) can be created as follows:

$$\Phi_{m+1}(s) = \Phi_m(s) + \lambda_1 Ł\left[\left(D^{\nu}\tilde{\Phi}_m(t) + \beta^{\nu}\tilde{\Phi}_m(t)\right)\right], \tag{14}$$

$$\Psi_{m+1}(s) = \Psi_m(s) + \lambda_2 Ł\left[\left(D^{\nu}\Psi_m(t) - \beta^{\nu}\tilde{\Phi}_m(t) + \mu^{\nu}\tilde{\Psi}_m(t)\right)\right], \tag{15}$$

where $\lambda_k\,(k=1,2)$ are the general Lagrange multipliers, which can be identified by variational theory [22]. The optimality conditions give the following:

$$\frac{\delta\tilde{\Phi}_{m+1}(s)}{\delta\tilde{\Phi}_m(s)}=0,\quad \frac{\delta\tilde{\Psi}_{m+1}(s)}{\delta\tilde{\Psi}_m(s)}=0,\quad \delta\tilde{\Phi}_n=0,\quad \delta\tilde{\Psi}_n=0,$$

where $\tilde{\Phi}_m$ and $\tilde{\Psi}_m$ are considered as restricted variations and the above equations give:

$$\lambda_1=\lambda_2=-\frac{s+\nu(1-s)}{s}.$$

Replacing λ_1,λ_2 into the functional equations (14)-(15) and using inverse Laplace transform to obtain the following iteration formula:

$$\Phi_{m+1}(t)=\Phi_m(t)-Ł^{-1}\left[\left(\frac{s+\nu(1-s)}{s}\right)Ł\left[D^{\nu}\Phi_m(t)+\beta^{\nu}\Phi_m(t)\right]\right],\qquad(16)$$

$$\Psi_{m+1}(t)=\Psi_m(t)-Ł^{-1}\left[\left(\frac{s+\nu(1-s)}{s}\right)Ł\left[D^{\nu}\Psi_m-\beta^{\nu}\Phi_m+\mu^{\nu}\Psi_m\right]\right].\qquad(17)$$

If the initial conditions of the system are satisfied, the initial approximations $\Phi_0(t)$ and $\Psi_0(t)$ can be arbitrarily chosen. Last but not least, we approximate $\Phi(t)$, and $\Psi(t)$ by the m-th terms, $\Phi_m(t)$, and $\Psi_m(t)$, respectively:

$$\Phi(t)=\lim_{m\to\infty}\Phi_m(t),\quad \Psi(t)=\lim_{m\to\infty}\Psi_m(t).\qquad(18)$$

6. Numerical applications

In this section, we present the approximate solution of the BECS. Considering the LVIM formula (16)-(17) with various values of ν, m with $\beta=0.02873$, $\mu=0.08442$, and starting conditions $\Phi_0=4$, $\Psi_0=0$ through Figures 1-4, we are now prepared to obtain an approximation of the solution to the model under study.

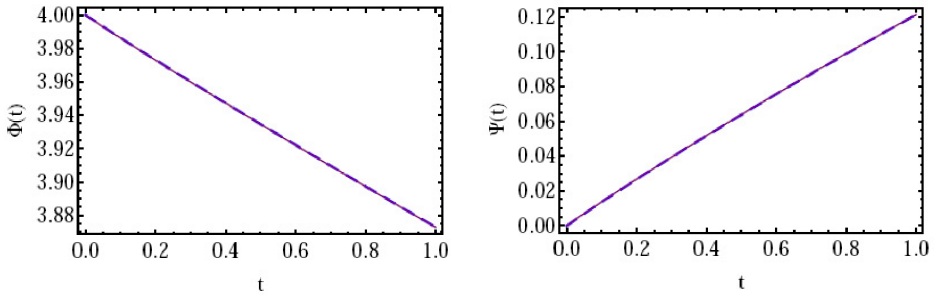

Fig. 1. Comparison the approximate and exact solutions with $\nu=0.97,$ and $m=4$

