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CHARACTERISTIC OF ION-ACOUSTIC WAVES DESCRIBED IN THE SOLUTIONS OF THE (3+1)-DIMENSIONAL GENERALIZED KORTEWEG-DE VRIES-ZAKHAROV-KUZNETSOV EQUATION

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Abstract. The generalized Korteweg-de Varies-Zakharov-Kuznetsov equation (gKdV-ZK) in (3+1)-dimension has been investigated in this research. This model is used to elucidate how a magnetic field affects the weak ion-acoustic wave in the field of plasma physics. To deftly analyze the wide range of wave structures, we utilized the modified extended tanh and the extended rational sinh-cosh methods. Hyperbolic, periodic, and traveling wave solutions are presented as the results. Consequently, solitary wave solutions are also attained. This study shows that the solutions reported here are distinctive when our findings are contrasted against well-known outcomes. Moreover, realized findings are figured out in 3-dimensional, 2-dimensional, and contour profile graphs for the reader to comprehend their dynamics due to parameter selection. According to the findings, we can conclude that the suggested computational techniques are simple, dynamic, and well-organized. These methods are very functional for numerical calculations of complex nonlinear problems. Our results include a fundamental starting point in understanding physical behavior and the structure of the studied systems.

MSC 2010: *00A05, 00A35, 97Mxx, 97Nxx, 97Pxx*
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1. Introduction

Non-linearity theories depend heavily on analytical solutions to nonlinear partial differential equations (NPDEs) which can be used to interpret natural phenomena in physical and applied studies such as fluid mechanics, hydrodynamics, mathematical physics, optics, elasticated media, chemical reactions, astrophysics, ecosystems, quantum theory, geology, plasma physics, wave propagation, and shallow water [1–5]. Several methods for obtaining solutions of nonlinear partial differential equations have been improved recently, for instance, the method of the inverse scattering [6], the Bernoulli sub-equation function methods [7,8], and the use of the Laplace transformation for the system which involves the Caputo fractional derivatives [9].

To investigate the ion-acoustic wave constructions in plasma physics, the (3+1)-dimensional gKdV-ZK equation is used which is taken in the following configuration [10,11],

$$u_t + \alpha u^2 u_x + \tau u_{xxx} + \ell \left(u_{yy} + u_{zz}\right)_x = 0. \tag{1}$$

In (1), the real constants α, τ and ℓ govern for the ambiguous allocation of nonlinear electromagnetic modes that characterize composites of hot isotherm, heat transfer in fluid, and cold immovable organisms. The $u(x,y,z,t)$ is the wave description that is characterized as the spatial and temporal different form in liquid illustrations. The (3+1)-dimensional (gKdV-ZK) and its variations versions are important nonlinear models appearing in physics and engineering. Since then, the search for the solution of the gKdV-ZK equation and its variations were studied intensively with respect to different methods, such as using the Lie symmetry approach [12], the bilinear transformation-model and the expansion version of the ϕ^6 method [13], the new modification of the extended direct algebraic method [14], implementing of the fractional direct algebraic methods and the extended direct algebraic method, to present solitary wave solutions. Furthermore, the stability analysis is discussed [15], the extended F-expansion technique and the expansion version of the $\exp(-\phi(\mathscr{F}))$ approach [16], the first integral method [17], through using Painlevé expansion, the Painlevé--Bäcklund transformations are obtained, and the bilinear forms of (1) have been derived [18]. The Bernoulli sub-ODE approach has been utilized [19]. The modified G'/G^2-expansion method, the $1/G'$-expansion technique, and the Kudryashov method have been applied [20]. The generalized exponential rational function method has been employed [21]. The new auxiliary equation method to obtain the closed forms of the traveling and solitary wave solutions has been implemented [22]. Considerable new and significant developments for investigating the ion-acoustic and traveling wave solutions for nonlinear partial differential equations using different types of mathematical methods were applied recently [23–30].

The algorithmic flow of this paper is as follows. In Section 1, the literature is reviewed related to the gKdV-ZK equation. In Section 2, the structures of the mentioned methods are discussed briefly. In Section 3, the suggested approaches are applied to the gKdV-ZK equation, and its obtained solutions are shown graphically, afterward, and in the last Section, the final thoughts have been addressed.

