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LOCALLY DEFINED OPERATORS ACTING $C^{\infty}(A)$ INTO $C^{0}(A)$

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Abstract. We investigate the locally defined operators, sometimes called operators with memory, that map the space $C^{\infty}(A)$ of continuously differentiable functions in the sense of Whitney defined on a compact subset $A \subset \mathbb{R}^n$ into the space of continuous functions defined on the same set A. Using the Whitney Extension Theorem, we give a representation formula for such operators stating that every local operator $K : C^{\infty}(A) \to C^{0}(A)$ is a generalized Nemytskii operator generated by some function $h : A \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$.

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1. Introduction

Let X be a topological space, Y, Z be arbitrary nonempty sets and let $\mathscr{G} = \mathscr{G}(X,Y)$ and $\mathscr{H} = \mathscr{H}(X,Z)$ stand for two classes of functions $\varphi : X \to Y$ and $\phi : X \to Z$, respectively. An operator $K : \mathscr{G} \to \mathscr{H}$ is said to be locally defined (or with memory), if two functions from the class $\mathscr{G}$ coincide on an open subset then the images of the functions through the operator K are equal on the same open set. A typical example is the Nemytskii composition operator

$$H(\varphi)(x) = h(x, \varphi(x)), \quad \varphi \in \mathscr{G} \quad (x \in X),$$

generated by some function h of two variables. It turns out that in some classes of functions it is the only operator that has this memory property (see for instance [1,2]).

However, in the case where the topological space X, the sets Y, Z, and the classes of functions $\mathscr{G}$ and $\mathscr{H}$ are more significant ones, this definition of Nemytskii operator is not sufficient to describe the possible forms of the locally defined operators $K : \mathscr{G} \to \mathscr{H}$. It was shown by Lichawski et al. [3] that if X is a real compact interval, $Y = Z = \mathbb{R}$, $\mathscr{G}$ is the class $C^m(X)$ of m-times continuously differentiable functions and $\mathscr{H} = C^0(X)$, then there exists a function $h : X \times \mathbb{R}^{m+1} \to \mathbb{R}$ such that

$$K(\varphi)(x) = h\left(x, \varphi(x), \varphi'(x), ..., \varphi^{(m)}(x)\right), \quad \varphi \in C^m(X), \, x \in X.$$

An analogous result was proved by Matkowski and Wróbel for Whitney differentiable functions [4].

The main result of this paper gives a representation formula for operators with memory of the type $K : \bigcap_{m\in\mathbb{N}} C^m(A) \to C^0(A)$. Here $C^m(A)$ denotes the class of m-times continuously differentiable functions in the Whitney sense. In particular, taking $n = 1$, and $A \subset \mathbb{R}$ as an interval, we obtain one of the results of [3] stating that the operator K must be of the form

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \ldots), \quad (x \in A), \quad \varphi \in C^\infty(A),$$

where the function $h : A \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is uniquely determined.

Closely related to the theory of operators with memory is the study of solutions of nonlinear integral equations of the Hammerstein type and the Volterra-Hammerstein type having application in many fields of science such as economics or physics where the memory effects are very important. More details can be found, for example, in the monographs [5, 6]. It is worth mentioning that the Whitney-differentiable functions are useful in the issues related to optimization problems [7].

2. Preliminaries

In this paper, the symbols $\mathbb{N}_0$, $\mathbb{R}$ denote, respectively, the set of nonnegative integers and the set of real numbers. Let $\mathbb{N}_0^n := \prod_{i=1}^{n} \mathbb{N}_0$ for $n \in \mathbb{N}$ and $\mathbb{R}^{\mathbb{N}} := \prod_{i=1}^{n} \mathbb{R}$.

For $n \in \mathbb{N}$, $j \in \mathbb{N}_0^n$, $j = (j_1, \ldots, j_n)$, and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we put

$$|j| := j_1 + \ldots + j_n, \;\; j! := j_1! \cdot \ldots \cdot j_n!, \quad x^j := x_1^{j_1} \cdot \ldots \cdot x_n^{j_n},$$

and

$$\|x\| := \sqrt{\sum_{i=1}^{n} x_i^2}.$$

We start with the following

Definition 1 ([18], cf. also [9]). *Let $A \subset \mathbb{R}^n$ be a nonempty set and let $m \in \mathbb{N}_0$. A family f of functions $f^j : A \to \mathbb{R}$ for $j \in \mathbb{N}_0^n, |j| \le m$, i.e.,*