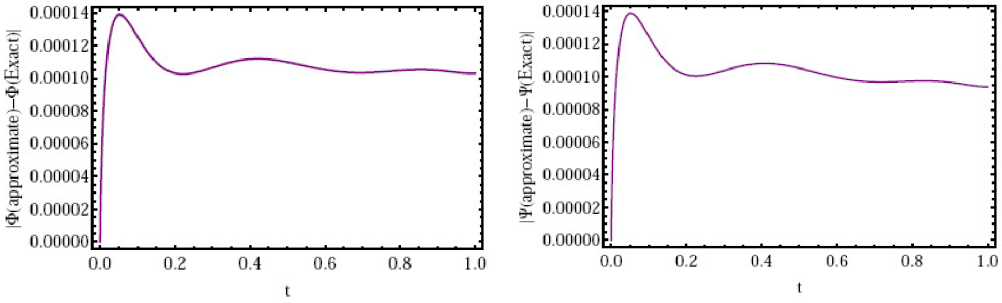

Fig. 2. The absolute error with $\nu = 0.95,$ and $m = 5$

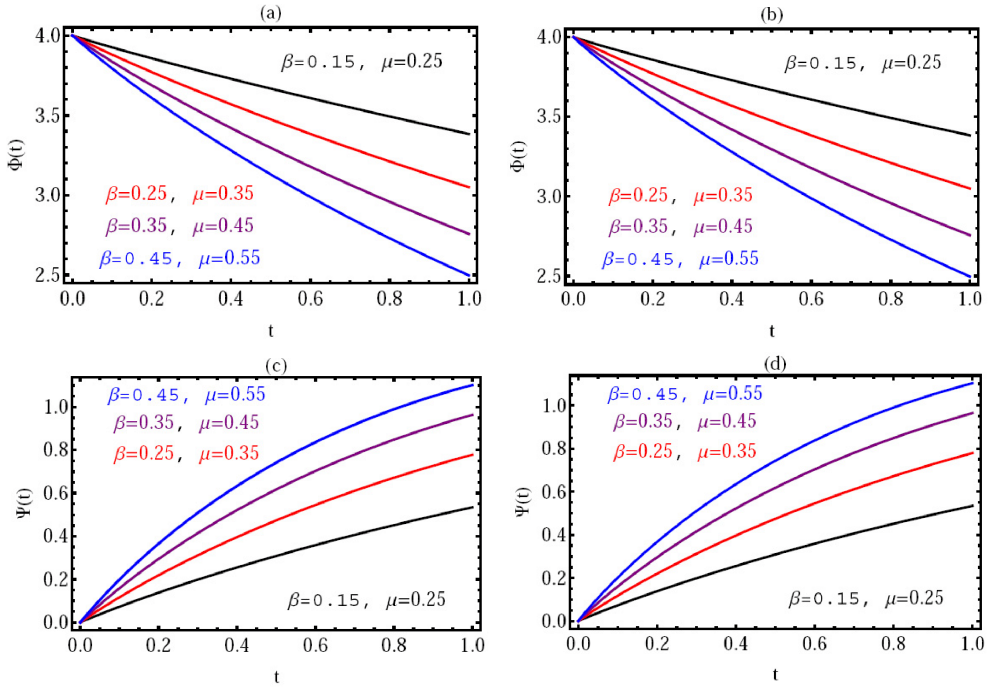

Fig. 3. Behavior of the approximate and exact solutions with distinct values of β and μ at $m = 5, \Phi_0 = 4$

By using the LVIM formula (16)-(17), if we start with $\Phi_0(t) = 4$, and $\Psi_0(t) = 0$, we can obtain:

$$\Phi_1(t) = 4 - \frac{0.429t^{\nu}}{\Gamma(1+\nu)}, \quad \Psi_1(t) = \frac{0.429t^{\nu}}{\Gamma(1+\nu)},$$

$$\Phi_2(t) = 4 - \frac{0.429t^{\nu}}{\Gamma(1+\nu)} + \frac{0.046t^{2\nu}}{\Gamma(1+2\nu)}, \quad \Psi_2(t) = \frac{0.429t^{\nu}}{\Gamma(1+\nu)} - \frac{0.136t^{2\nu}}{\Gamma(1+2\nu)},$$

$$\Phi_3(t) = 4 - \frac{0.429t^{\nu}}{\Gamma(1+\nu)} + \frac{0.046t^{2\nu}}{\Gamma(1+2\nu)} - \frac{0.002t^{3\nu}}{\Gamma(1+3\nu)},$$

$$\Psi_3(t) = \frac{0.429t^{\nu}}{\Gamma(1+\nu)} - \frac{0.136t^{2\nu}}{\Gamma(1+2\nu)} + \frac{0.012t^{3\nu}}{\Gamma(1+3\nu)},$$

and so on; similarly the rest of the components can be obtained. Here, in our calculations, we approximated $\Phi(t)$, and $\Psi(t)$, by $\Phi(t) \cong \Phi_6(t)$, and $\Psi(t) \cong \Psi_6(t)$, respectively.

