

ON ONE-DIMENSIONAL DIFFUSION PROCESSES WITH MOVING MEMBRANES

Bohdan Kopytko¹, Roman Shevchuk²

¹ *Department of Mathematics, Czestochowa University of Technology
Czestochowa, Poland*

² *Department of Mathematics, Lviv Polytechnic National University
Lviv, Ukraine*

bohdan.kopytko@pcz.pl, r.v.shevchuk@gmail.com

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Abstract. Using the method of the classical potential theory, we construct the two-parameter Feller semigroup associated, on the given interval of the real line, with the Markov process such that it is a result of pasting together, at some point of the interval, two ordinary diffusion processes given in sub-domains of this interval. It is assumed that the position on the line of boundary points of these sub-domains depends on the time variable. In addition, some variants of the general nonlocal boundary condition of Feller-Wentzell's type are given in these points. The resulting process can serve as a one-dimensional mathematical model of the physical phenomenon of diffusion in media with moving membranes.

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1. Introduction

The paper is devoted to the construction, using analytical methods, of the two-parameter Feller semigroup associated with a certain inhomogeneous Markov process on the bounded interval of the real line. This process occurs after pasting together, at some point of the interval, two diffusion processes given by their generating differential operators. It is assumed that the position on the real line of the boundary points of the considered domains is determined by given functions which, as well as the process itself, depend on the time variable. Some variants of the general boundary condition or conjugation condition of Feller-Wentzell's type for one-dimensional diffusion processes are supposed to be given in these points (see [1, 2]). This type of Markov process can serve as a one-dimensional mathematical model of the physical phenomenon of diffusion in media with membranes which are located at some points (cf. [3–5]). In our case, such points are the boundaries of the given interval of the line as well as the point of pasting together two given diffusion processes. We say that

the membranes placed at these points are moving because their position on the line changes and depends on the time variable.

In the paper we study in detail the case, when the boundary conditions on the outer boundaries of two domains in which the diffusion processes are given, correspond to the property of total reflection of a diffusion process. Furthermore, the conjugation condition defined on the common boundary of these domains corresponds to a partial reflection of a process in combination with the possibility of its leaving the boundary by jumps.

The investigation of the problem of construction of the two-parameter Feller semigroup associated with the needed Markov process leads to the nonlocal Feller-Wentzell initial-boundary value problem for the one-dimensional (with respect to the space variable) parabolic backward Kolmogorov equation with discontinuous coefficients. The problem, formulated in such a way, is considered for the first time in bounded curvilinear domains with non-smooth boundaries (cf. [3–6]). The classical solvability of this problem in the space of continuous functions is established here under some assumptions on its output data by the boundary integral equations method with the use of the fundamental solutions of the uniformly parabolic equations and the associated potentials.

Note that earlier in [4], a similar problem was considered for the case of non-moving membranes, and in [5] it was studied under the assumption that domains, where the diffusion processes are considered, are half-bounded intervals of the real line. We also mention the paper [7] where the approach to investigation of parabolic conjugation problems that we developed was applied for the first time to the study of the topical problem of the high-energy astrophysics concerned with solving a certain non-stationary kinetic equation that describes the acceleration of charged particles in the vicinity of strong shock waves. As for the application of other approaches to the construction of diffusion processes in domains of a finite-dimensional Euclidean space, given the diffusion coefficients and boundary conditions, they are reflected, in particular, in [8–10].

2. Main assumptions and problem setting

Consider in the strip

$$\Pi = [0, T] \times \mathbb{R} = \{(s, x) \in \mathbb{R}^2 : 0 \leq s \leq T; -\infty < x < \infty\}$$

($T > 0$ is fixed) two parabolic operators

$$\frac{\partial}{\partial s} + L_s^{(i)} \equiv \frac{\partial}{\partial s} + \frac{1}{2} b_i(s, x) \frac{\partial^2}{\partial x^2} + a_i(s, x) \frac{\partial}{\partial x}, \quad i = 1, 2. \quad (1)$$

Assume that the coefficients of the operators $L_s^{(i)}$ are real-valued functions which satisfy the following conditions:

1) there exist positive constants C_1 and C_2 such that

$$C_1 \leq b_i(s, x) \leq C_2, \quad (s, x) \in \Pi, \quad i = 1, 2;$$

2) in Π , the functions $b_i(s, x), a_i(s, x), i = 1, 2$, are bounded and continuous and belong to the Hölder class $H^{\frac{\lambda}{2}, \lambda}(\Pi)$, $0 < \lambda < 1$ (for the definition of Hölder classes, see [11, Ch. I, §1]).

In the strip Π , we consider two bounded domains:

$$D_t^{(i)} = \{(s, x) : 0 \leq s < t \leq T, \quad g_i(s) < x < g_{i+1}(s)\}, \quad i = 1, 2,$$

where $g_j(s), s \in [0, T], j = 1, 2, 3$, are given functions which satisfies the condition

3) $g_j \in H^{\frac{1+\lambda}{2}}([0, T]), j = 1, 2, 3$ (λ is the constant in the condition 2).