2. Descriptions of the methods

Usually, the construction structure of the applied methods relays on the following step

Step 1. Consider the following nonlinear partial differential equation:

$$\phi(\psi, \psi_x, \psi_t, \psi_y, \psi_z, \psi_{xzt}, \psi_{xx}, \psi_{yzt}, \psi_{xyzt}, \cdots) = 0, \tag{2}$$

where $\psi = \psi(x,y,z,t)$. Set

$$\psi(x,y,z,t) = \mathscr{U}(\mathscr{F}),\ \mathscr{F} = \delta_1 x + \delta_2 y + \delta_3 z + \delta_4 t, \tag{3}$$

where δ_i for $i = 1,2,3,4$ are arbitrary constants different from zero. If (3) is substituted in (2), then one obtains:

$$\mathscr{N}\left(\mathscr{U}, \mathscr{U}', \mathscr{U}'', \mathscr{U}''', \cdots\right) = 0, \tag{4}$$

where $\mathscr{U} = \mathscr{U}(\mathscr{F})$, $\mathscr{U}' = \dfrac{d\mathscr{U}}{d\mathscr{F}}$, $\mathscr{U}'' = \dfrac{d^2\mathscr{U}}{d\mathscr{F}^2}, \cdots$.

2.1. Structure of the extended rational sinh – cosh method (ERSCM)

This method is described in the following steps where step one is identically the same as the first step in the above

Step 2. Let (4) have solutions with the following forms:

$$\mathscr{U}(\mathscr{F}) = \frac{\gamma_0 \sinh(\mu\mathscr{F})}{\gamma_2 + \gamma_1 \cosh(\mu\mathscr{F})},\ \cosh(\mu\mathscr{F}) \neq -\frac{\gamma_2}{\gamma_1}, \tag{5}$$

or in the form:

$$\mathscr{U}(\mathscr{F}) = \frac{\gamma_0 \cosh(\mu\mathscr{F})}{\gamma_2 + \gamma_1 \sinh(\mu\mathscr{F})},\ \sinh(\mu\mathscr{F}) \neq -\frac{\gamma_2}{\gamma_1}, \tag{6}$$

where γ_0, γ_1 and γ_2 are purposeful coefficients to be determined later and μ is a non-zero wave number.

Step 3. Anonymous parameters can be discovered by replacing one of (5) or (6) into (4), assembling all the terms which have the same powers of $\cosh^\lambda(\mu\mathscr{F})$ or $\sinh^\lambda(\mu\mathscr{F})$ and equating to zero all the coefficients of $\cosh^\lambda(\mu\mathscr{F})$ or $\sinh^\lambda(\mu\mathscr{F})$ produces several algebraic equations. It is conceivable to find the solutions to the system of the algebraic equations utilizing various packages of symbolic computing tools.

Remark 1 Alternatively, one may gather and equalize to zero all the parameters for the same powers of $\cosh^{\lambda 1}(\mu\mathscr{F})\sinh^{\lambda 2}(\mu\mathscr{F})$, it generates a system of the algebraic equations, therefore solving the obtained system, where $\lambda_1, \lambda_2 = 0, 1, 2, \cdots n$. □

Step 4. By restoring the values of $\gamma_0, \gamma_1, \gamma_2$ and μ into (5) or (6), the solution of (4) can be found, combining parameters and re-installing found solutions, one can derive the solution for the (2).

2.2. Structure of the modified extended tanh method (METM)

The modified extended tanh method is abbreviated in the following steps where step one is the same as step one above.

Step 2. Assume that (4) has the following possible solution

$$\mathscr{U}(\mathscr{F}) = \mathscr{A}_0 + \sum_{i=1}^{m} (\mathscr{A}_i \mathscr{E}^i + \mathscr{B}_i \mathscr{E}^{-i}), \tag{7}$$

where $\mathscr{A}_0, \mathscr{A}_i, \mathscr{B}_i, i = 1, \cdots m$, are constants that have to be demonstrated later such that either $\mathscr{A}_m \neq 0$ or $\mathscr{B}_m \neq 0$, and the function $\mathscr{E}(\mathscr{F})$ satisfies the given Riccati ordinary differential equation:

$$\mathscr{E}'(\mathscr{F}) = \mathscr{B} + \mathscr{E}^2(\mathscr{F}), \tag{8}$$

solution of (8) are publicized as follows:
Family 1: The solutions of (8) have the following form for $\mathscr{B} < 0$

$$\mathscr{E}(\mathscr{F}) = -\sqrt{-\mathscr{B}} \tanh(\sqrt{-\mathscr{B}}\mathscr{F}), \text{ or } \mathscr{E}(\mathscr{F}) = -\sqrt{-\mathscr{B}} \coth(\sqrt{-\mathscr{B}}\mathscr{F}). \tag{9}$$

Family 2: The solutions of (8) have the following specifications for $\mathscr{B} > 0$

$$\mathscr{E}(\mathscr{F}) = \sqrt{\mathscr{B}} \tan(\sqrt{\mathscr{B}}\mathscr{F}), \text{ or } \mathscr{E}(\mathscr{F}) = -\sqrt{\mathscr{B}} \cot(\sqrt{\mathscr{B}}\mathscr{F}). \tag{10}$$

Family 3: When $\mathscr{B} = 0$, the solutions of (8) take the following form

$$\mathscr{E}(\mathscr{F}) = -\frac{1}{\mathscr{F}}. \tag{11}$$

Step 3. The positive integer m should be determined in (4) by applying the balancing principle between the greater power of the nonlinear terms and the highest order derivatives.