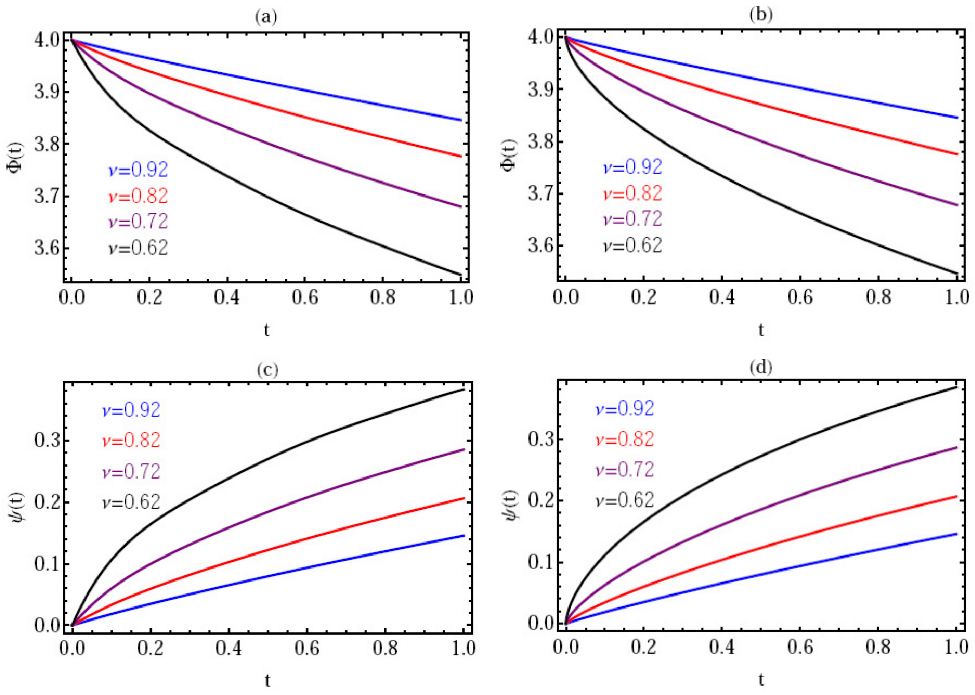

Fig. 4. Behavior of the approximate and exact solutions with distinct values of ν at $m = 5$

Figure 1 gives a comparison between the numerical and exact solutions at $\nu = 0.97$ with $m = 4$, and Figure 2 shows the absolute error at $\nu = 0.95$ with $m = 5$. The behavior of the numerical (a and c), and exact (b and d) solutions with $\nu = (0.92, 0.82, 0.72, 0.62)$ at $m = 5$ is shown in Figure 3. Finally, the behavior of the numerical (a and c), and exact solutions (b and d) with $\beta = 0.15, 0.25, 0.35, 0.45$, and $\mu = 0.25, 0.35, 0.45, 0.55$ with $m = 5$ is shown in Figure 4.

Based on the stability analysis which is presented in Section 4, the proposed LVIM scheme is unconditionally stable. However, we discussed and were satisfied with the stability of the method through the numerically simulated solutions presented. We evaluated the absolute error (AE) of the solution with different values of the order of the fractional derivative ν, and the order of approximation, m (see Fig. 2), and we also considered the parameters β, and μ also with various values to see its

effect on the behavior and stability of the solution (see Fig. 3). Additionally, from Figure 4, we can see the effect of the order of fractional derivatives is consistent with the natural behavior of the model. Furthermore, to provide further confirmation, we compared our obtained results with those obtained by using the Chebyshev spectral collocation method for the same model but with the Atangana-Baleanu-Caputo fractional derivative with non-singular kernel [8], by computing the relative error (RE) through Table 1 at $\nu = 0.9$, $\Phi_0 = 6$ and $\Psi_0 = 0$. As a result, the behavior of the numerical solution is heavily reliant on ν, β, and μ, proving that the recommended numerical approach can successfully solve the given system in the situation of this type of fractional derivatives. In addition, the given comparison demonstrates how thorough the approach suggested in this article is.

