

## A NUMERICAL SCHEME FOR TIME-FRACTIONAL FOURTH-ORDER REACTION-DIFFUSION MODEL

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**Abstract.** In fractional calculus, the fractional differential equation is physically and theoretically important. In this article an efficient numerical process has been developed. Numerical solutions of the time fractional fourth order reaction diffusion equation in the sense of Caputo derivative is obtained by using the implicit method, which is a finite difference method and is developed by increasing the number of iterations. The advantage of the implicit difference scheme is unconditionally stable. The stability analysis and convergency have been proven. A numerical example has been presented, and the validity of the method is supported by tables and graphics.

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### 1. Introduction

The fractional calculus is created by expanding the integer account. In cases where the integer sequence cannot make sense, a fractional analysis is needed. It extends integer orders derivatives and integrals to real and variable orders and it offers a more accurate and adequate definition of daily survival and memory properties. While fractional calculus has been seen as a more pure theory in the past, it has recently suffered a significant development.

The fractional differential equation is used today in many areas such as biology, physics, chemistry and engineering. Numerical methods have been developed to solve these problems. Guo et al. [1] used the finite difference method to solve a physical problem. Labora et al. [2] and Guo et al. [3] presented some numerical methods to solve mathematical modelling problems. Dabiri et al. studied in numerical solutions of a multi order fractional differential equation via the collocation method in [4]. Yuttanan [5] used a method to obtain numerical solution of a physical problem.

The fourth order reaction-diffusion equations have an important role in different field of science such as pattern formation bistable systems, propagation of domain walls in liquid crystals, traveling waves of reaction diffusion systems, thin beams, brain warpin, strain gradient elasticity and fluids on lung. Some of the studies on these topics are: Myers and Charpin [6, 7] studied in mathematical modelling with a biological and physical problem. Halpern et al. [8] proposed a theoretical study of surfactant. Tariq and Akram [9] used a numerical method to solve a fourth order differential equation. Toga [10] and Memoli et al. [11] studied in brain warping and brain imaging. By examining the models used in different fields, information about that model is obtained and its advance to that field is determined. In the literature there are many numerical methods that allow for solving fractional differential equations [12-14]. Du et al. [15] presented a local discontinuous Galerkin method. Liu et al. [16] used a finite element scheme, Abbaszadeh and Dehghan [17] used a collocation method. Zhuang and Liu used an implicit difference method [18], Tadjeran and Meerschaert [19] presented a numerical method to solve a fractional superdiffusive differential equation. The alternating directions implicit (ADI) approach, the Crank-Nicolson method and the Richardson extrapolation were used [19]. In [20], the Grünwald finite difference approximation is used for the fractional derivative with the ADI method and the locally one dimensional (LOD) method is used in [21-25].

In this study our aim is to examine the numerical solutions of the time fractional fourth order reaction diffusion equation [26]

$$\frac{\partial u(x, t)}{\partial t} + \frac{\partial^\beta u(x, t)}{\partial t^\beta} - \Delta u(x, t) + \Delta^2 u(x, t) + \gamma u(x, t) = f(x, t), \quad (1)$$

with the general initial and boundary conditions,

$$u(x, 0) = g(x), \quad x \in \bar{\xi} = \xi \cup \partial\xi,$$

$$u(x, t) = \Delta u(x, t) = 0, \quad x \in \partial\xi, \quad t > 0$$

where  $\Delta$  and  $\Delta^2$  is the Laplacian and bi-Laplacian operators with respect to the spatial variables,  $0 < \beta < 1$ ,  $\gamma$  is the positive constant,  $\xi$  represents a bounded region in  $\mathbb{R}^2$ ,  $\partial\xi$  for the boundary of  $\xi$ , and  $f(x, t)$  is a sufficiently smooth function.

In this work, the fourth order reaction diffusion equation has been made discrete using finite differences. The implicit difference method has been used to find the numerical solution in each  $(i, j)$  step. By using this method,  $\mathbf{AU} = \mathbf{B}$  equation system has obtained, where  $\mathbf{A}$  is the coefficients matrix,  $\mathbf{U}$  is the numerical results to be found, and  $\mathbf{B}$  is the known values matrix. By applying the implicit difference method for each step, the system has been solved and numerical solutions have been obtained.