We will use also the following notations:

$$I_{1s} = \{x : g_1(s) \leq x < g_2(s)\} = [g_1(s), g_2(s)),$$

$$I_{2s} = \{x : g_2(s) < x \leq g_3(s)\} = (g_2(s), g_3(s)],$$

$$I_{is}^\delta = \{y : y \in I_{is}, |y - g_2(s)| < \delta\}, \quad \delta > 0, \quad i = 1, 2,$$

$$\mathcal{C}_j = \{(s, g_j(s)) : s \in [0, T]\}, \quad j = 1, 2, 3,$$

$$I_s = I_{1s} \cup I_{2s}, \quad I_s^\delta = I_{1s}^\delta \cup I_{2s}^\delta, \quad D_t = D_t^{(1)} \cup D_t^{(2)},$$

$$\Delta_{\tilde{x}} f(\cdot, x) \equiv f(\cdot, x) - f(\cdot, \tilde{x}), \quad \Delta_{\tilde{t}} f(t, \cdot) \equiv f(t, \cdot) - f(\tilde{t}, \cdot),$$

\overline{G} denotes the closure of the set G and $C_b(\mathbb{R})$ is the space of bounded continuous functions on \mathbb{R} with the norm $\|\varphi\| = \sup |\varphi(x)|$. In what follows C and c are various positive constants independent of (s, x) , which will be used without specifying their values. Other notations will be explained immediately after their appearance in the text of the paper.

Operators (1) have the fundamental solutions (see [3, Ch. II, §1], [11, Ch. IV, §11], [12, Ch. I, §2])

$$\Gamma_i(s, x, t, y) = \Gamma_{i0}(s, x, t, y) + \Gamma_{i1}(s, x, t, y), \quad i = 1, 2, \quad 0 \leq s < t \leq T, \quad x, y \in \mathbb{R}, \quad (2)$$

where

$$\Gamma_{i0}(s, x, t, y) = [2\pi b_i(t, y)(t - s)]^{-\frac{1}{2}} \exp \left\{ -\frac{(y - x)^2}{2b_i(t, y)(t - s)} \right\},$$

$$\Gamma_{i1}(s, x, t, y) = \int_s^t d\tau \int_{\mathbb{R}} \Gamma_{i0}(s, x, \tau, z) Z_i(\tau, z, t, y) dz$$

Z_i can be found from the condition that the function Γ_i in (2) satisfies in (s, x) the equation $\frac{\partial u}{\partial s} + L_s^{(i)} u = 0$ and the estimates

$$|D_s^r D_x^p \Gamma_i(s, x, t, y)| \leq C(t-s)^{-\frac{1+2r+p}{2}} \exp\left\{-c \frac{(y-x)^2}{t-s}\right\}, \quad (3)$$

$$|D_s^r D_x^p \Gamma_{i1}(s, x, t, y)| \leq C(t-s)^{-\frac{1+2r+p-\lambda}{2}} \exp\left\{-c \frac{(y-x)^2}{t-s}\right\}, \quad (4)$$

hold, where λ is the constant in the condition 2. Here, r and p are the nonnegative integers for which $2r + p \leq 2$, D_s^r and D_x^p are the partial derivatives with respect to s of order r and with respect to x of order p , respectively. Note that for the function Γ_{i0} in (2), the inequality (3) holds when r and p are any nonnegative integers.

In the domain D_t ($t \in (0, T]$), we set the nonlocal parabolic conjugation problem of the Feller-Wentzell's type with respect to the unknown function $u(s, x, t)$ ($(s, x) \in \bar{D}_t$):

$$\frac{\partial u}{\partial s} + L_s^{(i)} u = 0, \quad (s, x) \in D_t^{(i)}, \quad i = 1, 2, \quad (5)$$

$$\lim_{s \uparrow t} u(s, x, t) = \varphi(x), \quad x \in \bar{I}_t, \quad (6)$$

$$\frac{\partial u}{\partial x}(s, g_{2i-1}(s), t) = 0, \quad 0 \leq s < t \leq T, \quad i = 1, 2, \quad (7)$$

$$u(s, g_2(s) - 0, t) = u(s, g_2(s) + 0, t), \quad 0 \leq s \leq t \leq T, \quad (8)$$

$$q_1(s) \frac{\partial u}{\partial x}(s, g_2(s) - 0, t) - q_2(s) \frac{\partial u}{\partial x}(s, g_2(s) + 0, t) + \int_{I_s} [u(s, g_2(s), t) - u(s, y, t)] \mu(s, dy) = 0, \quad 0 \leq s < t \leq T, \quad (9)$$

where φ , q_1 , q_2 are the given functions and μ is the given measure.

