

## SOLVING LINEAR FREDHOLM INTEGRO-DIFFERENTIAL EQUATION BY NYSTRÖM METHOD

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**Abstract.** The study of the solution's existence and uniqueness for the linear integro-differential Fredholm equation and the application of the Nyström method to approximate the solution is what we will present in this paper. We use the Neumann theorem to construct a sufficient condition that ensures the solution's existence and uniqueness of our problem in the Banach space  $C^1[a, b]$ . We have applied the Nyström method based on the trapezoidal rule to avoid adding other conditions in order to the approximation method's convergence. The Nyström method discretizes the integro-differential equation into solving a linear system. Only with the existence and uniqueness condition, we show the solution's existence and uniqueness of the linear system and the convergence of the numerical solution to the exact solution in infinite norm sense. We present two theorems to give a good estimate of the error. Also, to show the efficiency and accuracy of the Nyström method, some numerical examples will be provided at the end of this work.

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### 1. Introduction

The importance of the integro-differential equations is clearly demonstrated in physics, mathematical modelling, chemistry, biology, engineering, etc. [1–4]. This has led to the presence of different types and forms of equations. Therefore, an approximation and numerical processing is necessary. Most of the equations found, used and studied are the equations where the derivative of the unknown function is outside the integral sign, and scientists have built several methods to estimate the solution like the Homotopy perturbation method [5], Fractional order operational matrix methods the Adomain decomposition method [6] and another method using Legendre scaling functions [7,8], Embedded pseudo-Runge-Kutta methods [9],

Volterra-Runge-Kutta method [10], the iterative variational method [11] and the projection methods [12].

Contrary to existing research, the numerical method that we will present in this article is the Nyström method. However, this method was applied in the first state to approximate the solution of an integral equation and its theoretical framework is well given by Atkinson and Han in [13]. Recently, many others have relied on this method such as H. Guebbai et al. in [14, 15], H. Zhou and Q. Wang [16], S. Segni et al. [17], M. Ghiat et al. [18], S. Salah et al. [19], Lemita and Guebbai [20]: to give an estimation solution of the non-linear integro-differential Volterra equation with a continuous kernel, or a weakly singular kernel [21, 22] or to approximate a solution of a Fredholm linear system.

Our objective is to treat a new type of Fredholm integro-differential equation where the derivative is inside the integral sign. We construct an approximate solution based on the Nyström method, especially since this method in general is used for solving integral equations numerically. We focused on the study of this equation on two sides; the analytical and numerical side.

Let  $u \in C^1[a, b]$  be a solution of the following linear Fredholm integro-differential equation:

$$\forall x \in [a, b], \lambda u(x) = \int_a^b K_1(x, t)u(t) dt + \int_a^b K_2(x, t)u'(t) dt + f(x), \quad (1)$$

where  $\lambda \neq 0$  is a complex parameter,  $f \in C^1[a, b]$  and  $K_i$ , for  $i = 1, 2$  are given functions. We construct some conditions that ensures solution's existence and uniqueness of equation (1). We prove that those conditions are compatible with the Nystöme approximation method.

## 2. Analytical study

In the starting point of our work, we construct a condition that ensures the existence and uniqueness of the solution. Let  $X = C^1[a, b]$  be a Banach space with the norm:

$$\forall v \in X, \|v\|_X = \max_{a \leq x \leq b} |v(x)| + \max_{a \leq x \leq b} |v'(x)|.$$

We suppose that  $K_i$ , for  $i = 1, 2$  satisfied the next assumption:

$$(H_1) \left\| \begin{array}{l} \frac{\partial K_i}{\partial x}(x, t) \in C^0([a, b]^2, \mathbb{R}), \\ \max_{a \leq x, t \leq b} \left( |K_i(x, t)|, \left| \frac{\partial K}{\partial x_i}(x, t) \right| \right) \leq M_i. \end{array} \right\|$$

Using this assumption to define the linear operator  $T$  by:

$$T : X \longrightarrow X$$

$$v \longmapsto Tv(x) = \int_a^b K_1(x,t)v(t) dt + \int_a^b K_2(x,t)v'(t) dt, \quad \forall x \in [a,b]. \quad (2)$$