The main purpose of this article is to study a computational approach based on the implicit difference scheme to solve the time fractional fourth order reaction

diffusion equation. Compared to other methods, it has been observed that the calculation process is reduced while maintaining a high degree of accuracy. One of the most important advantages is that the presented method is unconditionally stable. The number of iterations in the approach to fractional derivatives and other derivatives in problem solving is of great importance when we look at the accuracy of the results. The article has been organized as follows after these ideas. In Section 2, the numerical method used is introduced, and the equation is made discrete. In Section 3, the stability of the method is investigated. In Section 4, numerical results are obtained by solving the fourth order reaction diffusion equation. The results obtained are supported by tables and graphs.

## 2. Numerical methods

In this section we define the basic ideas for the numerical solution of the time fractional reaction diffusion equation Eq. (1) by using implicit finite differences. The domain  $[0, l] \times [0, T]$  is divided into on  $N \times M$  mesh with  $h = \frac{l}{N}$  and  $dt = \Delta t = \frac{T}{M}$  respectively,  $x_i = ih$  for  $i = 1, 2, \dots, N$  is the  $i^{th}$  node. In the temporal dimension, the uniform step size  $\Delta t$  is used, thus  $t_j = j\Delta t$  is the time level for  $j^{th}$  step. The quantity  $u(x_i, t_j)$  represents the exact solution at  $(x_i, t_j)$  while  $u_i^j$  represents the numerical solution at  $(x_i, t_j)$ , the value of  $g$  at the grid point is denoted  $g(x_i, t_j) = g_i^j$ .

In Eq. (1), instead of the time fractional derivative, us  $g$  as the fractional partial derivative with Caputo sense of order  $\beta$  defined by [27]

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} = \frac{1}{\Gamma(n - \beta)} \int_0^t \frac{\partial u(x, s)}{\partial s} (t - s)^{-\beta} ds, \quad (2)$$

$$0 \leq t \leq T, \quad 0 < \beta < 1,$$

$$\frac{\partial^\beta u(x, t)}{\partial t^\beta} = \frac{du(x, t)}{dt}, \quad 0 \leq t \leq T, \quad \beta = 1. \quad (3)$$

where  $\Gamma(\cdot)$  is the Gamma function.

A discrete approximation to the fractional derivative Eq. (2) can be obtained by a formula as follows [27]

$$\frac{\partial^\beta u(x_i, t_j)}{\partial t^\beta} = \sigma_{\beta, k} \sum_{n=1}^j \omega_n^{(\beta)} (u_i^{j-n+1} - u_i^{j-n}) + O(\Delta t), \quad (4)$$

where  $\sigma_{\beta, k} = \frac{1}{\Gamma(1-\beta)} \frac{1}{1-\beta} \frac{1}{\Delta t^\beta}$  and  $\omega_j^{(\beta)} = j^{1-\beta} - (j-1)^{1-\beta}$ .

Hence,

$$\frac{\partial^\beta u(x_i, t_j)}{\partial t^\beta} = D_t^{(\beta)} u_i^j + O(\Delta t), \quad (5)$$

and the first order approximation method for the computation of Caputo's fractional derivative is then given by the following equation

$$D_t^{(\beta)} u_i^j = \sigma_{\beta,k} \sum_{n=1}^j \omega_n^{(\beta)} (u_i^{j-n+1} - u_i^{j-n}). \quad (6)$$

The finite difference approximation for the second order, fourth order spatial derivative can be stated as follows respectively

$$\frac{\partial^2 u(x_i, t_j)}{\partial x^2} = \frac{u(x_{i-1}, t_j) - 2u(x_i, t_j) + u(x_{i+1}, t_j))}{h^2}, \quad (7)$$

$$\frac{\partial^4 u(x_i, t_j)}{\partial x^4} = \frac{u(x_{i-2}, t_j) - 4u(x_{i-1}, t_j) + 6u(x_i, t_j) - 4u(x_{i+1}, t_j) + u(x_{i+2}, t_j))}{h^4}. \quad (8)$$