Assume that the functions φ , q_1 , q_2 and the measure μ satisfy the following conditions:

- 4) the function φ is defined on \mathbb{R} and belongs to the space $C_b(\mathbb{R})$;
- 5) the functions q_1 , q_2 are nonnegative, continuous and satisfy the inequality

$$q_1(s) + q_2(s) > 0, \quad s \in [0, T]; \quad (10)$$

- 6) $\mu(s, \cdot)$ is the nonnegative measure on I_s such that for any $\delta > 0$ and any bounded continuous function f on I_s , $i = 1, 2$, the integrals

$$F_f^{(i)}(s) = \int_{I_s^\delta} (y - g_2(s)) f(y) \mu(s, dy), \quad G_f^{(i)}(s) = \int_{I_s \setminus I_s^\delta} f(y) \mu(s, dy), \quad i = 1, 2,$$

are continuous on $[0, T]$ as functions of s .

The purposes of the following sections are to establish by the potential method the classical solvability of the problem (5)-(9) and to prove that the family of linear operators (which act in $C_b(\mathbb{R})$), defined by means of the solution of this problem, forms the two-parameter Feller semigroup associated with some inhomogeneous Markov process on \bar{I}_s .

3. Solving the nonlocal parabolic conjugation problem

In this section we solve by the method of potential theory the initial boundary-value problem (5)-(9).

Theorem 1 *Let the operators $L_s^{(i)}$, $i = 1, 2$, satisfy the conditions 1, 2 and the functions $g_j(s)$, $j = 1, 2, 3$, belong to the Hölder class $H^{\frac{1+\lambda}{2}}([0, T])$. If φ is bounded continuous on \mathbb{R} , q_i , $i = 1, 2$, and μ , respectively, satisfy the conditions 5 and 6, then there exists a unique classical solution of the problem (5)-(9) which is continuous in the closed domain \bar{D}_t . \square*

PROOF We first prove the existence of a solution $u(s, x, t)$. We find it in the form

$$u(s, x, t) = u_i(s, x, t) = u_{i0}(s, x, t) + u_{i1}(s, x, t), \quad (s, x) \in \bar{D}_t^{(i)}, \quad i = 1, 2, \quad (11)$$

where $u_{i0}(s, x, t)$ ($0 \leq s < t \leq T$, $x \in \mathbb{R}$) are the Poisson potentials

$$u_{i0}(s, x, t) = \int_{\mathbb{R}} \Gamma_i(s, x, t, y) \varphi(y) dy, \quad i = 1, 2, \quad (12)$$

and $u_{i1}(s, x, t)$ ($0 \leq s < t \leq T$, $x \in \mathbb{R}$) express as a sum of simple-layer potentials

$$u_{i1}(s, x, t) = \sum_{j=0}^1 \int_s^t \Gamma_i(s, x, \tau, g_{i+j}(\tau)) V_{2i-1+j}(\tau, t) d\tau, \quad i = 1, 2, \quad (13)$$

with the unknown densities V_k , $k = 1, 2, 3, 4$, to be found.

Suppose a priori that $V_k(\tau, t)$ are continuous for $0 \leq \tau < t \leq T$ and bounded by $C(t - \tau)^{-\mu}$, where $0 \leq \mu < 1$. These functions will be defined from the conditions (7)-(9). Consider first the boundary conditions (7). From the well-known theorem on the jump of the conormal derivative of a parabolic simple-layer potential (see [12, Ch. V, §1], [5, Formula (1.22)]), it is seen directly that these conditions lead to the equalities

$$V_{3i-2}(s, t) = \Psi_{3i-2}(s, t) + \sum_{j=0}^1 \int_s^t K_{ij}(s, \tau) V_{2i+j-1}(\tau, t) d\tau, \quad i = 1, 2, \quad (14)$$

where

$$\Psi_{3i-2}(s, t) = (-1)^{i-1} b_i(s, g_{2i-1}(s)) \frac{\partial u_{i0}}{\partial x}(s, g_{2i-1}(s), t), \quad i = 1, 2,$$

$$K_{ij}(s, \tau) = (-1)^{i-1} b_i(s, g_{2i-1}(s)) \frac{\partial \Gamma_i}{\partial x}(s, g_{2i-1}(s), \tau, g_{i+j}(\tau)), \quad i = 1, 2, j = 0, 1.$$

Using the conditions 1, 4 and the estimates (3) (with $r = 0, p = 1$), we find that $\Psi_{3i-2}(s, t)$ ($0 \leq s < t \leq T, i = 1, 2$) are continuous functions which satisfy the inequality

$$|\Psi_{3i-2}(s, t)| \leq C \|\varphi\| (t-s)^{-\frac{1}{2}}, \quad i = 1, 2. \quad (15)$$

Further, the integrals with kernels K_{10} and K_{21} coincide with direct values of conormal derivatives of the simple-layer potentials with an accuracy to bounded multipliers. Therefore, for these kernels, the inequality

$$|K_{ij}(s, t)| \leq C(\tau-s)^{-1+\frac{\lambda}{2}} \quad (16)$$

holds whenever $0 \leq s < \tau \leq t \leq T$ (see inequality (1.23) in [5]).