It is clear that,

$$\forall x \in [a,b], (Tv)'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)v(t) dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)v'(t) dt. \quad (3)$$

Then, the derivative  $u'$  is given implicitly by

$$\forall x \in [a,b], \lambda u'(x) = \int_a^b \frac{\partial K_1}{\partial x}(x,t)u(t) dt + \int_a^b \frac{\partial K_2}{\partial x}(x,t)u'(t) dt + f'(x). \quad (4)$$

**Theorem 1** Under the assumption  $(H_1)$  and if  $|\lambda| > 2(b-a)(M_1 + M_2)$ , then the equation (1) has a unique solution in  $X$ .  $\square$

PROOF To prove the solution's existence and uniqueness for (1), we write it under the form

$$(\lambda I - T)u = f. \quad (5)$$

We will demonstrate that  $(\lambda I - T)^{-1}$  exists and is bounded. First of all, we define  $\|T\|$  by

$$\|T\| = \sup_{\|u\|_X \leq 1} \|Tu\|_X. \quad (6)$$

Such that,

$$\|Tu\|_X = \max_{a \leq x \leq b} |Tu(x)| + \max_{a \leq x \leq b} |(Tu)'(x)|. \quad (7)$$

However, we have

$$\begin{aligned} |Tu(x)| &\leq \int_a^b |K_1(x,t)| |u(t)| dt + \int_a^b |K_2(x,t)| |u'(t)| dt, \\ &\leq (b-a)M_1 \max_{a \leq t \leq b} |u(t)| + (b-a)M_2 \max_{a \leq t \leq b} |u'(t)|, \\ &\leq (b-a)(M_1 + M_2)\|u\|_X, \end{aligned} \quad (8)$$

and

$$\begin{aligned} |(Tu)'(x)| &\leq \int_a^b \left| \frac{\partial K_1}{\partial x}(x,t) \right| |u(t)| dt + \int_a^b \left| \frac{\partial K_2}{\partial x}(x,t) \right| |u'(t)| dt, \\ &\leq (b-a)M_1 \max_{a \leq t \leq b} |u(t)| + (b-a)M_2 \max_{a \leq t \leq b} |u'(t)|, \\ &\leq (b-a)(M_1 + M_2)\|u\|_X. \end{aligned} \quad (9)$$

Substituting (8) and (9) in (7), we obtain

$$\|Tu\|_X \leq 2(b-a)(M_1 + M_2)\|u\|_X, \quad (10)$$

which implies that,

$$\|T\| \leq 2(b-a)(M_1 + M_2). \quad (11)$$

Using the fact that  $|\lambda| > 2(b-a)(M_1 + M_2)$ , we get  $\|T\| < |\lambda|$ . Therefore, according to theorem 1.2.11 in [23]  $sp(T) \subseteq \overline{B(0, \|T\|)}$ , where  $sp(T)$  is the spectrum set of  $T$ . Then

$$\left\{ \lambda \in \mathbb{C}^*, |\lambda| > 2(b-a)(M_1 + M_2) \right\} \subseteq re(T),$$

where,  $re(T)$  is the resolvent set of  $T$ . We select the Nuemann's theorem [24] to confirm that  $(\lambda I - t)^{-1}$  exists and

$$\|(\lambda I - T)^{-1}\| \leq \frac{1}{|\lambda| - \|T\|} \leq \frac{1}{|\lambda| - 2(b-a)(M_1 + M_2)}. \quad (12)$$

### 3. Numerical study ■

To search for an approximation solution of the system (1) and (4), we use the Nyström method which is based on the following numerical integration scheme:

$$\forall n \geq 1, \forall \phi \in C^0[a, b], \quad \int_a^b \phi(x) dx \approx \sum_{i=0}^n \omega_i \phi(x_i), \quad (13)$$

where,  $\{\omega_i\}_{0 \leq i \leq n}$  called quadrature weights such that  $W = \sup_{n \geq 1} \sum_{i=0}^n |\omega_i| < \infty$ ,

$h = \frac{(b-a)}{n}$  and  $x_i = a + ih$ , for  $i = 0, 1, 2, \dots, n$ , are points discretization of the interval  $[a, b]$ .