The time derivative can be stated with the forward difference scheme as follows

$$\frac{\partial u(x_i, t_j)}{\partial t} = \frac{u(x_i, t_j) - u(x_i, t_{j-1}))}{\Delta t}. \quad (9)$$

Substituting (6), (7), (8) and (9) into (1), we obtain,

$$\begin{aligned} U_{i-2}^j \left( \frac{1}{h^4} \right) + U_{i-1}^j \left( -\frac{1}{h^2} - \frac{4}{h^4} \right) + U_i^j \left( \frac{1}{\Delta t} + \frac{2}{h^2} + \frac{6}{h^4} + \gamma \right) + U_{i+1}^j \left( -\frac{1}{h^2} - \frac{4}{h^4} \right) + \\ + U_{i+2}^j \left( \frac{1}{h^4} \right) = f_i^j - \sigma_{\beta,k} \sum_{n=1}^j \omega_n^{(\beta)} (U_i^{j-n+1} - U_i^{j-n}) + \frac{1}{\Delta t} U_i^{j-1}. \end{aligned} \quad (10)$$

Equation (10) requires, at each time step, the solving of a triangular system of linear equations where the right-hand side utilizes all the history of the computed solution up to that time.

### 3. Stability analysis of the method

In this section we show that the given method is unconditionally stable. According to the Lax Equivalence Theorem [28], the numerical solution converges to the exact solution when  $h, \Delta t$  approaches to 0.

**Theorem 1.** The implicit numerical method Eq. (10), solution to Eq. (1) with  $0 < \beta < 1$  on the finite domain  $0 \leq x \leq 1$  with boundary conditions  $u(0, t) = u(1, t) = 0$  for all  $t \geq 0$ , is consistent and unconditionally stable [27].

**Proof.** To analyze the stability of the given method, we assume the solution of the form  $U_j^n = \xi_n e^{i\omega jh}$ ,  $i = \sqrt{-1}$ ,  $\omega$  is a real. Therefore Eq. (10) becomes

$$\begin{aligned} & \xi_n e^{i\omega(j-2)h} Q1 + \xi_n e^{i\omega(j-1)h} Q3 + \xi_n e^{i\omega jh} Q2 + \xi_n e^{i\omega(j+1)h} Q3 + \\ & + \xi_n e^{i\omega(j+2)h} Q1 = \xi_{n-1} e^{i\omega jh} - \sigma_{\beta,k} \sum_{j=1}^n \omega_j^{(\beta)} (\xi_{n-j+1} e^{i\omega jh} - \xi_{n-j} e^{i\omega jh}) \end{aligned} \quad (11)$$

where:  $Q1 = \frac{1}{h^4}$ ,  $Q2 = -\frac{1}{h^2} - \frac{4}{h^4}$ ,  $Q3 = \frac{1}{\Delta t} + \frac{2}{h^2} + \frac{6}{h^4} + \gamma$ .

Reordering the equation, we obtain,

$$\begin{aligned} & \xi_n e^{-2i\omega h} Q1 + \xi_n e^{-i\omega h} Q3 + \xi_n Q2 + \xi_n e^{i\omega h} Q3 + \xi_n e^{2i\omega h} Q1 = \\ & = \xi_{n-1} - \sigma_{\beta,k} \sum_{j=1}^n \omega_j^{(\beta)} (\xi_{n-j+1} - \xi_{n-j}). \end{aligned} \quad (12)$$

Simplifying and grouping like terms,

$$2\xi_n Q1 \cos 2\omega h + 2\xi_n Q3 \cos \omega h + \xi_n Q2 = \xi_{n-1} - \sum_{j=1}^n \sigma_{\beta,k} \omega_j^{(\beta)} (\xi_{n-j+1} - \xi_{n-j}). \quad (13)$$