In view of the condition 3, the estimate (3) and the inequalities

$$|g_i(\tau) - g_j(\tau)| \geq C > 0, \quad i \neq j, \tau \in [0, T], \quad (17)$$

$$\sigma^\nu e^{-c\sigma} \leq \text{const} \quad (0 \leq \sigma < \infty, 0 \leq \nu < \infty), \quad (18)$$

it is not difficult to show that the estimate (16) can be also applied to the kernels K_{11} and K_{20} .

Consider now the conjugation conditions (8) and (9). By requiring that the function u in (11) satisfies the condition (8), we get the equation

$$\sum_{i=1}^2 \sum_{j=0}^1 \int_s^t (-1)^{i-1} \Gamma_i(s, g_2(s), \tau, g_{i+j}(\tau)) V_{2i+j-1}(\tau, t) d\tau = \Phi_0(s, t), \quad (19)$$

where $\Phi_0(s, t) = u_{20}(s, g_2(s), t) - u_{10}(s, g_1(s), t)$.

The equation (19) is the Volterra integral equation of the first kind. In order to regularize it, we introduce the integro-differential operator \mathcal{E} which acts by the rule

$$\mathcal{E}(s, t)f = \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^t (\rho-s)^{-\frac{1}{2}} f(\rho, t) d\rho, \quad 0 \leq s < t \leq T. \quad (20)$$

After applying the operator \mathcal{E} in (20) to the both sides of the equation (19), we obtain

$$\sum_{i=1}^2 (-1)^{i-1} \frac{V_{i+1}(s, t)}{\sqrt{b_i(s, g_2(s))}} = -\Phi(s, t) + \sum_{i=1}^2 \sum_{j=0}^1 \int_s^t L_{ij}(s, \tau) V_{2i+j-1}(\tau, t) d\tau, \quad (21)$$

where

$$\Phi(s, t) = \frac{1}{\sqrt{2\pi}} \int_s^t (\rho - s)^{-\frac{3}{2}} [\Phi_0(\rho, t) - \Phi_0(s, t)] d\rho - \sqrt{\frac{2}{\pi}} (t - s)^{-\frac{1}{2}} \Phi_0(s, t),$$

$$L_{ij}(s, \tau) = (-1)^{i-1} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} \Gamma_i(\rho, g_2(\rho), \tau, g_{i+j}(\tau)) d\rho, \quad i + j \neq 2,$$

$$L_{ij}(s, \tau) = (-1)^{i-1} \sqrt{\frac{2}{\pi}} \frac{\partial}{\partial s} \int_s^\tau (\rho - s)^{-\frac{1}{2}} [\Gamma_{i1}(\rho, g_2(\rho), \tau, g_{i+j}(\tau)) + \Delta_{g_2(\rho)}^{g_2(\tau)} \Gamma_i(\rho, g_2(\rho), \tau, g_{i+j}(\tau))] d\rho, \quad i + j = 2.$$

Furthermore, Φ and L_{ij} , respectively, satisfy the inequalities (15) and (16).

Thus, like (7), the condition (8) ultimately leads to the Volterra integral equation of the second kind with the kernels that have integrable singularity.

The last integral equation, which connects all four functions V_k , $k = 1, 2, 3, 4$, can be obtained from the conjugation condition (9). By the already mentioned relation on the jump of conormal derivative of a simple-layer potential, we get

$$\sum_{i=1}^2 \frac{q_i(s)}{b_i(s, g_2(s))} V_{i+1}(s, t) = \Psi(s, t) + \sum_{i=1}^2 \sum_{j=0}^1 \int_s^t M_{ij}(s, \tau) V_{2i+j-1}(\tau, t) d\tau, \quad (22)$$

where

$$\begin{aligned} \Psi(s, t) &= \sum_{i=1}^2 \left[(-1)^i q_i(s) \frac{\partial u_{i0}}{\partial x}(s, g_2(s), t) + \int_{I_{is}} \Delta_y^{g_2(s)} u_{i0}(s, y, t) \mu(s, dy) \right], \\ M_{ij}(s, \tau) &= (-1)^i q_i(s) \frac{\partial \Gamma_i}{\partial x}(s, g_2(s), \tau, g_{i+j}(\tau)) + \int_{I_{is}} \Delta_y^{g_2(s)} \Gamma_i(s, y, \tau, g_{i+j}(\tau)) \mu(s, dy) \\ &= M_{ij}^{(1)}(s, \tau) + M_{ij}^{(2)}(s, \tau), \quad i = 1, 2, j = 0, 1. \end{aligned} \quad (23)$$

From the mean value theorem, the conditions 4-6 and the inequalities (3), (17), (18), it follows that the function Ψ allows the estimate (15), and that $M_{ij}^{(1)}(s, \tau)$ satisfies (16).