$$f = \{f^j \in \mathbb{R}^A : j \in \mathbb{N}_0^n, \quad |j| \le m\}$$

is said to be a Whitney jet (or a Whitney m-jet) on A, briefly $f \in C^m(A)$, if for all $j \in \mathbb{N}_0^n$, $|j| \le m$, $x_0 \in A$, and $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$

the inequalities $\|x-x_0\|<\delta$ *and* $\|y-x_0\|<\delta$ *imply that*

$$\left| f^j(x) - \sum_{|i|\le m-|j|} \frac{f^{j+i}(y)}{i!}(x-y)^i \right| \le \varepsilon \, \|x-y\|^{\,m-|j|}.$$

(Here $\mathbb{R}^A$ stands for the set of all real functions defined on A.)

Definition 2 ([8, p. 65]). *Let* $n\in\mathbb{N}$ *and let* $A\subset\mathbb{R}^n$ *be a nonempty set. A family of functions* $f^j : A\to\mathbb{R}$ *for* $j\in\mathbb{N}_0^n$, *i.e.,*

$$f=\{f^j\in\mathbb{R}^A:\ \ j\in\mathbb{N}_0^n\}$$

is said to be a Whitney ∞*-jet on* A, *briefly* $f\in C^\infty(A)$, *if for all* $m\in\mathbb{N}_0$ *a family* $f=\{f^j\in\mathbb{R}^A:\ \ j\in\mathbb{N}_0^n,\ |j|\le m\}$ *is a Whitney* m*-jet on* A.

Thus $f,g\in C^\infty(A)$ and $f=g$ imply that

$$f=\{f^j\in\mathbb{R}^A : j\in\mathbb{N}_0^n\},\quad g=\{g^j\in\mathbb{R}^A : j\in\mathbb{N}_0^n\},$$

and

$$f^j=g^j \quad \text{for all} \quad j\in\mathbb{N}_0^n.$$

Remark 1. *Assume that* $A\subset\mathbb{R}^n$ *is an open set and let* $f=\{f^j\in\mathbb{R}^A:\ \ j\in\mathbb{N}_0^n\}$. *Then* $f\in C^\infty(A)$ *if and only if the function* $f^{(0,\dots,0)}$ *is of the class* C^∞ *on* A *in the classical sense and*

$$f^j=\frac{\partial^{|j|}f^{(0,\dots,0)}}{\partial x_1^{j_1}\dots\partial x_n^{j_n}},\quad j\in\mathbb{N}_0^n.$$

Moreover, if a function $f^{(0,\dots,0)}$ *is of the class* C^∞ *on* A *and the limits of all partial derivatives of* f *at the boundary points of an open set* A *exist, then the unique extension of* $f^{(0,\dots,0)}$ *to the closure of* A *is considered as a function of class* C^∞ *in this closure.*

In the above cases, one can identify the function $f^{(0,\dots,0)}$ *with the whole class* f.

Remark 2. *Let* $n\in\mathbb{N}$ *and let* $A\subset\mathbb{R}^n$ *be a nonempty and compact set. Then* $f\in C^\infty(A)$ *if and only if for every* $m\in\mathbb{N}_0$ *the family* $f=\{f^j\in\mathbb{R}^A : j\in\mathbb{N}_0^n, |j|\le m\}$ *fulfils the following condition*

$$f^j(x)-\sum_{|i|\le m-|j|}\frac{f^{j+i}(y)}{i!}(x-y)^i = o\left(\|x-y\|^{m-|j|}\right) \quad \text{as} \quad \|x-y\|\to 0,$$

where $x,y\in A,\ |j|\le m$.

Let us recall the following Whitney Extension Theorem which plays a crucial role in our paper.

Theorem 1 ([8, p. 65]). *Let $n \in \mathbb{N}$ and a closed set $A \subset \mathbb{R}^n$ be fixed. If $f = \{f^j \in \mathbb{R}^A : j \in \mathbb{N}_0^n\}$ is a Whitney ∞–jet on A, then there exists a function $g : \mathbb{R}^n \to \mathbb{R}$ of the class C^∞ on $\mathbb{R}^n$ in the classical sense such that*

$$\frac{\partial^{|j|} g}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}(x) = f^j(x), \qquad x \in A, \quad j \in \mathbb{N}_0^n.$$

3. Main result

Let $J_i \subset \mathbb{R}$, $i = 1, \dots, n$, be open intervals. A set $J \subset \mathbb{R}^n$,

$$J = \prod_{i=1}^{n} J_i,$$

the Cartesian product of the intervals J_i, will be called an open interval in $\mathbb{R}^n$.