Applying (13) to the integral term of the system (1)-(4), we obtain

$$\begin{cases} \lambda u(x) = \sum_{j=0}^n \omega_j K_1(x, x_j) u(x_j) + \sum_{j=0}^n \omega_j K_2(x, x_j) u'(x_j) + f(x) + e_{1,n}(x), \\ \lambda u'(x) = \sum_{j=0}^n \omega_j \frac{\partial K_1}{\partial x}(x, x_j) u(x_j) + \sum_{j=0}^n \omega_j \frac{\partial K_2}{\partial x}(x, x_j) u'(x_j) + f'(x) + e_{2,n}(x), \end{cases}$$

where, local errors are given by:

$$\begin{aligned} e_{1,n}(x) &= \int_a^b \left[ K_1(x, t) u(t) + K_2(x, t) u'(t) \right] dt \\ &\quad - \sum_{j=0}^n \omega_j \left[ K_1(x, x_j) u(x_j) - K_2(x, x_j) u'(x_j) \right], \end{aligned} \quad (14)$$

$$e_{2,n}(x) = \int_a^b \left[ \frac{\partial K_1}{\partial x}(x,t)u(t) + \frac{\partial K_2}{\partial x}(x,t)u'(t) \right] dt - \sum_{j=0}^n \omega_j \left[ \frac{\partial K_1}{\partial x}(x,x_j)u(x_j) - \frac{\partial K_2}{\partial x}(x,x_j)u'(x_j) \right]. \quad (15)$$

We set  $x = x_i$  and we suppose that  $\forall x \in [a, b]$ ,  $\{e_{p,n}(x)\}_{1 \leq p \leq 2}$  are negligible to get the following linear approximation system with  $2n + 2$  equations:

$$\begin{cases} \lambda u_i = \sum_{j=0}^n \omega_j K_1(x_i, x_j) u_j + \sum_{j=0}^n \omega_j K_2(x_i, x_j) u'_j + f_i, \\ \lambda u'_i = \sum_{j=0}^n \omega_j \frac{\partial K_1}{\partial x}(x_i, x_j) u_j + \sum_{j=0}^n \omega_j \frac{\partial K_2}{\partial x}(x_i, x_j) u'_j + f'_j, \end{cases} \quad (16)$$

where  $f_i = f(x_i)$ ,  $f'_i = f'(x_i)$  and  $u_i, u'_i$  are the approximations of  $u(x_i)$  and  $u'(x_i)$  respectively.

### 3.1. System analysis

**Theorem 2** *If  $|\lambda| > 2W(M_1 + M_2)$ , then the system (16) has a unique solution.  $\square$*

PROOF To demonstrate the existence of the system solution (16), we apply the Banach fixed point theorem. For this reason, we write the system (16) in the following form:

$$U = A[U].$$

where,  $U = (u_0, u_1, \dots, u_n, u'_0, u'_1, \dots, u'_n)^t$  is a vector of  $\mathbb{R}^{2n+2}$  and

$$A[U] = \begin{cases} \frac{1}{\lambda} \left[ \sum_{j=0}^n \omega_j K_1(x_i, x_j) u_j + \sum_{j=0}^n \omega_j K_2(x_i, x_j) u'_j + f_i \right], & 0 \leq i \leq n, \\ \frac{1}{\lambda} \left[ \sum_{j=0}^n \omega_j \frac{\partial K_1}{\partial x}(x_{i-n-1}, x_j) u_j + \sum_{j=0}^n \omega_j \frac{\partial K_2}{\partial x}(x_{i-n-1}, x_j) u'_j + f'_i \right], & n+1 \leq i \leq 2n+2. \end{cases}$$

For  $\mathbb{R}^{2n+2}$ , we use the following norm:

$$\|U\|_{\mathbb{R}^{2n+2}} = \max_{0 \leq i \leq n} |u_i| + \max_{0 \leq i \leq n} |u'_i|.$$