Let us get down to studying the integral term $M_{ij}^{(2)}(s, \tau)$ on the right-hand side of (23). For any $\delta > 0$, we have

$$\begin{aligned} M_{ij}^{(2)}(s, \tau) &= \int_{I_{is} \setminus I_{is}^\delta} \Delta_y^{g_2(s)} \Gamma_i(s, y, \tau, g_{i+j}(\tau)) \mu(s, dy) \\ &+ \int_{I_{is}^\delta} \Delta_y^{g_2(s)} \Gamma_{i1}(s, y, \tau, g_{i+j}(\tau)) \mu(s, dy) + \int_{I_{is}^\delta} \Delta_y^{g_2(s)} \Gamma_{i0}(s, y, \tau, g_{i+j}(\tau)) \mu(s, dy) \\ &= M_{ij}^{(21)}(s, \tau) + M_{ij}^{(22)}(s, \tau) + M_{ij}^{(23)}(s, \tau). \end{aligned} \quad (24)$$

Applying the mean value theorem to the difference $\Delta_y^{g_2(s)} \Gamma_{i1}(s, y, \tau, g_{i+j}(\tau))$ in the expression for $M_{ij}^{(22)}$ and using the inequalities (3), (4) as well the condition 6, we find that for $0 \leq s < \tau \leq t \leq T$,

$$|M_{ij}^{(2k)}(s, \tau)| \leq C(\delta)(\tau - s)^{-1 + \frac{\lambda}{2}}, \quad k = 1, 2, \quad (25)$$

where $C(\delta)$ is some constant depending on δ .

Further, after writing the integrand in $M_{ij}^{(23)}$ in the form

$$-[2\pi b_i(\tau, g_{i+j}(\tau))(\tau - s)]^{-\frac{1}{2}} \int_0^1 \frac{\partial}{\partial \theta} \exp \left\{ -\frac{A(\theta, y, g_{i+j}(\tau), g_2(s))}{2b_i(\tau, g_{i+j}(\tau))(\tau - s)} \right\} d\theta,$$

where

$$A(\theta, y, g_{i+j}(\tau), g_2(s)) = (1 - \theta)(y - g_{i+j}(\tau))^2 + \theta(g_2(s) - g_{i+j}(\tau))^2,$$

we obtain

$$\begin{aligned} M_{ij}^{(23)}(s, \tau) &= \frac{g_{i+j}(\tau) - g_2(\tau)}{\sqrt{2\pi}[b_i(\tau, g_{i+j}(\tau))(\tau - s)]^{\frac{3}{2}}} \int_{I_{is}^\delta} (y - g_2(s)) \mu(s, dy) \\ &\times \int_0^1 \exp \left\{ -\frac{A(\theta, y, g_{i+j}(\tau), g_2(s))}{2b_i(\tau, g_{i+j}(\tau))(\tau - s)} \right\} d\theta - \frac{\pi}{[2\pi b_i(\tau, g_{i+j}(\tau))(\tau - s)]^{\frac{3}{2}}} \\ &\times \int_{I_{is}^\delta} (y - g_2(s))^2 \mu(s, dy) \int_0^1 \exp \left\{ -\frac{A(\theta, y, g_{i+j}(\tau), g_2(s))}{2b_i(\tau, g_{i+j}(\tau))(\tau - s)} \right\} d\theta \\ &= M_{ij}^{(231)}(s, \tau) + M_{ij}^{(232)}(s, \tau). \end{aligned} \quad (26)$$

Using the conditions 3, 6 and the inequalities (17), (18), we find that $M_{ij}^{(231)}(s, \tau)$ allows the inequality (25). For $M_{ij}^{(232)}(s, \tau)$, after applying the inequalities

$$A(\theta, y, g_{i+j}(\tau), g_2(s)) \geq \theta(1 - \theta)(y - g_2(s))^2 \quad (27)$$

and $\sigma e^{-c\sigma} \leq (c \cdot e)^{-\frac{1}{2}}$ ($c > 0$, $0 \leq \sigma < \infty$), we get ($0 \leq s < \tau \leq t \leq T$)

$$|M_{ij}^{(232)}(s, \tau)| \leq \frac{1}{2C_1} \left(\frac{\pi C_2}{2C_1 e} \right)^{\frac{1}{2}} \lambda_{is}^\delta (\tau - s)^{-1}, \quad i = 1, 2, \quad (28)$$

where C_1 and C_2 are the constants from 1), and by λ_{is}^δ we denote the integrals of the function $|y - g_2(s)|$ over I_{is}^δ with respect to the measure μ . We pay attention to the fact that the functions $M_{ij}^{(232)}(s, \tau)$ have non-integrable singularity at $\tau = s$.