Now we are in a position to introduce the definition of a locally defined operators of the type $K : C^\infty(A) \to C^0(A)$.

Definition 3. *Let $n \in \mathbb{N}$ and a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed. An operator $K : C^\infty(A) \to C^0(A)$ is said to be locally defined if for every open interval $J \subset \mathbb{R}^n$ and for all $\varphi, \psi \in C^\infty(A)$,*

$$\varphi|_{A \cap J} = \psi|_{A \cap J} \Rightarrow K(\varphi)|_{A \cap J} = K(\psi)|_{A \cap J},$$

where $\varphi|_{A \cap J} := \{\varphi^j|_{A \cap J} : \ j \in \mathbb{N}_0^n\}$.

We shall need the following two lemmas.

Lemma 1. *Let a nonempty and closed set $A \subset \mathbb{R}^n$ be fixed and let $K : C^\infty(A) \to C^0(A)$ be a locally defined operator. Then for every $x_0 \in A$ and for all $\varphi, \psi \in C^\infty(A)$, if*

$$\varphi^j(x_0) = \psi^j(x_0) \quad \text{for all} \quad j \in \mathbb{N}_0^n,$$

then

$$K(\varphi)(x_0) = K(\psi)(x_0).$$

By the Whitney Extension Theorem (Theorem 1) and Remark 2, the proof of the lemma is very similar to the proof of Lemma 1 of [4] and will be omitted .

Lemma 2 ([9, p. 11]). *Let $n \in \mathbb{N}$, $\delta > 0$, and a compact set $C \subset \mathbb{R}^n$ be fixed. Then there exists a function $g \in C^\infty(\mathbb{R}^n)$ and nonnegative reals $C_{(k_1,\dots,k_n)}$, $k_1, \dots, k_n \in \mathbb{N}_0$, such that*

1. *a function g is nonnegative and*

$$g(x) = \begin{cases} 0, & \text{for} \quad (x_1,\ldots,x_n) \in C, \\ 1, & \text{for} \quad (x_1,\ldots,x_n) \ \textit{such that} \ d((x_1,\ldots,x_n),C) \geq \delta; \end{cases}$$

2. *for all* $k_1,\ldots,k_n \in \mathbb{N}_0$ *and* $(x_1,\ldots,x_n) \in \mathbb{R}^n$

$$\left| \frac{\partial^{|k|} g}{\partial x_1^{k_1} \ldots \partial x_n^{k_n}}(x_1,\ldots,x_n) \right| \leq \frac{c_{(k_1,\ldots,k_n)}}{\delta^{k_1+\ldots+k_n}}.$$

(Here $d((x_1,\ldots,x_n),C)$ denotes the distance from $x = (x_1,\ldots,x_n)$ to C, i.e.,

$$d((x_1,\ldots,x_n),C) = \inf\{\|(x_1-c_1,\ldots,x_n-c_n)\| : (c_1,\ldots,c_n) \in C\}.)$$

The main result of the present paper reads as follows:

Theorem 2. *Let* $A \subset \mathbb{R}^n$ *be a compact set. If an operator* $K : C^\infty(A) \to C^0(A)$ *is locally defined then there exists a unique function* $h : A \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ *such that*

$$K(\varphi)(x) = h\left(x, \varphi^{(0,\ldots,0)}(x), \varphi^{(1,\ldots,0)}(x), \ldots, \varphi^{(0,\ldots,1)}(x), \ldots\right)$$

for all $\varphi \in C^\infty(A)$ *and* $x = (x_1,\ldots,x_n) \in A$.

Proof. To simplify the notations we present the proof in the case $n = 2$. (In general case the proof goes along the same lines.)

To construct the function $h : A \times \mathbb{R}^{\mathbb{N}} \mapsto \mathbb{R}$ put

$$\delta := \operatorname{diam} A, \quad C := \operatorname{cl}B((0,0),1),$$

where $\operatorname{cl}B((0,0),1)$ denotes a closed ball centered at $(0,0)$ and with the radius $r = 1$ in the Euclidean space and $\operatorname{diam}A$ denotes the diameter of the set A.