Step 4. Putting (7) and (8) in to (4) and combining all the phrases with the same power of $\mathscr{E}^i$ equating to zero for $i = 0, \mp 1, \mp 2, \cdots$, following the process will generate a system of algebraic equations that is solvable by using computer packages to obtain the values of the parameters.

Step 5. Contracted values of constants, and using equations (9)-(11), one can obtain the semi-analytic solution of (2).

3. Applications of the proposed methods

Applying wave transformation (3) to (1), one obtains the following:

$$\alpha\delta_1\mathscr{U}^2\mathscr{U}' + \left(\tau\delta_1^3 + \delta_2^2\delta_1\ell + \delta_3^2\delta_1\ell\right)\mathscr{U}''' + \delta_4\mathscr{U}' = 0, \tag{12}$$

and taking integration in (12) we assume $\mathscr{U}(\cdot)$ at the initial point is zero, then one ends up with

$$\alpha\delta_1\mathscr{U}^3 + 3\left(\tau\delta_1^3 + \delta_2^2\delta_1\ell + \delta_3^2\delta_1\ell\right)\mathscr{U}'' + 3\delta_4\mathscr{U} = 0. \tag{13}$$

The nonlinear ordinary differential equation observed in (13), is an approved equation to performing the balance principle between $\mathscr{U}''$ and $\mathscr{U}^3$.

3.1. Application of ERSCM to the aforementioned model

To solve (1) by using extended rational sinh-cosh method, assume that (13) has the solution of the following form:

$$\mathscr{U}(\mathscr{F}) = \frac{\gamma_0\cosh(\mu\mathscr{F})}{\gamma_1\sinh(\mu\mathscr{F}) + \gamma_2},\ \sinh(\mu\mathscr{F}) \neq -\frac{\gamma_2}{\gamma_1}, \tag{14}$$

where γ_i for $i = 0,1,2$ are parameters that should be $\gamma_0 \neq 0,\ \gamma_1^2 + \gamma_2^2 \neq 0,$ and also μ is a non-zero wave number. Subsequently, the first and second derivatives of the assumed trial solution in (14) with respect to $\mathscr{F}$ are taken in the following format

$$\mathscr{V}' = \frac{\gamma_0\mu\left(\gamma_2\sinh(\mu\mathscr{F}) - \gamma_1\right)}{\left(\gamma_1\sinh(\mu\mathscr{F}) + \gamma_2\right)^2}, \tag{15}$$

and

$$\mathscr{V}'' = \frac{\gamma_0\mu^2\cosh(\mu\mathscr{F})\left(-\gamma_2\gamma_1\sinh(\mu\mathscr{F}) + 2\gamma_1^2 + \gamma_2^2\right)}{\left(\gamma_1\sinh(\mu\mathscr{F}) + \gamma_2\right)^3}. \tag{16}$$

Subbing equations (14)-(16) into the transformed model (13), one gets the following

$$\begin{aligned} &\alpha\gamma_0^2\delta_1\sinh^2(\mu\mathscr{F}) + \alpha\gamma_0^2\delta_1 - 3\tau\gamma_1\gamma_2\delta_1^3\mu^2\sinh(\mu\mathscr{F}) + 6\tau\gamma_1^2\delta_1^3\mu^2 \\ &+ 3\tau\gamma_2^2\delta_1^3\mu^2 - 3\gamma_1\gamma_2\delta_2^2\delta_1\mu^2\ell\sinh(\mu\mathscr{F}) - 3\gamma_1\gamma_2\delta_3^2\delta_1\mu^2\ell\sinh(\mu\mathscr{F}) \\ &+ 6\gamma_1^2\delta_2^2\delta_1\mu^2\ell + 3\gamma_2^2\delta_2^2\delta_1\mu^2\ell + 6\gamma_1^2\delta_3^2\delta_1\mu^2\ell + 3\gamma_2^2\delta_3^2\delta_1\mu^2\ell \\ &+ 3\gamma_1^2\delta_4\sinh^2(\mu\mathscr{F}) + 6\gamma_1\gamma_2\delta_4\sinh(\mu\mathscr{F}) + 3\gamma_2^2\delta_4 = 0. \end{aligned} \tag{17}$$