Let  $U$  and  $V \in \mathbb{R}^{2n+2}$ , then

$$\|A[U] - A[V]\|_{\mathbb{R}^{2n+2}} \leq \frac{2W(M_1 + M_2)}{|\lambda|} \|U - V\|_{\mathbb{R}^{2n+2}},$$

Using the fact that  $|\lambda| > 2W(M_1 + M_2)$  and the Banach fixed point theorem, we conclude that the system (16) has a unique solution. ■

### 3.2. Error analysis

Our goal is to choose a numerical quadrature that provides that the error estimation converges to 0. We select the formula of trapezoidal rules to ensure this convergence, see [13] and [25]. It also provides  $\sum_{j=0}^n |\omega_j| = (b - a)$ , which makes the analytical and numerical study compatible. In this rule, we have  $\forall n \geq 1$

$$\left\| \begin{array}{l} h = \frac{(b-a)}{n}, \\ \omega_0 = \omega_n = \frac{h}{2}, \\ \omega_1 = \omega_2 = \dots = \omega_{n-1} = h, \\ x_i = a + (i-1)h, \quad i = 0, 1, \dots, n. \end{array} \right.$$

First, we define the continuity module  $\kappa_0$  by

$$\forall h > 0, \forall v \in C^0[a, b], \kappa_0(v, h) = \sup_{|x-y| \leq h} |v(x) - v(y)|,$$

and the continuity module  $\kappa_1$  as follows

$$\forall h > 0, \forall v \in C^1[a, b], \kappa_1(v, h) = \kappa_0(v, h) + \kappa_0(v', h),$$

**Theorem 3** Let  $e_n = \left( e_{1,n}(x_0), e_{1,n}(x_1), \dots, e_{1,n}(x_n), e_{2,n}(x_0), e_{2,n}(x_1), \dots, e_{2,n}(x_n) \right)^t$  be a vector in  $\mathbb{R}^{2n+2}$ , then:

$$\|e_n\|_{\mathbb{R}^{2n+2}} \leq (b-a) \sum_{p=1}^2 M_p \left[ \max_{0 \leq i \leq n} \kappa_1(K_{p,x_i}, h) \|u\|_X + 2 \kappa_1(u, h) \right], \quad (17)$$

where,  $K_{p,x_i} = K_p(x_i, \cdot)$ , for  $p = 1, 2$  and  $i = 0, 1, \dots, n$ . □

PROOF For  $i = 0, 1, \dots, n$ , we have

$$\begin{aligned} |e_{1,n}(x_i)| &= \left| \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} K_1(x_i, t) u(t) dt - \frac{h}{2} \left[ K_1(x_i, x_{j+1}) u(x_{j+1}) + K_1(x_i, x_j) u(x_j) \right] \right. \\ &\quad \left. + \int_{x_j}^{x_{j+1}} K_2(x_i, t) u'(t) dt - \frac{h}{2} \left[ K_2(x_i, x_{j+1}) u'(x_{j+1}) + K_2(x_i, x_j) u'(x_j) \right] \right|, \\ |e_{2,n}(x_i)| &= \left| \sum_{j=0}^{n-1} \int_{x_j}^{x_{j+1}} \frac{\partial K_1}{\partial x}(x_i, t) u(t) dt - \frac{h}{2} \left[ \frac{\partial K_1}{\partial x}(x_i, x_{j+1}) u(x_{j+1}) + \frac{\partial K_1}{\partial x}(x_i, x_j) u(x_j) \right] \right. \\ &\quad \left. + \int_{x_j}^{x_{j+1}} \frac{\partial K_2}{\partial x}(x_i, t) u'(t) dt - \frac{h}{2} \left[ \frac{\partial K_2}{\partial x}(x_i, x_{j+1}) u'(x_{j+1}) + \frac{\partial K_2}{\partial x}(x_i, x_j) u'(x_j) \right] \right|. \end{aligned}$$