By Lemma 2, there exists a nonnegative function $g_1 \in C^\infty(\mathbb{R}^2)$ and the constants $c_{(k_1,k_2)} \geq 0$; $k_1,k_2 \in \mathbb{N}_0$, such that

$$g_1(x_1,x_2) = \begin{cases} 0, & \text{for} \quad (x_1,x_2) \in C, \\ 1, & \text{for} \quad (x_1,x_2) \ \text{such that}\ d((x_1,x_2),C) \geq \delta \end{cases} \tag{1}$$

and for all $k_1,k_2 \in \mathbb{N}_0, (x_1,x_2) \in \mathbb{R}^2$,

$$\left| \frac{\partial^{k_1+k_2} g_1}{\partial x_1^{k_1} \partial x_2^{k_2}}(x_1,x_2) \right| \leq \frac{c_{(k_1,k_2)}}{\delta^{k_1+k_2}}. \tag{2}$$

Put

$$g := 1 - g_1$$

and

$$S_{(i_1,i_2)} := \max\left\{c_{(l_1,l_2)} : l_1 = 0,\dots,i_1-1;\ l_2 = 0,\dots,i_2-1\right\}, \tag{3}$$

for $i_1, i_2 \in \mathbb{N}_0$. Fix arbitrarily $(z_1,z_2) \in A$, $y_{(i_1,i_2)} \in \mathbb{R}$, $i_1,i_2 \in \mathbb{N}_0$, and choose $\lambda_{(i_1,i_2)} \in (-1,1)$, $i_1,i_2 \in \mathbb{N}_0$, such that for all $k_1 = 0,\dots,i_1-1$, $k_2 = 0,\dots,i_2-1$, the following inequalities

$$\left(1+\lambda_{(i_1,i_2)}\right)^{k_1+k_2} < \left(2^{i_1+i_2} S_{(i_1,i_2)} \left|y_{(i_1,i_2)}\right| \delta^{i_1+i_2-k_1-k_2}\right)^{-1} \tag{4}$$

hold true. Let us define the function $g_{(i_1,i_2)} : \mathbb{R}^2 \mapsto \mathbb{R},\ i_1,i_2 \in \mathbb{N}_0$, by the formula

$$g_{(i_1,i_2)}(x_1,x_2) := \frac{y_{(i_1,i_2)}}{i_1!i_2!}(x_1-z_1)^{i_1}(x_2-z_2)^{i_2} g\left(\lambda_{(i_1,i_2)}(x_1-z_1,x_2-z_2)\right). \tag{5}$$

Then for all $(x_1,x_2) \in \mathbb{R}^2$, $i_1,i_2 \in \mathbb{N}_0$, and $k_1 = 0,\dots,i_1-1$, $k_2 = 0,\dots,i_2-1$, we have

$$\left|\frac{\partial^{k_1+k_2} g_{(i_1,i_2)}}{\partial x_1^{k_1}\partial x_2^{k_2}}(x_1,x_2)\right| = \left|\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} y_{(i_1,i_2)}\binom{k_1}{l_1}\binom{k_2}{l_2}[(i_1-k_1+l_1)!(i_2-k_2+l_2)!]^{-1}\right.$$

$$\left.\cdot(x_1-z_1)^{i_1-k_1+l_1}(x_2-z_2)^{i_2-k_2+l_2}\lambda_{(i_1,i_2)}^{l_1+l_2}\frac{\partial^{l_1+l_2} g}{\partial x_1^{l_1}\partial x_2^{l_2}}\left(\lambda_{(i_1,i_2)}(x_1-z_1,x_2-z_2)\right)\right|$$

$$\leq \left|\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2} y_{(i_1,i_2)}\binom{k_1}{l_1}\binom{k_2}{l_2}(x_1-z_1)^{i_1-k_1+l_1}(x_2-z_2)^{i_2-k_2+l_2}\lambda_{(i_1,i_2)}^{l_1+l_2}\frac{\partial^{l_1+l_2} g}{\partial x_1^{l_1}\partial x_2^{l_2}}\right.$$