Consider the system of equations (21), (22). Solving it with respect to V_2 and V_3 , and attaching to the obtained equations two more equations from the system (14), we finally obtain ($0 \leq s < t \leq T$)

$$V_i(s, t) = \Psi_i(s, t) + \sum_{j=1}^4 \int_s^t N_{ij}(s, \tau) V_j(\tau, t) d\tau, \quad i = 1, 2, 3, 4, \quad (29)$$

where $\Psi_1(s, t)$ and $\Psi_4(s, t)$ are defined in (14),

$$\Psi_i(s, t) = d_{i-1}(s) \left[\Psi(s, t) + \frac{(-1)^{i-1} q_{4-i}(s)}{\sqrt{b_{4-i}(s, g_2(s))}} \Phi(s, t) \right], \quad i = 2, 3,$$

$$d_{i-1}(s) = \frac{b_{i-1}(s, g_2(s)) \sqrt{b_{4-i}(s, g_2(s))}}{q_1(s) \sqrt{b_2(s, g_2(s))} + q_2(s) \sqrt{b_1(s, g_2(s))}}, \quad i = 2, 3,$$

$$N_{13}(s, \tau) = N_{14}(s, \tau) = N_{41}(s, \tau) = N_{42}(s, \tau) \equiv 0, \quad N_{11}(s, \tau) = K_{10}(s, \tau)$$

$$N_{12}(s, \tau) = K_{11}(s, \tau), \quad N_{43}(s, \tau) = K_{20}(s, \tau), \quad N_{44}(s, \tau) = K_{21}(s, \tau),$$

$$N_{ij}(s, \tau) = d_{i-1}(s) \left[M_{1,j-1}(s, \tau) + \frac{(-1)^i q_{4-i}(s)}{\sqrt{b_{4-i}(s, g_2(s))}} L_{1,j-1}(s, \tau) \right], \quad i = 2, 3, j = 1, 2,$$

$$N_{ij}(s, \tau) = d_{i-1}(s) \left[M_{2,j-3}(s, \tau) + \frac{(-1)^i q_{4-i}(s)}{\sqrt{b_{4-i}(s, g_2(s))}} L_{2,j-3}(s, \tau) \right], \quad i = 2, 3, j = 3, 4.$$

Combining (15), (16), (25), (28) and using the inequalities (10) and

$$|d_j(s)| \leq \frac{C_2}{q_0} \left(\frac{C_2}{C_1} \right)^{\frac{1}{2}}, \quad s \in [0, T], j = 1, 2; \quad q_0 = \min_{s \in [0, T]} (q_1(s) + q_2(s)) > 0,$$

we find that $\Psi_i(s, t)$ ($i = 1, 2, 3, 4$) and $N_{ij}(s, \tau)$ ($i = 1, 4; j = 1, 2, 3, 4$) satisfy, respectively, the estimates (15) and (16), and that the absolute values of $N_{ij}(s, \tau)$

($i = 2, 3; j = 1, 2, 3, 4$) are bounded by $C(\delta)(\tau - s)^{-1 + \frac{\lambda}{2}} + m(\delta)(\tau - s)^{-1}$, where

$$m(\delta) = \left(\frac{C_2}{C_1}\right)^2 \frac{\pi}{q_0} \max_{s \in [0, T]} (\lambda_{1s}^\delta + \lambda_{2s}^\delta).$$

The main point of the following discussion is to prove the possibility of applying the ordinary method of successive approximations to the system of Volterra integral equations of the second kind (29). Thus, we look for the solution of the system of equations (29) of the form

$$V_i(s, t) = \sum_{k=0}^{\infty} V_i^{(k)}(s, t), \quad i = 1, 2, 3, 4 \quad (30)$$

where ($i = 1, 2, 3, 4, k = 1, 2, \dots$)

$$V_i^{(0)}(s, t) = \Psi_i(s, t), \quad V_i^{(k)}(s, t) = \sum_{j=1}^4 \int_s^t N_{ij}(s, \tau) V_j^{(k-1)}(\tau, t) d\tau.$$

Let us show that the series (30) converges when $0 \leq s < t \leq T$. For the functions $V_i^{(0)}(s, t) \equiv \Psi_i(s, t)$, $i = 1, 2, 3, 4$, we already have the estimate with the right-hand side $C_0 \|\varphi\| (t - s)^{-\frac{1}{2}}$ (see (15)), where C_0 is some fixed positive constant.