It can be reduced to,

$$\xi_n [2Q1 \cos 2\omega h + 2Q3 \cos \omega h + Q2] = \xi_{n-1} - \sum_{j=1}^n \sigma_{\beta,k} \omega_j^{(\beta)} (\xi_{n-j+1} - \xi_{n-j}), \quad (14)$$

or

$$\xi_n = \frac{\xi_{n-1} - \sum_{j=1}^n \sigma_{\beta,k} \omega_j^{(\beta)} (\xi_{n-j+1} - \xi_{n-j})}{2Q1 \cos 2\omega h + 2Q3 \cos \omega h + Q2}. \quad (15)$$

From Eq. (15), we observe that

$$\xi_1 \leq \xi_0. \quad (16)$$

Since

$$2Q1 \cos 2\omega h + 2Q3 \cos \omega h + Q2 \geq 1 \text{ for all } \omega, h \text{ and } \gamma.$$

Using Eq. (16) and the positiveness of the coefficients, Eq. (4) follows that

$$\xi_j \leq \xi_{j-1}, \quad j = 1, \dots, n-1.$$

This shows that  $\xi_n \leq \xi_{n-1} \leq \xi_{n-2} \leq \dots \leq \xi_1 \leq \xi_0$ .

Thus  $\xi_n = |U_j^n| \leq \xi_0 = |U_j^0| \leq |g_j|$ , which entails  $\|U^n\|_{l_2} \leq \|g\|_{l_2}$  and we have stability.

**Remark.** For  $\beta = 1$ , the numerical scheme is reduced to the convergent implicit algorithm for the equation [28].

The method presented with the procedures performed is stable and consistent according to reference [27].

#### 4. Numerical experiments

In this section, the accuracy of the proposed method has been shown by giving a numerical example and evaluates the effect of various  $dt$  and  $h$  values. The studies were carried out with an intel core i5 processor computer, using the Maple 13 program.

The maximum norm order is calculated by the following

$$L_\infty = \max_{1 \leq j \leq N-1} |U(x_j, T) - u(x_j, T)|.$$

Consider the following equation [26]

$$\frac{\partial u(x, t)}{\partial t} + \frac{\partial^\beta u(x, t)}{\partial t^\beta} + \frac{\partial^4 u(x, t)}{\partial x^4} + \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) = \left( 2t + t^2 + \frac{2t^{2-\beta}}{\Gamma(3-\beta)} \right) \sin x \quad (17)$$

$(x, t) \in [0, 2\pi] \times [0, T]$ .

The exact solution is  $u(x, t) = t^2 \sin x$ . The initial and boundary conditions are  $u(2\pi, t) = u(1, t) = 0$ ,  $u(x, 0) = 0$ .

Table 1. The maximum norm errors for a different value of  $\beta$  at a different time step

$dt$	$\beta = 0.35$		$\beta = 0.95$	
	$L_\infty$	CPU Time	$L_\infty$	CPU Time
1/500	5.79546e-3	0.6 s	5.26704e-3	1.3 s
1/1000	1.27661e-3	0.6 s	1.14407e-3	0.7 s
1/5000	3.87768e-5	0.7 s	3.67928e-5	0.7 s

Table 2. The absolute errors for a different value of  $N$  at  $T = 100$ 

$N$	Absolute Error		
	$\beta = 0.2$	CPU Time	$\beta = 0.002$
10	2.92406e-4	0.7 s	2.96625e-4
20	7.92612e-5	1.7 s	7.76293e-5
30	1.10219e-5	2.7 s	1.07381e-5
40	2.61512e-6	10 s	2.54561e-6
50	8.54232e-7	18 s	8.31333e-7

Table 3. The maximum norm errors for a different time and spatial steps

$h$	$\beta = 0.4$	
	$dt$	$L_\infty$
1/10	1/100	8.41470e-1
1/20	1/400	5.25919e-2
1/30	1/900	9.99092e-3
1/40	1/1600	3.28699e-3

Table 4. The absolute errors for  $N = 10$ ,  $\beta = 0.04$  with various  $t$  with CPU time 2.4 s