In (17), start to collect all the terms that have the same powers of $\sinh^\tau(\mu\mathscr{F})$ where $\tau = 0,1,2$. Make the summation of the coefficients of $\sinh^\tau(\mu\mathscr{F})$ equal to zero. After summarizing the process, a set of algebraic equations will be obtained as the following:

$$\left.\begin{array}{l}\alpha\gamma_0^2\delta_1+6\tau\gamma_1^2\delta_1^3\mu^2+3\tau\gamma_2^2\delta_1^3\mu^2+6\gamma_1^2\delta_2^2\delta_1\mu^2\ell+3\gamma_2^2\delta_2^2\delta_1\mu^2\ell\\ +6\gamma_1^2\delta_3^2\delta_1\mu^2\ell+3\gamma_2^2\delta_3^2\delta_1\mu^2\ell+3\gamma_2^2\delta_4=0,\\ -3\tau\gamma_1\gamma_2\delta_1^3\mu^2-3\gamma_1\gamma_2\delta_2^2\delta_1\mu^2\ell-3\gamma_1\gamma_2\delta_3^2\delta_1\mu^2\ell+6\gamma_1\gamma_2\delta_4=0,\\ \alpha\gamma_0^2\delta_1+3\gamma_1^2\delta_4=0.\end{array}\right\} \tag{18}$$

Solving system (18) with the aid of the computers software packages, obtaining the following parameters.
Case 1. Obtained parameters in solving algebraic system (18) are in the following

$$\gamma_2=-i\gamma_1;\alpha=-\frac{3\gamma_1^2\mu^2\left(\tau\delta_1^2+\delta_2^2\ell+\delta_3^2\ell\right)}{2\gamma_0^2};\delta_4=\frac{1}{2}\delta_1\mu^2\left(\tau\delta_1^2+\delta_2^2\ell+\delta_3^2\ell\right). \tag{19}$$

Substituting parameters (19) gathering with (14) into (13), the solution of (1) has been derived as follows

$$u_{S1}=\frac{\gamma_0\cosh\left(\mu\left(\frac{1}{2}\delta_1\mu^2t\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)+\delta_1x+\delta_2y+\delta_3z\right)\right)}{\gamma_1\left(\sinh\left(\mu\left(\frac{1}{2}\delta_1\mu^2t\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)+\delta_1x+\delta_2y+\delta_3z\right)\right)+i\right)}. \tag{20}$$

Characteristics of the real and imagined part of the solution existing in (20) have been presented in the following graphs where $\mu=\frac{1}{2}$; $\ell=\frac{2}{5}$; $\tau=\frac{1}{3}$; $\gamma_0=\frac{7}{2}$; $\gamma_1=\frac{1}{3}$; $\delta_1=-\frac{3}{2}$; $\delta_2=\frac{2}{3}$; $\delta_3=\frac{5}{2}$; $y=\frac{2}{3}$; $z=\frac{1}{3}$, and $-10\le x\le 10$.

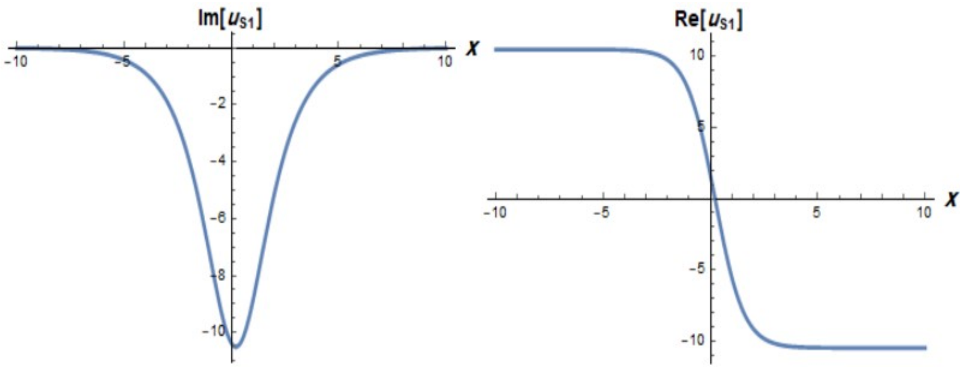

Fig. 1. 2D graphs of (20) where $-10\le x\le 10$, $t=\frac{3}{2}$

Case 2. The following coefficients are the outcomes from solving the system (18)

$$\gamma_0=-\frac{i\sqrt{3}\gamma_1\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\delta_1}};\gamma_2=-i\gamma_1;\delta_3=\frac{\sqrt{-\tau\delta_1^3\mu^2-\delta_2^2\delta_1\mu^2\ell+2\delta_4}}{\sqrt{\delta_1}\mu\sqrt{\ell}}. \tag{21}$$