But for all  $t \in [x_j, x_{j+1}]$ ,

$$\begin{aligned}
& \left| \int_{x_j}^{x_{j+1}} K_1(x_i, t)u(t) dt - \frac{h}{2} \left[ K_1(x_i, x_{j+1})u(x_{j+1}) + K_1(x_i, x_j)u(x_j) \right] \right| \\
& \leq h \left[ \max_{0 \leq i \leq n} \kappa_0(K_{1_{x_i}}, h) \|u\|_X + M_1 \kappa_0(u, h) \right], \\
& \left| \int_{x_j}^{x_{j+1}} K_2(x_i, t)u'(t) dt - \frac{h}{2} \left[ K_2(x_i, x_{j+1})u'(x_{j+1}) + K_2(x_i, x_j)u'(x_j) \right] \right| \\
& \leq h \left[ \max_{0 \leq i \leq n} \kappa_0(K_{2_{x_i}}, h) \|u\|_X + M_2 \kappa_0(u', h) \right], \\
& \left| \int_{x_j}^{x_{j+1}} \frac{\partial K_1}{\partial x}(x_i, t)u(t) dt - \frac{h}{2} \left[ \frac{\partial K_1}{\partial x}(x_i, x_{j+1})u(x_{j+1}) + \frac{\partial K_1}{\partial x}(x_i, x_j)u(x_j) \right] \right| \\
& \leq h \left[ \max_{0 \leq i \leq n} \kappa_0 \left( \frac{\partial K_{1_{x_i}}}{\partial x}, h \right) \|u\|_X + M_1 \kappa_0(u, h) \right], \\
& \left| \int_{x_j}^{x_{j+1}} \frac{\partial K_2}{\partial x}(x_i, t)u'(t) dt - \frac{h}{2} \left[ \frac{\partial K_2}{\partial x}(x_i, x_{j+1})u'(x_{j+1}) + \frac{\partial K_2}{\partial x}(x_i, x_j)u'(x_j) \right] \right| \\
& \leq h \left[ \max_{0 \leq i \leq n} \kappa_0 \left( \frac{\partial K_{2_{x_i}}}{\partial x}, h \right) \|u\|_X + M_2 \kappa_0(u', h) \right].
\end{aligned}$$

Which gives

$$\begin{aligned}
|e_{1,n}(x_i)| & \leq (b-a) \left[ M_1 \max_{0 \leq i \leq n} \kappa_0(K_{1_{x_i}}, h) + M_2 \max_{0 \leq i \leq n} \kappa_0(K_{2_{x_i}}, h) \right] \|u\|_X \\
& \quad + (b-a)(M_1 + M_2) \kappa_1(u, h), \\
|e_{2,n}(x_i)| & \leq (b-a) \left[ M_1 \max_{0 \leq i \leq n} \kappa_0 \left( \frac{\partial K_{1_{x_i}}}{\partial x}, h \right) + M_2 \max_{0 \leq i \leq n} \kappa_0 \left( \frac{\partial K_{2_{x_i}}}{\partial x}, h \right) \right] \|u\|_X \\
& \quad + (b-a)(M_1 + M_2) \kappa_1(u, h).
\end{aligned}$$

Finally

$$\begin{aligned}
& \max_{0 \leq i \leq n} |e_{1,n}(x_i)| + \max_{0 \leq i \leq n} |e_{2,n}(x_i)| \\
& \leq (b-a) \left[ M_1 \max_{0 \leq i \leq n} \kappa_1(K_{1_{x_i}}, h) + M_2 \max_{0 \leq i \leq n} \kappa_1(K_{2_{x_i}}, h) \right] \|u\|_X \\
& \quad + 2(b-a)(M_1 + M_2) \kappa_1(u, h).
\end{aligned}$$

Then

$$\|e_n\|_{\mathbb{R}^{2n+2}} \leq (b-a) \sum_{p=1}^2 M_p \left[ \max_{0 \leq i \leq n} \kappa_1(K_{p_{x_i}}, h) \|u\|_X + 2 \kappa_1(u, h) \right]. \quad \blacksquare$$

To demonstrate the effectiveness of our applied method and to prove the convergence of the approached solution, we will state the following theorem:

**Theorem 4** Let  $\varepsilon_n$ , the error defined by:

$$\varepsilon_n = \max_{0 \leq i \leq n} |u(x_i) - u_i| + \max_{0 \leq i \leq n} |u'(x_i) - u'_i|, \quad (18)$$