$$\left.\cdot\left(\lambda_{(i_1,i_2)}(x_1-z_1,x_2-z_2)\right)\right|.$$

Hence, by (1), (2) and (3), we get

$$\left|\frac{\partial^{k_1+k_2} g_{(i_1,i_2)}}{\partial x_1^{k_1}\partial x_2^{k_2}}(x_1,x_2)\right| \leq \left|y_{(i_1,i_2)}\right| \sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2}\delta^{i_1+i_2-k_1-k_2+l_1+l_2}\lambda_{(i_1,i_2)}^{l_1+l_2}\frac{c_{(l_1,l_2)}}{\delta^{l_1+l_2}}$$

$$\leq |y_{(i_1,i_2)}| S_{(i_1,i_2)}\delta^{i_1+i_2-k_1-k_2}\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_2}\binom{k_1}{l_1}\binom{k_2}{l_2}\lambda_{(i_1,i_2)}^{l_1+l_2}$$

$$= |y_{(i_1,i_2)}| S_{(i_1,i_2)}\delta^{i_1+i_2-k_1-k_2}\left(1+\lambda_{(i_1,i_2)}\right)^{k_1+k_2}$$

and finally, by (4),

$$\left|\frac{\partial^{k_1+k_2} g_{(i_1,i_2)}}{\partial x_1^{k_1}\partial x_2^{k_2}}(x_1,x_2)\right| < \frac{1}{2^{i_1+i_2}}, \quad k_1 = 0,\dots,i_1-1, \quad k_2 = 0,\dots,i_2-1.$$

Thus we have shown that for arbitrary $i_1,i_2 \in \mathbb{N}_0$, the function $g_{(i_1,i_2)}$ defined by (5) satisfies the inequalities

$$\left\|\frac{\partial^{k_1+k_2} g_{(i_1,i_2)}}{\partial x_1^{k_1}\partial x_2^{k_2}}\right\|_{\infty} < \frac{1}{2^{i_1+i_2}}, \quad k_1 = 0,\dots,i_1-1, \quad k_2 = 0,\dots,i_2-1,$$

where $\|\cdot\|_{\infty}$ denotes the supremum norm on $C^0(\mathbb{R}^2)$. It follows that the function $P_{z_1,z_2,y_{(0,0)},y_{(1,0)},\dots} : A \mapsto \mathbb{R}$ defined by

$$P_{z_1,z_2,y_{(0,0)},y_{(1,0)},\dots}(x_1,x_2) := \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty} g_{(i_1,i_2)}(x_1,x_2), \quad (x_1,x_2) \in A,$$

is of the class C^{∞} in the Whitney sense on the set A and (cf. [8, p. 408] and Remark 1)

$$\frac{\partial^{i_1+i_2} P_{z_1,z_2,y_{(0,0)},y_{(1,0)},\dots}}{\partial x_1^{i_1}\partial x_2^{i_2}}(z_1,z_2) = y_{(i_1,i_2)}, \quad i_1,i_2 \in \mathbb{N}_0.$$

Taking

$$h(z_1,z_2,y_{(0,0)},y_{(1,0)},\dots) := K(P_{z_1,z_2,y_{(0,0)},y_{(1,0)}},\dots)(z_1,z_2),$$

and applying Lemma 1, we infer that

$$K(\varphi)(z_1,z_2) = K\left(P_{z_1,z_2,\varphi^{(0,0)}(z_1,z_2),\varphi^{(1,0)}(z_1,z_2)},\dots\right)(z_1,z_2)$$

$$= h\left(z_1,z_2,\varphi^{(0,0)}(z_1,z_2),\varphi^{(1,0)}(z_1,z_2),\dots\right), \quad (z_1,z_2) \in A$$

for an arbitrary function $\varphi \in C^{\infty}(A)$, which proves the desired representation formula. Since the uniqueness of h is obvious, the proof is completed. ∎

Remark 4. Taking in the above theorem $n = 1$ and for $A \subset \mathbb{R}$ as a closed interval, one gets Theorem 2 of [3].

4. Conclusions

Applying the Whitney Extension Theorem, we prove that every operator with memory which acts the space of functions $\varphi = \{\varphi^j \in \mathbb{R}^A : j \in \mathbb{N}_0^n\}$ of the class C^{∞}

in the Whitney sense defined on a closed set $A \subset \mathbb{R}^n$ into the space of continuous functions on A is of the form

$$K(\varphi)(x) = h\left(x, \varphi^{(0,\ldots,0)}(x), \varphi^{(1,\ldots,0)}(x), \ldots, \varphi^{(0,\ldots,1)}(x), \ldots\right),$$

generated by some function $h : A \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. This generalizes one of the main results of [3].

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