Next, let us fix a sufficiently small $\delta = \delta_0$ so that $m_0 = m(\delta_0) < 1$. By induction on k , we establish, upon using successively (23)-(28), the condition 1), the relation

$$\int_s^t (t - \tau)^{-\frac{1}{2}} (\tau - s)^{-\frac{3}{2}} e^{-\frac{\theta(1-\theta)(y-g_2(s))^2}{2C_2(\tau-s)}} d\tau = \left(\frac{2\pi C_2}{\theta(1-\theta)(t-s)}\right)^{\frac{1}{2}} \frac{1}{|y - g_2(s)|},$$

and the estimate (15) (for $V_i^{(0)}(s, t)$), that

$$|V_i^{(k)}(s, t)| \leq C_0 \|\varphi\| (t - s)^{-\frac{1}{2}} \sum_{n=0}^k \binom{k}{n} h_{s,t}^{(k-n)} m_0^n, \quad (31)$$

where

$$h_{s,t}^{(k)} = \frac{(C\Gamma(\frac{\lambda}{2}))^k \Gamma(\frac{1}{2})}{\Gamma(\frac{1+k\lambda}{2})} (t - s)^{\frac{k\lambda}{2}}$$

($\Gamma(\sigma)$ is the gamma function). Using (31), we obtain

$$\sum_{k=0}^{\infty} |V_i^{(k)}(s, t)| \leq C_0 \|\varphi\| (t - s)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{[C\Gamma(\frac{\lambda}{2})]^k \Gamma(\frac{1}{2})}{\Gamma(\frac{1+k\lambda}{2}) (1 - m_0)^{k+1}} (t - s)^{\frac{k\lambda}{2}}. \quad (32)$$

From the inequality (32), it follows that the series (30) converges and that the functions $V_i(s, t)$ ($i = 1, 2, 3, 4$) are continuous for $0 \leq s < t \leq T$, and they allow

the inequality (15). Thus, the formula (30) represents the unique solution of the system of integral equations (29).

Having proved that $u(s, x, t)$ satisfies (8)-(9), we now get down to proving (5) and (6). From the properties of the fundamental solution $\Gamma_i(s, x, t, y)$, $i = 1, 2$, the estimate (3) (with $r = p = 0$) and the condition 4), it follows that the function $u_{i0}(s, x, t)$, $i = 1, 2$, is continuous in $\bar{D}_t^{(i)}$, satisfies in $D_t^{(i)}$ the equation (5), the initial condition (6) as well as the inequality

$$|u_{i0}(s, x, t)| \leq C\|\varphi\|, \quad i = 1, 2. \quad (33)$$

Further, by (3) and (15) (with Ψ_{3i-2} replaced by V_i , $i = 1, 2, 3, 4$), we obtain

$$u_{i1}(s, x, t) \leq C\|\varphi\| \int_s^t (\tau - s)^{-\frac{1}{2}} (t - \tau)^{-\frac{1}{2}} d\tau \leq C\|\varphi\|, \quad i = 1, 2. \quad (34)$$

Hence the function $u_{i1}(s, x, t)$ is bounded in $\bar{D}_t^{(i)}$ and satisfies in $D_t^{(i)}$ the equation (5) with zero initial condition $\lim_{s \rightarrow t} u_{i1}(s, x, t) = 0$ ($x \in I_t$). Additionally, we check that this condition holds for $x \in \bar{I}_t$. Thus, the existence of a classical solution of the problem (5)-(9) is proved. The proof of uniqueness is a repetition of the proof of Theorem 2.2 in [5] with obvious changes.

The proof of Theorem 1 is now complete. ■

4. Construction of a diffusion process with moving membranes

In this section we prove that the solution $u(s, x, t) \equiv T_{st}\varphi(x)$ of the problem (5)-(9) can be considered as the two-parameter operator semigroup describing a certain Markov process with trajectories in curvilinear domain \bar{D}_T .

Having the solution $u(s, x, t)$ of the problem (5)-(9), we define the two-parameter family of linear operators (T_{st}) , $0 \leq s \leq t \leq T$ in $C_b(\mathbb{R})$. For $0 \leq s < t \leq T$, $x \in \bar{I}_s$ and $\varphi \in C_b(\mathbb{R})$, we put

$$T_{st}\varphi(x) = T_{st}^{(0i)}\varphi(x) + T_{st}^{(1i)}\varphi(x), \quad x \in \bar{I}_{is}, \quad i = 1, 2, \quad (35)$$

where $T_{st}\varphi(x) = u(s, x, t)$, $T_{st}^{(0i)}\varphi(x) = u_{i0}(s, x, t)$, $T_{st}^{(1i)}\varphi(x) = u_{i1}(s, x, t)$, the functions u , u_{i0} and u_{i1} , $i = 1, 2$, are defined by the formulas (11), (12) and (13), (30), respectively. Furthermore, $T_{ss} = E$, where E is the identity operator, and $T_{st}\varphi(x)$ satisfies the estimate (33) for all $(s, x) \in \bar{D}_t$.