Assuming that  $|\lambda| > 2(b-a)(M_1 + M_2)$ . Then  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . □

PROOF we have

$$|\lambda| |u(x_i) - u_i| \leq |e_{1,n}(x_i)| + (b-a)(M_1 + M_2)\varepsilon_n, \quad (19)$$

$$|\lambda| |u'(x_i) - u'_i| \leq |e_{2,n}(x_i)| + (b-a)(M_1 + M_2)\varepsilon_n. \quad (20)$$

For (19)-(20), we obtain:

$$\varepsilon_n \leq \frac{b-a}{|\lambda| - 2(b-a)(M_1 + M_2)} \|e_n\|_{\mathbb{R}^{2n+2}}. \quad (21)$$

When  $n \rightarrow \infty$ ,  $h = \frac{b-a}{n} \rightarrow 0$  and by theorem (3)  $\lim_{n \rightarrow \infty} \|e_n\|_{\mathbb{R}^{2n+2}} = 0$ , then  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$ . ■

#### 4. Numerical examples

To show the effectiveness of the suggested methods, we apply this approach to the following numerical example:

$$\forall x \in [0, 1], \lambda u(x) = \int_0^1 \frac{u(t)}{\sqrt{2+t^3 + \frac{1}{1+x}}} dt + \int_0^1 \frac{u'(t)}{x+t^4+1} dt + f(x), \quad (22)$$

where,  $\lambda = 4$ ,  $u(x) = x^2$  and

$$f(x) = 4x^2 - \frac{\arctan\left(\frac{1}{\sqrt{x+1}}\right)}{\sqrt{x+1}} - \frac{2}{3} \left( \sqrt{3 + \frac{1}{1+x}} - \sqrt{2 + \frac{1}{1+x}} \right). \text{ We also have}$$

$$\max_{0 \leq x, t \leq 1} \left( |K_1(x, t)|, \left| \frac{\partial K_1}{\partial x}(x, t) \right| \right) \leq \frac{4}{5} \text{ and } \max_{0 \leq x, t \leq 1} \left( |K_2(x, t)|, \left| \frac{\partial K_2}{\partial x}(x, t) \right| \right) \leq 1.$$

Then,  $|\lambda| = 4 > \frac{18}{5}$ .

We give another example

$$\forall x \in [0, 1], \lambda u(x) = \int_0^1 \frac{u(t)}{e^x + e^t} dt + \int_0^1 \frac{u'(t)}{1+x+e^{2t}} dt + f(x), \quad (23)$$

where,  $\lambda = 2$ ,  $u(x) = e^x$  and



Table 1. The error between the exact and approximation solution of equation (22)

n	10	100	500	1000
$\max_{0 \leq i \leq n}  u(x_i) - u_i $	7.0610e-05	7.0506e-07	2.8202e-08	7.0505e-09
$\max_{0 \leq i \leq n}  u'(x_i) - u'_i $	1.6370e-04	1.6353e-06	6.5410e-08	1.6352e-08
$\epsilon_n$	2.3431e-04	2.3403e-06	9.3612e-08	2.3403e-08
time (seconds)	0.023258	0.049113	0.774680	3.641881

$f(x) = \log(e^x + 1) - \log(e^1 + e^x) + 2e^x - \arctan\left(\frac{e^1}{\sqrt{x+1}}\right) - \frac{\arctan\left(\frac{1}{\sqrt{x+1}}\right)}{\sqrt{x+1}}$ . We prove that

$$\max_{0 \leq x, t \leq 1} \left( |K_1(x, t)|, \left| \frac{\partial K_1}{\partial x}(x, t) \right| \right) \leq \frac{1}{3} \text{ and } \max_{0 \leq x, t \leq 1} \left( |K_2(x, t)|, \left| \frac{\partial K_2}{\partial x}(x, t) \right| \right) \leq \frac{1}{2}.$$

Then,  $|\lambda| = 2 > \frac{10}{6}$ .