$x$	Absolute errors		
	$t = 0.5$	$t = 0.05$	$t = 0.025$
$\pi/5$	8.28135e-2	4.43491e-4	6.57908e-5
$2\pi/5$	1.73365e-1	1.32387e-3	2.80524e-4
$3\pi/5$	1.65684e-1	5.67746e-4	7.20005e-5
$4\pi/5$	8.32047e-2	3.20027e-5	8.00687e-6
$\pi$	3.25026e-2	2.96058e-4	4.41725e-5
$6\pi/5$	1.36783e-1	4.96374e-4	6.54587e-5
$7\pi/5$	1.18799e-1	5.19179e-4	6.36023e-5
$8\pi/5$	1.52214e-1	3.28350e-4	3.59694e-5
$9\pi/5$	4.64995e-2	1.54730e-5	5.21205e-6

The results show that the numerical results obtained by the application of this method are close to the exact solutions. Absolute error values in special cases are examined in each table. It was observed that the errors got smaller as the number of iterations increased.

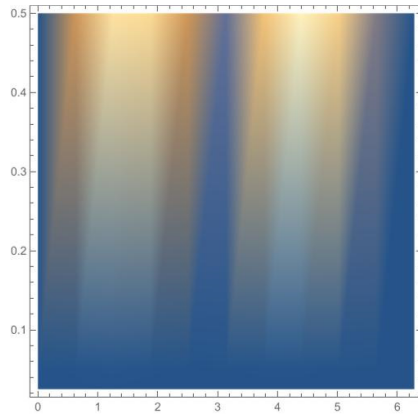


Fig. 1. Density graph of absolute errors for  $N = 10$ ,  $\beta = 0.04$

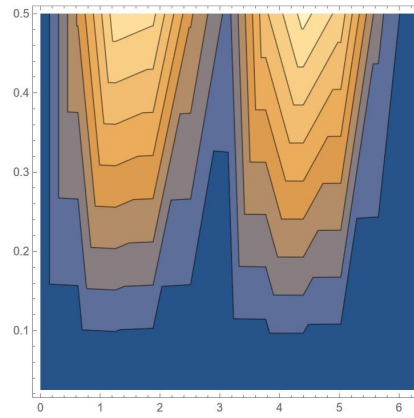


Fig. 2. Contour graph of absolute errors for  $N = 10$ ,  $\beta = 0.04$

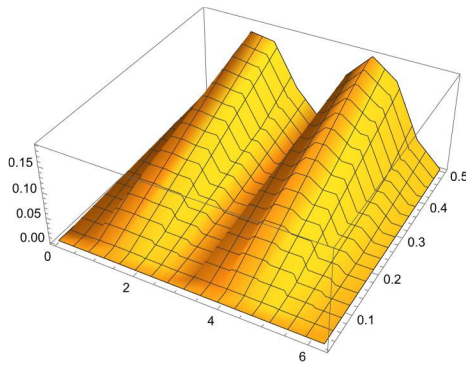


Fig. 3. 3D graph of absolute errors for  $N = 10$ ,  $\beta = 0.04$

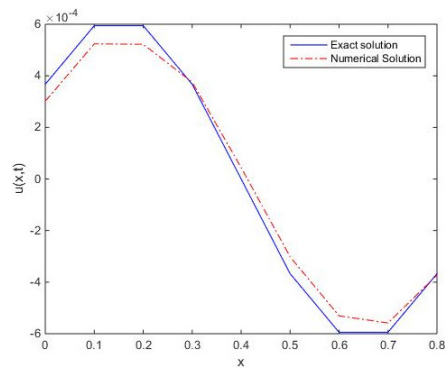


Fig. 4. Comparison of numerical and exact solutions for  $N = 10$ ,  $\beta = 0.04$ ,  $dt = 1/2000$