Theorem 2 *Assume that the coefficients of operators $L_s^{(i)}$ the functions g_j, q_i ($i = 1, 2, j = 1, 2, 3$) and the measure μ satisfy the conditions of Theorem 1. Then, the two-parameter operator family (T_{st}) , defined by (35), describes the inhomogeneous Feller process in \mathbb{R} , the trajectories of which are located in curvilinear domain \bar{D}_T . In $\bar{D}_T \setminus (\mathcal{C}_i \cup \mathcal{C}_{i+1})$, the trajectories of this process can be treated as the trajecto-*

ries of the diffusion process generated by $L_s^{(i)}$ ($i = 1, 2$), and at the points of curves \mathcal{C}_j ($j = 1, 2, 3$), they behave according to the boundary conditions in (7) and (9), respectively. \square

PROOF We first show that the family of operators (T_{st}) defined by (35) is the two-parameter Feller semigroup. To do this, we note that the semigroup property can be established using the assertion of Theorem 1 on the uniqueness of the solution of the problem (5)-(9) by repeating almost verbatim the proof of the analogous fact in [5, Section 3].

Further, combining the schemes used to prove lemmas in [4, Section 3] and [5, Section 3], we get the following result:

Lemma 1 *If $\varphi \in C_b(\mathbb{R})$ and $\varphi(x) \geq 0$ for all $x \in \mathbb{R}$, then $T_{st}\varphi(x) \geq 0$ for all $0 \leq s \leq t \leq T$, $x \in \bar{I}_s$.* \square

Now, we can easily finish the proof of Theorem 2. Putting $\varphi_0(x) \equiv 1$ in (35), we find by direct calculations that $T_{st}\varphi_0(x) \equiv 1$. From this and from Lemma 1, it follows that the operators T_{st} are contractive, i.e., they do not increase the norm of element.

Combining the above properties of the operator family T_{st} , we conclude (see [13, Ch. II, §1], [3, Ch. I]) that T_{st} is a two-parameter Feller semigroup associated with the desired Markov process.

The proof of Theorem 2 is complete. \blacksquare

5. Conclusions

In the paper, using analytical methods, we construct the two-parameter Feller semigroup associated with the inhomogeneous Markov process on the given interval of the real line with variable boundaries, which occurs after pasting together two given diffusion processes on this interval. The needed semigroup is obtained by the methods of potential theory with the use of the solution of the corresponding Feller-Wentzell initial-boundary value problem for one-dimensional (with respect to the space variable) parabolic backward Kolmogorov equation with discontinuous coefficients to which the output problem is reduced.

The peculiarity of this problem is that one of the two its conjugation conditions is non-classical and contains the non-local term of the integral type. Furthermore, the problem, formulated in such a way, is studied for the first time in bounded curvilinear domains, whose boundaries are non-smooth functions of time variable. The Markov process constructed in the described way can serve as a mathematical model of the physical phenomenon of diffusion in media with moving membranes.

References

- [1] Wentzell, A.D. (1956). Semigroups of operators that correspond to a generalized differential operator of second order. *Dokl. Akad. Nauk SSSR (N.S.)*, 111(2), 269-272.

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- [2] Langer, H., & Schenk, W. (1983). Knotting of one-dimensional Feller processes. *Math. Nachr.*, 113, 151-161.
 - [3] Portenko, M.I. (1995). *Diffusion processes in media with membranes*. Institute of Mathematics of the NAS of Ukraine.
 - [4] Kopytko, B.I., & Shevchuk, R.V. (2013). Diffusions in one-dimensional bounded domains with reflection, absorption and jumps at the boundary and at some interior point. *J. Appl. Math. Comput. Mech.*, 12(1), 55-68.
 - [5] Kopytko, B.I., & Shevchuk, R.V. (2020). One-dimensional diffusion processes with moving membrane: partial reflection in combination with jump-like exit of process from membrane. *Electron. J. Probab.*, 25, 1-21.
 - [6] Kamynin, L.I. (1964). A boundary-value problem in the theory of heat conduction with a non-classical boundary condition. *Comput. Math. Math. Phys.*, 4(6), 33-59.
 - [7] Petruk, O., & Kopytko, B. (2016). Time-dependent shock acceleration of particles. Effect of the time-dependent injection, with application to supernova remnants. *Monthly Notices of the Royal Astronomical Society*, 462(3), 3104-3114.
 - [8] Pilipenko, A.Yu. (2012). On the Skorokhod mapping for equations with reflection and possible jump-like exit from a boundary. *Ukrainian Math. J.*, 63(9), 1415-1432.
 - [9] Taira, K. (2014). *Semigroups, Boundary Value Problems and Markov Processes*. Springer Monographs in Mathematics.
 - [10] Arendt, W., Kunkel, S., & Kunze, M. (2016). Diffusion with nonlocal boundary conditions. *J. Funct. Anal.*, 270(7), 2483-2507.
 - [11] Ladyzhenskaya, O.A., Solonnikov, V.A., & Ural'tseva, N.N. (1967). *Linear and quasilinear equations of parabolic type*. Nauka.
 - [12] Friedman, A. (2013). *Partial differential equations of parabolic type*. Dover Publications.
 - [13] Dynkin, E.B. (1963). *Markov processes*. Fiz.-Mat. Lit.