The parameters in (21) along with using (14) were substituted into (13), and the result was the following solution of (1):

$$u_{S2} = \frac{\sqrt{3}\sqrt{\delta_4}\cosh\left(\delta_4\mu t+\delta_1\mu x+\delta_2\mu y+\frac{z\sqrt{2\delta_4-\delta_1\mu^2\left(\tau\delta_1^2+\delta_2^2\ell\right)}}{\sqrt{\delta_1}\sqrt{\ell}}\right)}{\sqrt{\alpha}\sqrt{\delta_1}\left(1+i\sinh\left(\delta_4\mu t+\delta_1\mu x+\delta_2\mu y+\frac{z\sqrt{2\delta_4-\delta_1\mu^2\left(\tau\delta_1^2+\delta_2^2\ell\right)}}{\sqrt{\delta_1}\sqrt{\ell}}\right)\right)}. \quad (22)$$

Case 3. The findings of solving the algebraic system (18) are listed below:

$$\gamma_0 = -\frac{i\sqrt{3}\gamma_1\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\delta_1}}; \gamma_2 = 0; \tau = \frac{-2\delta_1\delta_2^2\mu^2\ell-2\delta_1\delta_3^2\mu^2\ell+\delta_4}{2\delta_1^3\mu^2}. \quad (23)$$

The parameters in (23) along with (14) were substituted into (13), and the outcomes are the following solution of (1):

$$u_{S3} = -\frac{i\sqrt{3}\sqrt{\delta_4}\coth\left(\mu\left(\delta_4 t+\delta_1 x+\delta_2 y+\delta_3 z\right)\right)}{\sqrt{\alpha}\sqrt{\delta_1}}. \quad (24)$$

3.2. Application of METM to the aforementioned model

For solving the model stated in (1) through the use of the METM, firstly applying the balance principle in (13) between $\mathscr{U}''$ and $\mathscr{U}^3$ one immediately gets $m = 1$. Thereafter (7) becomes

$$\mathscr{U} = \mathscr{A}_0+\mathscr{A}_1\mathscr{E}+\frac{\mathscr{B}_1}{\mathscr{E}}, \quad (25)$$

wherein $\mathscr{A}_0, \mathscr{A}_1, \mathscr{B}_1$ are constants to be determining afterward, $\mathscr{A}_1 \neq 0$ or $\mathscr{B}_1 \neq 0$, here $\mathscr{E}(\mathscr{K})$ is a function satisfies the Riccati ODE that given in (9).
Take the first and second derivatives of (25) respectively, where (8) is considered, then one obtains the following:

$$\mathscr{U}' = \mathscr{A}_1\mathscr{B}+\mathscr{A}_1\mathscr{E}^2-\frac{\mathscr{B}\mathscr{B}_1}{\mathscr{E}^2}-\mathscr{B}_1, \quad (26)$$

and

$$\mathscr{U}'' = 2\mathscr{A}_1\mathscr{B}\mathscr{E}+2\mathscr{A}_1\mathscr{E}^3+\frac{2\mathscr{B}^2\mathscr{B}_1}{\mathscr{E}^3}+\frac{2\mathscr{B}\mathscr{B}_1}{\mathscr{E}}. \quad (27)$$

Inserting equations (25-27) gathering with (11) into (13) after simplification collect all the coefficients of $\mathscr{E}^i$ where $i = 0, 1, \cdots, 6$, and equating the coefficients of the same power of $\mathscr{E}^i$ to zero. Obtaining is a system of algebraic equations for the parameters as follows:

$$\left.\begin{array}{l}
Eq1:\mathscr{B}_1\delta_1\left(6\mathscr{B}^2\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)+\alpha\mathscr{B}_1^2\right)=0;\\
Eq2:3\mathscr{B}_1\left(\delta_1\left(\alpha\mathscr{A}_0^2+\alpha\mathscr{A}_1\mathscr{B}_1+2\mathscr{B}\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)\right)+\delta_4\right)=0;\\
Eq3:3\mathscr{A}_1\left(\delta_1\left(\alpha\mathscr{A}_0^2+\alpha\mathscr{A}_1\mathscr{B}_1+2\mathscr{B}\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)\right)+\delta_4\right)=0;\\
Eq4:\mathscr{A}_0\left(\alpha\delta_1\left(6\mathscr{A}_1\mathscr{B}_1+\mathscr{A}_0^2\right)+3\delta_4\right)=0;\\
Eq5:\mathscr{A}_1\delta_1\left(\alpha\mathscr{A}_1^2+6\tau\delta_1^2+6\left(\delta_2^2+\delta_3^2\right)\ell\right)=0;\\
Eq6:3\alpha\mathscr{A}_0\mathscr{B}_1^2\delta_1=0;\\
Eq7:3\alpha\mathscr{A}_0\mathscr{A}_1^2\delta_1=0.
\end{array}\right\} \tag{28}$$

The system encountered in (28) is solvable by employing computer programs, and the following cases are obtained:

Case 1. Captured parameters in solving (28) are in the following:

$$\begin{aligned}
&\mathscr{A}_1=-\frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\mathscr{B}}\sqrt{\delta_1}};\mathscr{B}_1=-\frac{i\sqrt{\frac{3}{2}}\sqrt{\mathscr{B}}\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\delta_1}};\\
&\mathscr{A}_0=0;\tau=\frac{\delta_4-4\mathscr{B}\delta_1\left(\delta_2^2+\delta_3^2\right)\ell}{4\mathscr{B}\delta_1^3}.
\end{aligned} \tag{29}$$

For the coefficients in (29) and for the different sign of $\mathscr{B}$, the following sub-cases of the solutions are gotten:

Case 1.1. Where $\mathscr{B}<0$ and performing $\mathscr{E}=-\sqrt{-\mathscr{B}}\tanh\left(\sqrt{-\mathscr{B}}\mathscr{K}\right)$ as a solution of the Riccati ODE stated in (11), the obtained solution of (1) becomes

$$u_{11}=\frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}\left(\mathscr{B}-\mathscr{B}\tanh^2\left(\sqrt{-\mathscr{B}}\mathscr{F}\right)\right)\coth\left(\sqrt{-\mathscr{B}}\mathscr{F}\right)}{\sqrt{\alpha}\sqrt{-\mathscr{B}}\sqrt{\mathscr{B}}\sqrt{\delta_1}}. \tag{30}$$

Case 1.2. If $\mathscr{B}>0$, and $\mathscr{E}=-\sqrt{\mathscr{B}}\cot\left(\sqrt{\mathscr{B}}\mathscr{K}\right)$ denoted as a solution of the Riccati ODE that exists in (11), then obtained solution of (1) becomes

$$u_{12}=\frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}\tan\left(\sqrt{\mathscr{B}}\mathscr{F}\right)\left(\mathscr{B}\cot^2\left(\sqrt{\mathscr{B}}\mathscr{F}\right)+\mathscr{B}\right)}{\sqrt{\alpha}\mathscr{B}\sqrt{\delta_1}}. \tag{31}$$

The behavior of the solution that exists in (31), has been demonstrated by the following three-dimension figure and contour surface plot where the current parameters are determined $\alpha=-\frac{4}{5}$; $\delta_1=-\frac{4}{3}$; $\delta_2=\frac{5}{3}$; $\delta_3=-\frac{7}{3}$; $\delta_4=-\frac{8}{3}$; $\mathscr{B}=\frac{1}{10}$; $y=-\frac{3}{5}$; $z=-\frac{1}{2}$,

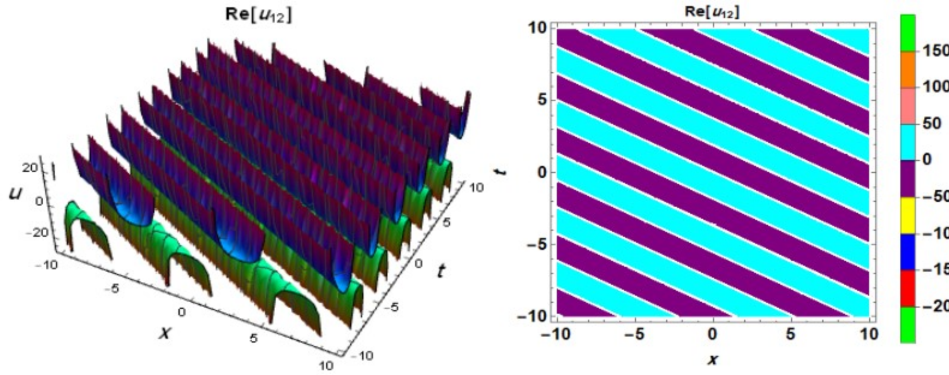

Fig. 2. 3D figure and contour surface plot of (31) where $-10 \leq x \leq 10,\ -10 \leq t \leq 10$

Case 1.3. Zero value of $\mathscr{B}$ is impossible so that in this case obtained solution of (1) is only trivial solution.