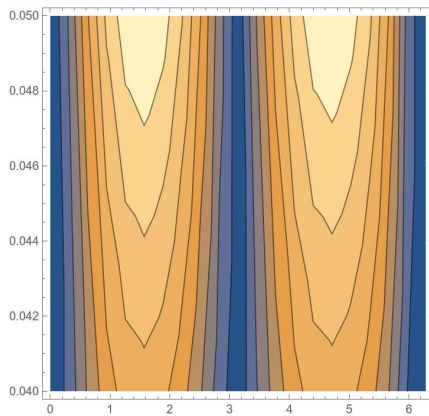


Fig. 5. Contour graph of absolute errors for  $N = 20$ ,  $\beta = 0.02$

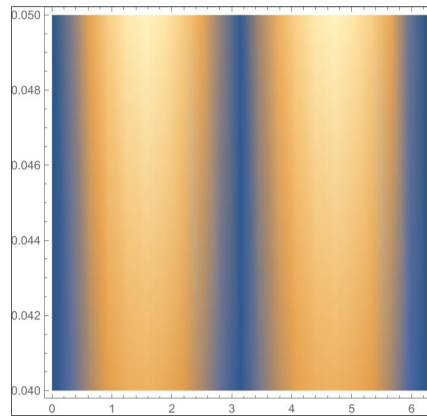


Fig. 6. Density graph of absolute errors for  $N = 20$ ,  $\beta = 0.02$



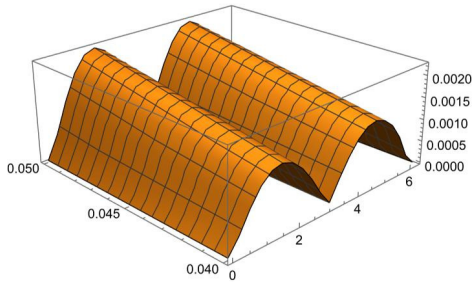


Fig. 7. 3D graph of absolute errors for  $N = 20$ ,  $\beta = 0.02$

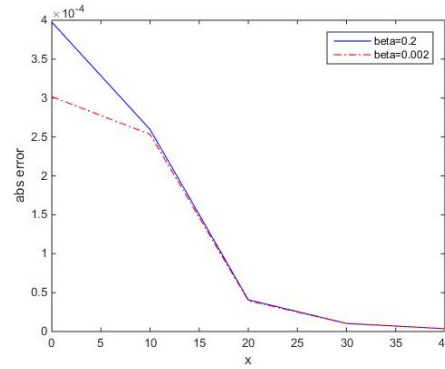


Fig. 8. Absolute errors for different  $\beta$  values

In Table 1, the maximum norms for different time intervals in different fractional orders have been compared. It has been observed that the errors decrease as the time interval value gets smaller. In Table 2, absolute errors have been compared for different  $N$  values and for various  $\beta$  values. It has been observed that the errors get smaller as the  $N$  value increases, that is, as the spatial size is divided into smaller pieces. For the fractional order given in Table 3, the maximum norm has been calculated for different  $h$  and  $dt$  values. In Table 4, absolute errors at different time steps have been calculated for  $N = 10$ ,  $\beta = 0.04$ .

In Figure 1 and 2, the density and contour graphs of absolute errors, respectively have been indicated when  $N = 10$ ,  $\beta = 0.04$  at various spatial spaces. 3D graph of absolute error for some value of  $N$  and  $\beta$  has shown in Figure 3. In Figure 4, numerical and exact solutions have been compared for different  $x$  values when  $N = 10$ ,  $\beta = 0.04$ ,  $dt = 1/2000$ . Contour, density and 3D graphs of absolute errors have been indicated in Figures 5, 6 and 7 respectively, when  $N = 20$ ,  $\beta = 0.02$ . In Figure 8, the absolute errors for different values for  $\beta$  have been shown.

## 5. Conclusions

The types of fractional reaction diffusion equations help to understand phenomena in different fields such as biological, chemical and physical. The most important advantage of numerical methods in these areas is that they contribute to the understanding of complex equations and the mathematical analysis of systems. For example, it aids in optimizing processes and predicts the properties of reactions in chemical applications. It provides the opportunity to visualize the geometric structure of molecules by modeling structures that are difficult to analyze.

In this study, the finite difference method has been used to obtain the numerical solutions of the time-fractional model. It is seen that the absolute error between the numerical solutions and the exact solutions decreases as the number of iterations

increases. This model is suitable for use in many areas and helps to make physical interpretations.

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