Table 2. The error between the exact and approximation solution of equation (23)

n	10	100	500	1000
$\max_{0 \leq i \leq n}  u(x_i) - u_i $	1.0479e-04	1.0468e-06	4.1871e-08	1.0468e-08
$\max_{0 \leq i \leq n}  u'(x_i) - u'_i $	5.8898e-05	5.8873e-07	2.3549e-08	5.8873e-09
$\epsilon_n$	1.6368e-04	1.6355e-06	6.5420e-08	1.6355e-08
time (seconds)	0.071628	0.048551	0.502111	3.834747

We applied the method proposed to approximate the solution of (22) and (23), and we calculated the error between the approximation and exact solution which is given by:

$$\epsilon_n = \max_{0 \leq i \leq n} |u(x_i) - u_i| + \max_{0 \leq i \leq n} |u'(x_i) - u'_i|.$$

Table 1 presented the numerical results of equation (22) and Table 2 presented the numerical results of equation (23).

For better observation on the effectiveness of our proposed method, we plot the approximate and exact solutions of (22) in Figure 1 and the approximate and exact solutions of (23) in Figure 2 in different collocation points  $x_i$ . Figures 1 and 2 clearly prove the accuracy of our numerical method proposed in this paper.

## 5. Conclusion

In this article, we have studied a different integro-differential equation than the existing ones. We proposed the Nyström technique to obtain an approach solution of linear Fredholm integro-differential equation. This method helped us to reformulate an equation to a linear system by block. Before applying this estimation, we posed some assumptions at the beginning to provide the existence and uniqueness of the

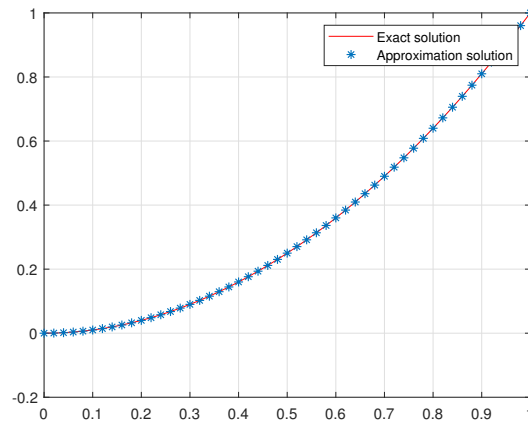


Fig. 1. The approximate and exact solutions with the Nyström method for equation (22),  $n = 50$

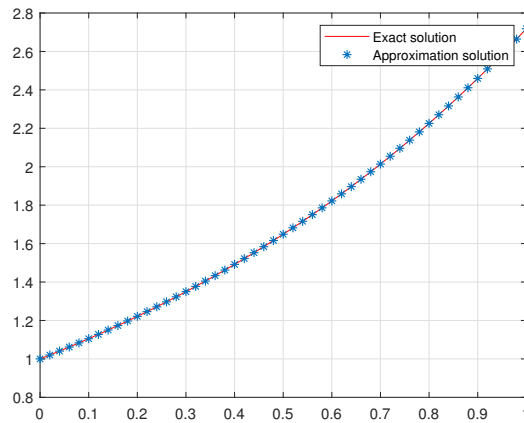


Fig. 2. The approximate and exact solutions with the Nyström method for equation (23),  $n = 50$

solution on the one hand and on the other hand, to affirm that the approximate solution converges to the exact solution in the space  $C^1[a, b]$ . We have constructed an error bound of the Nyström method for a function in  $C^1[a, b]$ . At the end, the comparison of the numerical results has shown that the method presented is an interesting tool for finding a numerical solution to the linear Fredholm's integro-differential equation.

As perspectives, we will apply the iterative methods to avoid the calculation of inverse matrix of system (16), especially when the size of this matrix is large. Other works in progress are developed around these kind of equations. We are developing an analytical and numerical study on the equation (1) in the Sobolev space  $W^{1,1}(a, b)$ . Also, we have the possibility to apply a new projection method based on Legendre polynomials in the Sobolev space  $H^1(a, b)$  or cubic b-spline functions and we take as a consideration all different cases of the kernels.

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