Case 2. Through solving the system (28) the following parameters are obtained:

$$\begin{aligned} &\mathscr{A}_1 = -\frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\mathscr{B}}\sqrt{\delta_1}}; \mathscr{B}_1 = -\frac{i\sqrt{\frac{3}{2}}\sqrt{\mathscr{B}}\sqrt{\delta_4}}{\sqrt{\alpha}\sqrt{\delta_1}}; \\ &\mathscr{A}_0 = 0; \delta_3 = \frac{\sqrt{\delta_4 - 4\mathscr{B}\delta_1\left(\tau\delta_1^2 + \delta_2^2\ell\right)}}{2\sqrt{\mathscr{B}}\sqrt{\delta_1}\sqrt{\ell}}. \end{aligned} \tag{32}$$

The existed coefficients in (32) and the sign of the constant $\mathscr{B}$ play a key role in creating the following sub-cases of the solutions to (1).

Case 2.1. If $\mathscr{B} < 0$ and using $\mathscr{E} = -\sqrt{-\mathscr{B}}\tanh\left(\sqrt{-\mathscr{B}}\mathscr{K}\right)$ as a solution of the Riccati ODE that has appeared in (9) then one gets the following solution of (1)

$$u_{21} = \frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}\left(\mathscr{B} - \mathscr{B}\tanh^2\left(\sqrt{-\mathscr{B}}\mathscr{F}\right)\right)\coth\left(\sqrt{-\mathscr{B}}\mathscr{F}\right)}{\sqrt{\alpha}\sqrt{-\mathscr{B}}\sqrt{\mathscr{B}}\sqrt{\delta_1}}. \tag{33}$$

Case 2.2. If $\mathscr{B} > 0$ using $\mathscr{E} = -\sqrt{\mathscr{B}}\cot\left(\sqrt{\mathscr{B}}\mathscr{K}\right)$ as a solution of the Riccati ODE that given in (11), then the following solution of (1) has been captured:

$$u_{22} = \frac{i\sqrt{\frac{3}{2}}\sqrt{\delta_4}\tan\left(\sqrt{\mathscr{B}}\mathscr{F}\right)\left(\mathscr{B}\cot^2\left(\sqrt{\mathscr{B}}\mathscr{F}\right) + \mathscr{B}\right)}{\sqrt{\alpha}\mathscr{B}\sqrt{\delta_1}}. \tag{34}$$

Inserting the values $\alpha = \frac{1}{100}$; $\delta_1 = \frac{1}{5}$; $\delta_2 = \frac{2}{3}$; $\delta_4 = -\frac{1}{5}$; $\mathscr{B} = \frac{1}{10}$; $y = \frac{3}{5}$; $z = \frac{3}{2}$; $\ell = \frac{1}{2}$; $\tau = \frac{3}{4}$, into (34), the outcomes are the following figures:

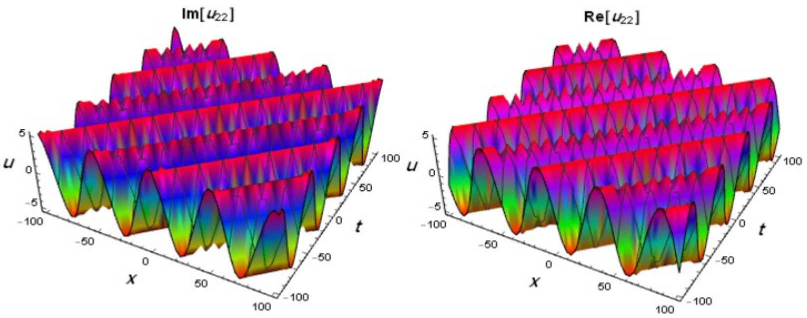

Fig. 3. 3D figures of (34) where $-100 \le x \le 100,\ -100 \le t \le 100$

Case 2.3. A zero value of $\mathscr{B}$ is impossible because it makes some of the coefficients undefined, hence, in this case, (1) has only a trivial solution.