Blood alcohol concentration is a commonly known and widely used measure of the state of intoxication a person may be in, and this measure is used for legal or medical purposes. BAC is expressed as a percentage of alcohol in the blood in units of mass of alcohol per unit of blood density or mass of alcohol per mass of blood, and this varies from one country to another. For example, in North America, the alcohol concentration is expressed as 0.10 (0.10% or one-tenth of a percent) meaning there are 0.10 grams of alcohol per deciliter of blood [23].

The range of alcohol concentration in the blood is (0, 0.5). For example, if it is in the range (0.20, 0.29), this leads to coma, inability to understand, weak sensations, and the possibility of a loss of consciousness. But if it exceeds 0.5, it causes a high risk of poisoning, including the possibility of death. As is clear from Figures 1, 3, and 4, the concentration of alcohol in the blood is in the safe range, and this is considered another criterion to measure the efficiency and accuracy of the proposed method and to achieve the stability and convergence in the resulting solutions.

Table 1. Comparison of the relative error for numerical solutions by two different methods

	RE of method [8]		RE of present method	
t	$\Phi(t)$	$\Psi(t)$	$\Phi(t)$	$\Psi(t)$
0.0	2.159753E-04	3.456123E-06	5.741025E-07	2.321045E-08
0.1	6.852320E-05	3.025874E-06	1.021345E-07	3.123456E-08
0.2	5.852014E-05	2.652413E-05	2.654123E-06	5.963258E-07
0.3	3.132145E-05	3.980021E-06	5.321412E-07	8.956542E-08
0.4	3.852014E-04	2.012365E-06	3.632584E-07	5.123054E-07
0.5	7.952147E-05	0.014785E-06	3.321004E-07	3.696325E-09
0.6	1.852140E-05	2.963258E-05	0.954127E-06	1.321456E-07
0.7	8.654123E-05	2.012345E-05	4.321456E-06	3.012547E-07
0.8	0.753654E-05	1.150210E-06	3.021456E-07	3.110253E-07
0.9	1.852014E-05	0.321470E-05	2.123456E-06	2.524142E-07
1.0	3.951023E-06	3.852140E-05	2.951423E-07	3.321456E-07

7. Conclusions

The Laplace VIM is a potent technique and semi-analytical approach that can deal with FDEs that are linear or nonlinear. We implemented this procedure to numerically

solve the fractional blood ethanol concentration system. The suggested scheme is a more dependable method that converges more quickly than those found in the literature. The findings demonstrate that the technique used is efficient and cost-effective for obtaining the rough solutions of the suggested model. By including new words drawn from the solution's constituent parts, we can also regulate and reduce the absolute error. To demonstrate the viability of the suggested method, approximate solutions with various values of ν, m, and the AE are computed. Additionally, we demonstrate a connection between the approximations of the solution and the iterations of the series of successive approximations employed in the proof of a fixed point. The key benefit is that the suggested method simplifies the computational procedures and is effective for nonlinear dynamical systems analysis. Using real data, it has been demonstrated that the CF-fractional operator can estimate the concentration of alcohol in human blood much more accurately than the integer order version of the model. It has also been observed that this fractional operator performs equally well as that of its integer order model. We intend to deal with these models in the future, but on a larger scale, by generalizing this research to include a modified proposed method or additional types of fractional derivatives. The Mathematica software program is used to perform numerical simulation operations.

References

[1] Kilbas, S.G., & Marichev, O.I. (1993). *Fractional Integrals and Derivatives: Theory and Applications*. Yverdon: Gordon & Breach.

[2] Adel, M., Sweilam, N.H., Khader, M.M., Ahmed, S.M., Ahmad, H., & Botmart, T. (2022). Numerical simulation using the non-standard weighted average FDM for the 2Dim variable-order Cable equation. *Results in Physics*, *39*, 105682. DOI: 10.1016/j.rinp.2022.105682.

[3] Adel, M., Khader, M.M., & Algelany, S. (2023). High-dimensional chaotic Lorenz system: Numerical treated using Changhee polynomials of the Appell type. *Fractal and Fractional*, *7*(5), 398.

[4] Mirza, I.A., & Vieru, D. (2017). Fundamental solutions to advection-diffusion equation with time-fractional Caputo-Fabrizio derivative. *Computer and Mathematics with Applications*, *73*(1), 1-10.

[5] Caputo, M., & Fabrizio, A. (2015). A new definition of fractional derivative without singular kernel. *Progress in Fractional Differentiation & Applications*, *1*(2), 1-13.

[6] Aguilar, J.G. (2017). Space-time fractional diffusion equation using a derivative with nonsingular and regular kernel. *Physica A: Statistical Mechanics and its Applications*, *465*, 562-572.

[7] Khan, M.A., Ullah, S., Okosun, K.O., & Shah, K. (2018). A fractional order pine wilt disease model with Caputo-Fabrizio derivative. *Advances in Difference Equations*, *410*, 1-18.

[8] Khader, M.M., & Saad, K.M. (2020). Numerical treatment for studying the blood ethanol concentration systems with different forms of fractional derivatives. *International Journal of Modern Physics C*, *31*(3), 2050044. DOI: 10.1142/S0129183120500448

[9] Ludwin, C. (2011). Blood alcohol content. *Undergraduate Journal of Mathematical Modeling*, *3*(2), 1-10.

[10] Qureshi, S., Yusuf, A., Shaikh, A.A., Inc, M., & Baleanu, D. (2019). Fractional modeling of blood ethanol concentration system with real data application. *Chaos*, *29*, 1-15.

[11] Sweilam, N.H., & Al-Bar, F. (2007). Variational iteration method for coupled nonlinear Schrödinger equations. *Computers and Mathematics with Applications*, *54*(8), 993-999.

[12] Iqbal, J., Shabbir, K., & Guran, L. (2021). Semi-analytical solutions of some nonlinear-time fractional models using variational iteration Laplace transform method. *Journal of Function Spaces*, *2021*, 8345682.

[13] Morales-Delgado, V.F., Gomez-Aguilar, J.F., Yépez-Martinez, H., Baleanu, D., Jimenez, R.F., & Olivares-Peregrino, V.H. (2016). Laplace homotopy analysis method for solving linear PDEs using a fractional derivative with and without kernel singular. *Advances in Difference Equations*, *2016*, 164.

[14] Karakoc, S.G., & Ali, K.K. (2021). Theoretical and computational structures on solitary wave solutions of Benjamin Bona Mahony-Burgers equation. *Tbilisi Mathematical Journal*, *14*(2), 33-50.

[15] Zhang, J., Wang, J., & Zhou, Y. (2020). Numerical analysis for time-fractional Schrödinger equation on two space dimensions. *Advances in Difference Equations*, *53*, 1-16.

[16] Zhang, T., & Yongkun, L. (2022). Exponential Euler scheme of multi-delay Caputo-Fabrizio fractional-order differential equations. *Applied Mathematics Letters*, *124*, 107709.

[17] Dong, Y., Tang, X., & Yuan, Y. (2020). Principled reward shaping for reinforcement learning via Lyapunov stability theory. *Neurocomputing*, *393*, 83-90.

[18] Li, L., & Chen, W. (2020). Exponential stability analysis of quaternion-valued neural networks with proportional delays and linear threshold neurons: Continuous-time and discrete-time cases. *Neurocomputing*, *381*, 152-166. DOI: 10.1016/j.neucom.2019.09.051.

[19] Kreyszig, E. (1991). *Introductory Functional Analysis with Applications*. Hoboken: John Wiley Sons.

[20] He, J.H. (1998). Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Computer Methods in Applied Mechanics and Engineering*, *167*(2), 57-68.

[21] Iqbal, J., Shabbir, K., & Guran, L. (2022). Stability analysis and computational interpretation of an effective semi-analytical scheme for fractional order non-linear partial differential equations. *Fractal and Fractional*, *6*, 293. DOI: 10.3390/fractalfract6070393.

[22] Jafari, H., & Alipoor, A. (2011). A new method for calculating the general Lagrange multiplier in the variational iteration method. *Numerical Methods for PDEs*, *27*(4), 996-1001.

[23] *https://en.wikipedia.org/wiki/Blood_alcohol_content#cite_note−Med*2019 − 1.