Case 3. Obtained parameters in solving (28) are in the following:

$$\begin{aligned}&\mathscr{A}_1 = -\frac{\sqrt{6}\sqrt{-\tau\delta_1^2 - \left(\delta_2^2+\delta_3^2\right)\ell}}{\sqrt{\alpha}}; \mathscr{B}_1 = \frac{i\sqrt{6}\mathscr{B}\sqrt{\tau\delta_1^2 + \left(\delta_2^2+\delta_3^2\right)\ell}}{\sqrt{\alpha}};\\ &\mathscr{A}_0 = 0; \delta_4 = -2\mathscr{B}\delta_1\left(-3i\sqrt{-\left(\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell\right)^2} + \tau\delta_1^2 + \left(\delta_2^2+\delta_3^2\right)\ell\right).\end{aligned} \tag{35}$$

For the coefficients in (35), and for the different sign of $\mathscr{B}$, the following subcases of the solutions are obtained.

Case 3.1. Where $\mathscr{B} < 0$ and performing $\mathscr{E} = -\sqrt{-\mathscr{B}}\coth\left(\sqrt{-\mathscr{B}}\mathscr{K}\right)$ as a solution of the Riccati ODE stated in (11) the obtained solution of (1) becomes

$$u_{31} = -\frac{\sqrt{6}\tanh\left(\sqrt{-\mathscr{B}}\mathscr{F}\right)\left(\mathscr{B}\mathscr{Z}\coth^2\left(\sqrt{-\mathscr{B}}\mathscr{F}\right) + i\mathscr{B}\mathscr{R}\right)}{\sqrt{\alpha}\sqrt{-\mathscr{B}}}, \tag{36}$$

herein $\mathscr{R} = \sqrt{-\tau\delta_1^2 - \left(\delta_2^2+\delta_3^2\right)\ell},\ \mathscr{Z} = \sqrt{\tau\delta_1^2 + \left(\delta_2^2+\delta_3^2\right)\ell}.$

Case 3.2. If $\mathscr{B} > 0$, and $\mathscr{E} = \sqrt{\mathscr{B}}\tan\left(\sqrt{\mathscr{B}}\mathscr{K}\right)$ denoted as a solution of the Riccati ODE that exist in (11), then the obtained solution of (1) becomes

$$u_{32} = \frac{\sqrt{6}\cot\left(\sqrt{\mathscr{B}}\mathscr{F}\right)\left(-\mathscr{B}\mathscr{R}\tan^2\left(\sqrt{\mathscr{B}}\mathscr{F}\right) + i\mathscr{B}\mathscr{Z}\right)}{\sqrt{\alpha}\sqrt{\mathscr{B}}}, \tag{37}$$

where $\mathscr{R} = \sqrt{-\tau\delta_1^2 - \left(\delta_2^2+\delta_3^2\right)\ell},\ \mathscr{Z} = \sqrt{\tau\delta_1^2 + \left(\delta_2^2+\delta_3^2\right)\ell}.$

Case 3.3. If $\mathscr{B}=0$ and $\mathscr{E}=-\frac{1}{\mathscr{F}}$ is a solution of the given Riccati ODE, then obtained solution of (1) take the following form:

$$u_{33}=\frac{\sqrt{6}\left(\sqrt{-\tau\delta_1^2-\left(\delta_2^2+\delta_3^2\right)\ell}-i\mathscr{B}\mathscr{F}^2\sqrt{\tau\delta_1^2+\left(\delta_2^2+\delta_3^2\right)\ell}\right)}{\sqrt{\alpha}\mathscr{F}}. \tag{38}$$

4. Results and conclusion

In the present investigation, both the extended rational sinh-cosh method and the modified extended tanh technique are applied for the first time to the (3+1)-dimensional non-linear generalized Korteweg-de Varies-Zakharov-Kuznetsov equation, intended for investigating some exact traveling wave solutions. Obtained solutions are in the form of complex trigonometric, osculating, hyperbolic trigonometric, and rational functions. Obtained solutions are shown graphically in multi-dimensions in addition to observing wave characteristics optically. The research results demonstrate the efficacy, utility, and reliability of the employed techniques in determining the accurate solution of the identified equation, and they will be used with other mathematical models. The outcomes also demonstrate that the obtained solutions' free parameters have a massive impact on the waveform and its functionality, which might be utilized for illustrating many different new and sophisticated characteristics that arise in several scientific disciplines.

Compared to the existing studies in the literature, outcomes are new and original. Specifically, our study deals with the feature of the ion-acoustic waves described in the solutions of the studied model, which provided a great reason to do this research. Physical structures of the solutions have been described by the anti-kink wave soliton discovered on the right-hand side of Figure 1, and the dark soliton on the left-hand side of Figure 1, periodic and traveling waves appear in Figures 2 and 3 respectively. Additionally, using the symbolic computational software tools, all the acquired solutions have been verified by inserting them back into the relevant equation.

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